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A FIXED POINT THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Let H be a real Hilbert space, C a closed convex subset of H, T a selfmapping of C. Let $A_n x$ denote the nth term of the Cesàro transform of the sequence of iterates $\{T^k x\}$. Baillon [1] proved that, if T is a nonexpansive selfmapping of C which has a fixed point, then $\{A_n x\}$ converges weakly to a fixed point of T. This result has been extended to strongly regular matrices by Brézis and Browder [2], Bruck [3], and Reich [9]. In a recent paper [6] the Baillon result was extended to asymptotically nonexpansive mappings. In this paper the result of [6] is extended to a wide class of matrix methods.

A mapping T is said to be asymptotically nonexpansive over C if, for each $x, y \in C$,

$$\|T^i x - T^i y\| \leqq (1+\alpha_i)\|x-y\|, \qquad i=1, 2, \cdots, \tag{1}$$

where $\lim_i \alpha_i = 0$.

An infinite matrix $A=(a_{nk})$ is called regular if it is limit-preserving over c, the space of convergent sequences. Necessary and sufficient conditions for regularity are: (i) $\|A\|=\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$; (ii) $\lim_n a_{nk}=0$ for $k=0, 1, 2, \cdots$, and (iii) $\lim_n t_n = 1$, where $t_n=\sum_{k=0}^{\infty} a_{nk}$. Let X be a locally convex space. A sequence $\{x_n\} \subset X$ is said to be almost convergent, written ac, if there exists a point $s \in X$ such that $\lim_n \sum_{k=0}^{n-1} x_{k+i}/n = s$, uniformly in i. A matrix A will be called strongly regular if, in addition to satisfying conditions (i) and (iii) for regularity, it also satisfies (ii′) $\lim_n \sum_k |a_{nk} - a_{n,k+1}| = 0$. A is called triangular if all its entries above the main diagonal are zero.

THEOREM. *Let C be a closed convex subset of a real Hilbert space H, T an asymptotically nonexpansive selfmap of C such that $\{T^n z\}$ is bounded for each $z \in C$. Let A be a strongly regular matrix. Define $A_n x = \sum_{k=0}^{\infty} a_{nk} T^k x$. Then, for each $x \in C$, $\{A_n x\}$ converges weakly to a fixed point p, which is the asymptotic center of $\{T^n x\}$.*

The proofs of Lemmas 2 and 3 of [6] are independent of the matrix A involved. So, to prove the Theorem, it is sufficient to show that Lemma 1 of [6] is true for each strongly regular matrix A; i.e., there exists a positive integer

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K_0 such that, for each $m \geqq K_0$, there exists a positive integer N_m satisfying

(2) $$\|A_n x - T^m A_n x\| < \varepsilon \qquad \text{for all } n \geqq N_m .$$

A regular matrix A will be called a hump matrix if each row contains a hump, and the heights of the humps tend to zero; i.e., for each n there exists an integer $p(n)$ such that $a_{nk} \leqq a_{n,k+1}$ for $0 \leqq k < p$ and $a_{nk} \geqq a_{n,k+1}$ for $k \geqq p$, and $\lim_n \sup_k |a_{nk}| = 0$.

The proof of (2) will make use of the following Lemma, which is an improvement of Lemma 1.1 of [3].

LEMMA. Let X *be a sequentially complete space. Then the following are equivalent:*

(i) *A sequence* $x \in X$ *is ac,*
(ii) $\lim_n \sum_k a_{nk} x_k$ *exists for every strongly regular matrix* A,
(iii) $\lim_n \sum_k a_{nk} x_k$ *exists for every regular hump matrix* A,
(iv) $\lim_n \sum_k a_{nk} x_k$ *exists for every regular trianglar hump matrix* A *with nonnegative entries and row sums one.*

The equivalence of (i) and (ii) comes from [3]. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear, since a regular hump matrix is also strongly regular. In [10] it was shown that (iii) $\Rightarrow$ (i), but the proof there uses matrices satisfying (iv). Therefore (iv) $\Rightarrow$ (i).

From the Lemma, it is sufficient to establish (2) for triangular regular hump matrices with nonnegative entries and row sums one.

For any $u \in H$,

$$\|A_n x - u\|^2 = \|\sum_{k=0}^{n} a_{nk} T^k x - u\|^2 = \|\sum_{k=0}^{n} a_{nk}(T^k x - u)\|^2$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{n} a_{nk} a_{nj} (T^k x - u,\ T^j x - u) .$$

Since H is a real Hilbert space, $2(T^k x - u,\ T^j x - u) = \|T^k x - u\|^2 + \|T^j x - u\|^2 - \|T^k x - T^j x\|^2$, so that

(3) $$2\|A_n x - u\|^2 = 2\sum_{k=0}^{n} a_{nk} \|T^k x - u\|^2 - \sum_{k=0}^{n} \sum_{j=0}^{n} a_{nk} a_{nj} \|T^k x - T^j x\|^2 .$$

Substituting $u = A_n x$ in (3) yields

(4) $$\sum_{k=0}^{n} \sum_{j=0}^{n} a_{nk} a_{nj} \|T^k x - T^j x\|^2 = 2\sum_{k=0}^{n} a_{nk} \|T^k x - A_n x\|^2 .$$

Put (4) in (3) and set $u = T^k A_n x$ to get, for $k \leqq n$,

$$\|A_nx-T^kA_nx\|^2=\sum_{j=0}^{k-1}a_{nj}\|T^jx-T^kA_nx\|^2$$

$$+\sum_{j=k}^{n}a_{nj}\|T^jx-T^kA_nx\|^2-\sum_{j=0}^{n}a_{nj}\|T^jx-A_nx\|^2 .$$

Using (1), and the fact that $a_{nk}\geqq 0$, it then follows that

$$\|A_nx-T^kA_nx\|^2\leqq\sum_{j=0}^{k-1}a_{nj}\|T^jx-T^kA_nx\|^2 \tag{5}$$

$$+(2\alpha_k+\alpha_k{}^2)\sum_{j=0}^{n-k}a_{n,j+k}\|T^jx-A_nx\|^2$$

$$+\sum_{j=0}^{n-k}(a_{n,j+k}-a_{nj})\|T^jx-A_nx\|^2$$

$$=I_1+I_2+I_3 ,\qquad \text{say} .$$

By hypothesis $\{T^jx\}$ is bounded for each $x\in C$. Let $M=\sup\{\|T^jx\| : j=0, 1, 2, \cdots\}$. Since A is nonnegative with row sums one, $\|A_nx\|\leqq M$.

Since A is a hump matrix, $a_{n,j+1}-a_{nj}\leqq 0$ for $j\geqq p$.

An estimate will first be found for I_3.

Case I. Suppose $p\leqq k<n-k$. Then

$$I_3\leqq\sum_{j=0}^{p}(a_{n,j+k}-a_{nj})\|T^jx-A_nx\|^2$$

$$\leqq\sum_{j=0}^{p}(a_{np}-a_{n0})\|T^jx-A_nx)\|^2\leqq 4M^2ka_{np} .$$

Case II. $k<p\leqq n-k$. Then

$$I_3\leqq\sum_{j=0}^{p-k}(a_{n,j+k}-a_{nj})\|T^jx-A_nx\|^2+\sum_{j=p-k+1}^{p}(a_{n,j+k}-a_{np}+a_{np}-a_{nj})\|T^jx-A_nx\|^2$$

$$\leqq 4M^2[\sum_{j=0}^{p-k}(a_{n,j+k}-a_{nj})+0+ka_{np}-\sum_{j=p-k+1}^{p}a_{nj}]$$

$$=4M^2[\sum_{j=k}^{p}a_{nj}-\sum_{j=0}^{p}a_{nj}+ka_{np}]=4M^2[-\sum_{j=0}^{k-1}a_{nj}+ka_{np}]\leqq 4M^2ka_{np} .$$

Case III. $n-k<p\leqq n$. Then

$$I_3=\sum_{j=0}^{p-k}(a_{n,j+k}-a_{nj})\|T^jx-A_nx\|^2+\sum_{j=p-k+1}^{n-k}(a_{n,j+k}-a_{np}+a_{np}-a_{nj})\|T^jx-A_nx\|^2 .$$

Proceeding as in Case II again yields $I_3\leqq 4M^2ka_{np}$.

Case IV. $p \leqq n-k<k$. Then

$$I_3 \leqq \sum_{j=0}^{p} (a_{n,j+k}-a_{nj})\|T^j x - A_n x\|^2,$$

and, following the same argument as Case II, one obtains $I_3 \leqq 4M^2 k a_{np}$.

Case V. $n-k<p\leqq k$. Then

$$I_3 = \sum_{j=0}^{n-k} (a_{n,j+k}-a_{np}+a_{np}-a_{nj})\|T^j x - A_n x\|^2$$

$$\leqq 4M^2[0+(n-k+1)a_{np} - \sum_{j=0}^{n-k} a_{nj}] \leqq 4M^2 k a_{np}.$$

Case VI. $n-k<k<p\leqq n$. Then I_3 is the same as in Case V. Thus, in all cases,

$$I_3 \leqq 4M^2[\sum_{j=0}^{k-1} a_{nj} + k a_{np}].$$

Now fix $\varepsilon>0$, and choose K_0 so that, for all $k \geqq K_0$, $2\alpha_k+\alpha_k^2<\varepsilon^2/12M^2$. Then, for all $k\geqq K_0$,

$$I_2 \leqq 4(2\alpha_k+\alpha_k^2)M^2 \sum_{j=0}^{n} a_{n,j+k} \leqq \varepsilon^2/3.$$

Since A is a regular hump matrix, $\lim_n a_{np}=0$ and A has a zero column limits. For each $m\geqq K_0$ choose N_m so that, for $n\geqq N_m$, $I_1<\varepsilon^2/3$ and $I_3<\varepsilon^2/3$, and the theorem is proved.

Remarks 1. A closed convex subset C of a real Hilbert space H is said to have the fixed point property for a family F of selfmaps of C if, for every $T\in F$, T has a fixed point. Ray [8] has shown that C has the fixed point property for nonexpansive maps if and only if C is bounded. This result is extendable to asymptotically nonexpansive mappings. The necessity follows by using the same example as in [8], since every nonexpansive mapping is asymptotically nonexpansive. For the sufficiency, assume that C is bounded. Then, from [4, Theorem 1], T has a fixed point in C. Consequently, the hypothesis, in the Theorem of this paper, that $\{T^n z\}$ be bounded for $z\in C$ is a natural and necessary one.

2. There are many strongly regular nonnegative matrices for which the Theorem applies. For example the Cesàro matrices of order $\alpha>0$; the Euler matrices; all Nörlund matrices with $p_{n+1}\geqq p_n$; and all weighted mean matrices with $p_{n+1}\leqq p_n$; all Nörlund matrices with $p_{n+1}\leqq p_n$ and $P_n\to\infty$; and by all weighted mean methods with $p_{n+1}\geqq p_n$ and $\lim_n p_n/P_n=0$. For definitions and basic properties of these methods the reader may consult [5]. The Chebyshev method (see [7] also satisfies the Theorem.

3. Since T is not assumed to be linear, one obtains a collection of nonlinear

Ergodic theorems by simply adding the restriction that $t_n=1$ for each n.

4. Theorem 3 of [6] has a natural extension for integral operators.

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