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AN EXTREMAL PROBLEM ON THE CLASSICAL CARTAN DOMAINS

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1. This paper is concerned with the following extremal problem: Let D be a bounded domain in the $2n$-dimensional Euclidean space $\boldsymbol{C}^n$ of n complex variables $z=(z_1, \cdots, z_n)$. Denote by $\mathcal{F}(D)$ the family of holomorphic mappings from D into the unit hyperball B_n in $\boldsymbol{C}^n$. It is required to find the precise value

$$M(z_0, D)=\sup_{f\in\mathcal{F}(D)}\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=z_0}\right| \qquad (z_0\in D),$$

where $\left(\frac{\partial f}{\partial z}\right)$ denotes the Jacobian matrix of f:

$$\left(\frac{\partial f}{\partial z}\right)=\begin{pmatrix} \frac{\partial f_1}{\partial z_1}\cdots\frac{\partial f_1}{\partial z_n} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial f_n}{\partial z_1}\cdots\frac{\partial f_n}{\partial z_n}\end{pmatrix}, \quad f=(f_1, \cdots, f_n).$$

If $w=h(z)$ is a biholomorphic mapping from D_1 onto D_2 and $w_0=h(z_0)$, then

$$M(z_0, D_1)=M(w_0, D_2)\left|\det\left(\frac{\partial h}{\partial z}\right)_{z=z_0}\right|,$$

namely, the quantity $M(z, D)$ is a relative invariant. Hence for a bounded homogeneous domain D it is sufficient to find the value $M(z_0, D)$ for a fixed point z_0 in D.

The automorphism of B_n which transforms a point $a=(a_1, \cdots, a_n)$ into the origin is given in the form

$$\varphi(z: a)=\mu(z-a)(I-\bar{a}'z)^{-1}U^{-1},$$

where $|\mu|^2=(1-a\bar{a}')^{-1}$ and $U'\bar{U}=(I-a'\bar{a})^{-1}$. Here I is the identity matrix and $\bar{A}$ denotes the conjugate matrix of A and A' the transposed matrix of A. Since

$$\left|\det\left(\frac{\partial\varphi}{\partial z}\right)_{z=a}\right|=(1-a\bar{a}')^{-(n+1)2}\geqq 1,$$

as far as $M(z_0, D)$ is concerned, we can replace $\mathcal{F}(D)$ by the subfamily $\mathcal{F}_{z_0}(D)$ of

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mappings which transform the point z_0 into the origin.

Carathéodory [2] proved that for the polydisc $P_n=\{(z_1, \cdots, z_n): |z_j|<1, j=1, \cdots, n\}$

$$M(0, P_n)=n^{-n/2}.$$

We shall find the value $M(0, D)$ for the classical Cartan domains.

By a classical Cartan domain we understand a domain of one of the following four types:

$$R_{\mathrm{I}}(r, s)=\{Z=(z_{jk}): I-Z\bar{Z}'>0, \text{ where } Z \text{ is an } r\times s \text{ matrix}\}, \quad (r\leqq s),$$

$$R_{\mathrm{II}}(p)=\{Z=(z_{jk}): I-Z\bar{Z}'>0, \text{ where } Z \text{ is a symmetric matrix of order } p\},$$

$$R_{\mathrm{III}}(q)=\{Z=(z_{jk}): I-Z\bar{Z}'>0, \text{ where } Z \text{ is a skew-symmetric matrix of order } q\},$$

$$R_{\mathrm{IV}}(n)=\{z=(z_1, \cdots, z_n): 1+|zz'|^2-2z\bar{z}'>0, 1-|zz'|>0\}.$$

Obviously,

$$R_{\mathrm{I}}(r, s)\subset \boldsymbol{C}^{rs}, \qquad R_{\mathrm{II}}(p)\subset \boldsymbol{C}^{p(p+1)/2},$$

$$R_{\mathrm{III}}(q)\subset \boldsymbol{C}^{q(q-1)/2}, \qquad R_{\mathrm{IV}}(n)\subset \boldsymbol{C}^{n}.$$

Instead of $R_{\mathrm{II}}(p)$ we consider the following modified domain:

$$\hat{R}_{\mathrm{II}}(p)=\{Z=(z_{jk}): z_{jk}=\sqrt{2}\, x_{jk} \ (j\neq k),\ z_{jj}=x_{jj}, \text{ where } X=(x_{jk})\in R_{\mathrm{II}}(p)\}.$$

We shall prove the following theorem:

THEOREM

$$M(0, R_{\mathrm{I}}(r, s))=r^{-rs/2}, \tag{1.1}$$

$$M(0, \hat{R}_{\mathrm{II}}(p))=2^{-p(p-1)/4}M(0, R_{\mathrm{II}}(p))=p^{-p(p+1)/4}, \tag{1.2}$$

$$M(0, R_{\mathrm{III}}(q))=\left[\frac{q}{2}\right]^{-q(q-1)/4}, \tag{1.3}$$

where $\left[\frac{q}{2}\right]$ *denotes the integral part of the number* $\frac{q}{2}$.

$$M(0, R_{\mathrm{IV}}(n))=1. \tag{1.4}$$

Now, we consider the modified domains:

$$R_{\mathrm{I}}^{0}(r, s)=\{Z: \sqrt{r}\, Z\in R_{\mathrm{I}}(r, s)\} \tag{1.5}$$

(1.6) $$R_{\mathrm{II}}^{0}(p)=\{Z:\sqrt{p}\,Z\in\hat{R}_{\mathrm{II}}(p)\},$$

(1.7) $$R_{\mathrm{III}}^{0}(q)=\{Z:\sqrt{\left[\frac{q}{2}\right]}Z\in R_{\mathrm{III}}(q)\},$$

(1.8) $$R_{\mathrm{IV}}^{0}(n)=R_{\mathrm{IV}}(n).$$

E. Cartan [3] proved that, if $n\neq 16, 27$, every irreducible bounded symmetric domain D in $\boldsymbol{C}^n$ is biholomorphically equivalent to a domain of one of the classical Cartan domains. Hence there exists a biholomorphic mapping f from D onto a domain of one of the domains (1.5)$\sim$(1.8) such that $f(0)=0$, here we may assume that D contains the origin. Since these four domains are contained in the unit hyperball (see Lemma in § 2) and since

$$M(0, R_v^0)=1 \qquad (v=\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}),$$

it follows that f is an extremal mapping, i. e.,

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|=M(0, D).$$

2. Let D be a bounded domain in $\boldsymbol{C}^n$. We denote by $\rho(z_0, D)$ the greatest lower bound of the radii of hyperballs $\{z=(z_1, \cdots, z_n): |z_1-z_1^0|^2+\cdots+|z_n-z_n^0|^2<\rho^2\}$, $z_0=(z_1^0, \cdots, z_n^0)$, containing D. By appealing to methods of Hua (see [4]) we are able to compute the value of $\rho(0, D)$ for the classical Cartan domains.

LEMMA

(2.1) $$\rho(0, R_{\mathrm{I}}(r, s))=\sqrt{r},$$

(2.2) $$\rho(0, \hat{R}_{\mathrm{II}}(p))=\rho(0, R_{\mathrm{II}}(p))=\sqrt{p},$$

(2.3) $$\rho(0, R_{\mathrm{III}}(q))=\sqrt{\left[\frac{q}{2}\right]},$$

(2.4) $$\rho(0, R_{\mathrm{IV}}(n))=1.$$

Proof. Let $Z\in R_{\mathrm{I}}(r, s)$. According to a result of Hua (see [4]) there exist two unitary matrices U and V of orders r and s, respectively, such that

$$W=UZV=\begin{pmatrix}\zeta_1 & 0\cdots 0 & 0\cdots 0\\ 0 & \zeta_2\cdots 0 & 0\cdots 0\\ \cdots & \cdots & \cdots\\ 0 & 0\cdots\zeta_r & 0\cdots 0\end{pmatrix}$$

and $W\in R_{\mathrm{I}}(r, s)$. Since $W\in R_{\mathrm{I}}(r, s)$, it follows that $|\zeta_j|<1$ $(j=1, \cdots, r)$. We arrange the elements of the matrices Z and W in the form of vectors in $\boldsymbol{C}^{rs}$

$$z=(z_{11}, \cdots, z_{1s}, \cdots, z_{r1}, \cdots, z_{rs}),$$

$$w=(w_{11}, \cdots, w_{1s}, \cdots, w_{r1}, \cdots . w_{rs}).$$

Then by the relation $W=UZV$ we have

$$w=zU'\times V$$

where $U'\times V$ is the Kronecker product of matrices U' and V. Since $U'\times V$ is also a unitary matrix of order rs, we have

$$\|z\|^2=\|w\|^2=|\zeta_1|^2+\cdots+|\zeta_r|^2<r,$$

where $\|z\|^2=|z_{11}|^2+\cdots+|z_{1s}|^2+\cdots+|z_{r1}|^2+\cdots+|z_{rs}|^2$. Hence

$$\rho(0,\ R_{\mathrm{I}}(r,\ s))\leqq\sqrt{r}.$$

On the other hand, for arbitrary complex numbers $\zeta_1, \cdots, \zeta_r$ such that $|\zeta_j|<1$ ($j=1, \cdots, r$), the point

$$Z=(z_{jk}),\quad z_{jk}=\begin{cases}\zeta_j & (j=k)\\ 0 & (j\neq k)\end{cases}$$

belongs to $R_{\mathrm{I}}(r, s)$ and, therefore, (2.1) follows.

Let $Z\in\hat{R}_{\mathrm{II}}(p)$ and set

$$X=(x_{jk}),\quad x_{jk}=\begin{cases}z_{jj} & (j=k)\\ \dfrac{1}{\sqrt{2}}z_{jk} & (j\neq k)\end{cases}.$$

Then $X\in R_{\mathrm{II}}(p)$. Again, by [4], there exists a unitary matrix U of order p such that

$$Y=UXU'=\begin{pmatrix}\zeta_1 & 0\cdots 0\\ 0 & \zeta_2\cdots 0\\ \cdots & \cdots\\ 0 & 0\cdots\zeta_p\end{pmatrix}$$

and $Y\in R_{\mathrm{II}}(p)$. Obviously, $|\zeta_j|<1$ ($j=1, \cdots, p$). We arrange the elements of the matrix Z in the form of a vector in $\boldsymbol{C}^{p(p+1)/2}$

$$z=(z_{11}, \cdots, z_{1p}, z_{22}, \cdots, z_{2p}, \cdots, z_{pp}).$$

On the other hand we arrange the elements of the matrices X and Y in the form of vectors in $\boldsymbol{C}^{p^2}$

$$x=(x_{11}, \cdots, x_{1p}, \cdots, x_{p1}, \cdots, x_{pp}),$$

$$y=(y_{11}, \cdots, y_{1p}, \cdots, y_{p1}, \cdots, y_{pp}).$$

By the relation $Y=UXU'$ we have $y=xU'\times U'$. Since $U'\times U'$ is a unitary matrix of order p^2, we have

$$\|z\|^2=\|x\|^2=\|y\|^2=|\zeta_1|^2+\cdots+|\zeta_p|^2<p.$$

Further, if $\xi_1, \cdots, \xi_p$ are complex numbers such that $|\xi_j|<1$ $(j=1, \cdots, p)$, then the point

$$Z=(z_{jk}), \quad z_{jk}=\begin{cases} \xi_j & (j=k) \\ 0 & (j\neq k) \end{cases}$$

belongs to $\hat{R}_{\mathrm{II}}(p)$. Thus it follows that

$$\rho(0, \hat{R}_{\mathrm{II}}(p))=\sqrt{p}.$$

Similarly we have

$$\rho(0, R_{\mathrm{II}}(p))=\sqrt{p}.$$

For each $Z\in R_{\mathrm{III}}(q)$ there exists a unitary matrix U of order q such that

$$W=UZU'=\begin{pmatrix} 0 & \zeta_1 \\ -\zeta_1 & 0 \end{pmatrix}\dotplus \cdots \dotplus\begin{pmatrix} 0 & \zeta_m \\ -\zeta_m & 0 \end{pmatrix} \qquad (q=2m)$$

or

$$W=UZU'=\begin{pmatrix} 0 & \zeta_1 \\ -\zeta_1 & 0 \end{pmatrix}\dotplus \cdots \dotplus\begin{pmatrix} 0 & \zeta_m \\ -\zeta_m & 0 \end{pmatrix}\dotplus 0 \qquad (q=2m+1),$$

and $W\in R_{\mathrm{III}}(q)$ (see [4]). Hence we obtain (2.3).

The last equality (2.4) is obvious.

3. We turn now to the proof of the theorem. We first prove (1.1). For $Z=(z_{jk})\in R_{\mathrm{I}}(r, s)$ we arrange the elements of Z in the form

$$z=(z_{11}, \cdots, z_{1s}, \cdots, z_{r1}, \cdots, z_{rs}).$$

Let f be a mapping belonging to the family $\mathcal{F}_0(R_{\mathrm{I}}(r, s))$. We set

$$f=(f_{11}, \cdots, f_{1s}, \cdots, f_{r1}, \cdots, f_{rs}),$$

$$f_{jk}(z)=a_{11}^{(jk)}z_{11}+\cdots+a_{1s}^{(jk)}z_{1s}+\cdots+a_{r1}^{(jk)}z_{r1}+\cdots+a_{rs}^{(jk)}z_{rs}+(\text{higher powers}).$$

Then

$$\left(\frac{\partial f}{\partial z}\right)_{z=0}=\begin{pmatrix} a_{11}^{(11)}\cdots a_{1s}^{(11)}\cdots a_{r1}^{(11)}\cdots a_{rs}^{(11)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{11}^{(rs)}\cdots a_{1s}^{(rs)}\cdots a_{r1}^{(rs)}\cdots a_{rs}^{(rs)} \end{pmatrix}.$$

There exists a unitary matrix U of order rs such that

$$U\left(\frac{\partial f}{\partial z}\right)_{z=0}=\begin{pmatrix} c_{11}^{(11)} & c_{12}^{(11)} & \cdots c_{rs}^{(11)} \\ 0 & c_{12}^{(12)} & \cdots c_{rs}^{(12)} \\ \cdots & \cdots & \cdots \\ 0 & 0 & \cdots c_{rs}^{(rs)} \end{pmatrix}.$$

We consider the mapping $g=\varphi\circ f$, where φ is the automorphism of B_{rs} defined by the linear transformation $w=zU'$. The mapping g belongs to $\mathcal{F}_0(R_{\mathrm{I}}(r, s))$ and

$$(3.1)\qquad \left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right| = \left|\det\left(\frac{\partial g}{\partial z}\right)_{z=0}\right| = |c_{11}^{(11)} c_{12}^{(12)} \cdots c_{rs}^{(rs)}|.$$

Let σ be a one-to-one mapping from $\{1, \cdots, r\}$ into $\{1, \cdots, s\}$. We take a unitary matrix $V=(v_{\alpha\beta})$ of order rs such that $v_{1\beta}=e^{i\theta_j}/\sqrt{r}$ for $\beta=(j-1)s+\sigma(j)$ $(j=1, \cdots, r)$ and $v_{1\beta}=0$ for the other β's, where $\theta_1, \cdots, \theta_r$ are arbitrary real numbers. We denote by ϕ the automorphism of B_{rs} defined by $w=zV'$. The mapping

$$h=\phi\circ g=(h_{11}, \cdots, h_{1s}, \cdots, h_{r1}, \cdots, h_{rs})$$

belongs to $\mathcal{F}_0(R_{\mathrm{I}}(r, s))$. We have the expansion

$$(3.2)\qquad \begin{aligned} &h_{11}(z)=b_{11}z_{11}+ \cdots +b_{1s}z_{1s}+ \cdots +b_{r1}z_{r1}+ \cdots +b_{rs}z_{rs}+(\text{higher powers}),\\ &b_{j\sigma(j)}=\frac{1}{\sqrt{r}}\{e^{i\theta_1}c_{j\sigma(j)}^{(1\sigma(1))}+e^{i\theta_2}c_{j\sigma(j)}^{(2\sigma(2))}+ \cdots +e^{i\theta_j}c_{j\sigma(j)}^{(j\sigma(j))}\} \qquad (j=1, \cdots, r). \end{aligned}$$

Let $\alpha_1, \cdots, \alpha_r$ be arbitrary complex numbers such that $|\alpha_j|=1$ $(j=1, \cdots, r)$. If $|\zeta|<1$, then the point

$$Z=(z_{jk}), \quad z_{jk}=\begin{cases} \alpha_j\zeta & (k=\sigma(j)) \\ 0 & (k\neq\sigma(j)) \end{cases}$$

belongs to $R_{\mathrm{I}}(r, s)$. Hence the function

$$\tilde{h}(\zeta)=h_{11}(z)=\{b_{1\sigma(1)}\alpha_1+\cdots+b_{r\sigma(r)}\alpha_r\}\zeta+(\text{higher powers})$$

is holomorphic in $|\zeta|<1$ and satisfies the conditions $|\tilde{h}(\zeta)|<1$, $\tilde{h}(0)=0$. Therefore, by Schwarz lemma,

$$(3.3)\qquad |b_{1\sigma(1)}\alpha_1+ \cdots +b_{r\sigma(r)}\alpha_r|\leqq 1.$$

Since θ_j and α_j are arbitrary, we have, by (3.2) and (3.3),

$$|c_{1\sigma(1)}^{(1\sigma(1))}|+|c_{2\sigma(2)}^{(2\sigma(2))}|+ \cdots +|c_{r\sigma(r)}^{(r\sigma(r))}|\leqq\sqrt{r}.$$

Therefore we obtain

$$(3.4)\qquad |c_{11}^{(11)}|+\cdots+|c_{1s}^{(1s)}|+\cdots+|c_{r1}^{(r1)}|+\cdots+|c_{rs}^{(rs)}|\leqq\sqrt{r}\,s.$$

Now, from (3.1) and (3.4) we have

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|\leqq r^{-rs/2}.$$

On the other hand, it follows from the Lemma that the mapping

$$w_{jk}=\frac{1}{\sqrt{r}}z_{jk} \qquad (j=1, \cdots, r;\ k=1, \cdots, s)$$

belongs to $\mathcal{F}_0(R_{\mathrm{I}}(r, s))$. Therefore, (1.1) follows.

4. Next we prove (1.2). For $Z=(z_{jk})\in\hat{R}_{\text{II}}(p)$ we arrange the elements of Z in the form of a vector in $\boldsymbol{C}^{p(p+1)/2}$

$$z=(z_{11},\cdots,z_{1p},z_{22},\cdots,z_{2p},\cdots,z_{pp}).$$

Let f be a mapping in $\mathfrak{F}_0(\hat{R}_{\text{II}}(p))$. We set

$$f=(f_{11},\cdots,f_{1p},f_{22},\cdots,f_{2p},\cdots,f_{pp}),$$

$$f_{jk}(z)=a_{11}^{(jk)}z_{11}+\cdots+a_{1p}^{(jk)}z_{1p}+a_{22}^{(jk)}z_{22}+\cdots+a_{2p}^{(jk)}z_{2p}+\cdots+a_{pp}^{(jk)}z_{pp}$$
$$+(\text{higher powers}).$$

We may assume that $\left(\frac{\partial f}{\partial z}\right)_{z=0}$ is a triangular matrix of order $p(p+1)/2$:

$$\left(\frac{\partial f}{\partial z}\right)_{z=0}=\begin{pmatrix} a_{11}^{(11)} & a_{12}^{(11)}\cdots a_{pp}^{(11)} \\ 0 & a_{12}^{(12)}\cdots a_{pp}^{(12)} \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ 0 & 0\quad\cdots a_{pp}^{(pp)} \end{pmatrix}.$$

Hence

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|=|a_{11}^{(11)}\cdots a_{1p}^{(1p)}a_{22}^{(22)}\cdots a_{2p}^{(2p)}\cdots a_{pp}^{(pp)}|. \tag{4.1}$$

We consider a modified mapping F which maps $\hat{R}_{\text{II}}(p)$ into B_{p^2}:

$$F=(F_{11},\cdots,F_{1p},\cdots,F_{p1},\cdots,F_{pp}),$$

$$F_{jk}=F_{kj}=\frac{1}{\sqrt{2}}f_{jk}\ (j<k),\quad F_{jj}=f_{jj}.$$

We first consider the case that p is even, i.e., $p=2m$. Denote by $\boldsymbol{S}_m$ the set of all one-to-one mappings σ from $\{1,\cdots,p\}$ onto itself such that

$$\sigma(j)\neq j,\quad \sigma\circ\sigma(j)=j\quad(j=1,\cdots,p).$$

Let $\sigma\in\boldsymbol{S}_m$ and let $j_1,\cdots,j_m$ be the natural numbers such that

$$1=j_1<j_2<\cdots<j_m<p,\quad j_v<\sigma(j_v)\quad(v=1,\cdots,m).$$

We take a unitary matrix $V=(v_{\alpha\beta})$ of order p^2 such that $v_{1\beta}=e^{i\theta_j}/\sqrt{p}$ for $\beta=(j-1)p+\sigma(j)$ $(j=1,\cdots,p)$ and $v_{1\beta}=0$ for the other β's, and denote by ϕ the automorphism of B_{p^2} given by V. The mapping

$$G=\phi\circ F=(G_{11},\cdots,G_{1p},\cdots,G_{p1},\cdots,G_{pp})$$

maps $\hat{R}_{\text{II}}(p)$ into B_{p^2}. Since

$$G_{11}=\frac{1}{\sqrt{2p}}\{(e^{i\theta_{j_1}}+e^{i\theta_{\sigma(j_1)}})f_{j_1\sigma(j_1)}+\cdots+(e^{i\theta_{j_m}}+e^{i\theta_{\sigma(j_m)}})f_{j_m\sigma(j_m)}\},$$

we have the expansion

$$G_{11}(z)=b_{11}z_{11}+\cdots+b_{1p}z_{1p}+b_{22}z_{22}+\cdots+b_{2p}z_{2p}+\cdots+b_{pp}z_{pp}$$
$$+(\text{higher powers}),$$

(4.2)
$$b_{j_v\sigma(j_v)}=\frac{1}{\sqrt{2p}}\{(e^{i\theta_{j_1}}+e^{i\theta_{\sigma(j_1)}})a_{j_v\sigma(j_v)}^{(j_1\sigma(j_1))}+\cdots+(e^{i\theta_{j_v}}+e^{i\theta_{\sigma(j_v)}})a_{j_v\sigma(j_v)}^{(j_v\sigma(j_v))}\},$$
$$(v=1,\cdots,m).$$

Let $\alpha_1,\cdots,\alpha_m$ be arbitrary complex numbers such that $|\alpha_v|=1$ $(v=1,\cdots,m)$. If $|\zeta|<1$, the point $Z=(z_{jk})$ such that $z_{j_v\sigma(j_v)}=z_{\sigma(j_v)j_v}=\sqrt{2}\alpha_v\zeta$ for $v=1,\cdots,m$ and $z_{jk}=0$ for the other j, k belongs to $\hat{R}_{\text{II}}(p)$. Hence the function

$$\tilde{G}(\zeta)=G_{11}(z)=\sqrt{2}(b_{j_1\sigma(j_1)}\alpha_1+\cdots+b_{j_m\sigma(j_m)}\alpha_m)\zeta+(\text{higher powers})$$

is holomorphic in $|\zeta|<1$ and satisfies the conditions $|\tilde{G}(\zeta)|<1$, $\tilde{G}(0)=0$. Thus we have

$$\sqrt{2}|b_{j_1\sigma(j_1)}\alpha_1+\cdots+b_{j_m\sigma(j_m)}\alpha_m|\leqq 1. \tag{4.3}$$

Since θ_j and α_v are arbitrary, we have, by (4.2) and (4.3),

$$2(|a_{j_1\sigma(j_1)}^{(j_1\sigma(j_1))}|+\cdots+|a_{j_m\sigma(j_m)}^{(j_m\sigma(j_m))}|)\leqq\sqrt{p}. \tag{4.4}$$

Further we take a unitary matrix $V_0=(v_{\alpha\beta}^0)$ of order p^2 such that $v_{1\beta}^0=e^{i\theta_j}/\sqrt{p}$ for $\beta=(j-1)p+j$ $(j=1,\cdots,p)$ and $v_{1\beta}^0=0$ for the other β's, and we consider the point

$$Z=(z_{jk}),\quad z_{jk}=\begin{cases}\alpha_j\zeta & (j=k)\\ 0 & (j\neq k)\end{cases},$$

where $|\alpha_j|=1$ $(j=1,\cdots,p)$, $|\zeta|<1$. Then we have the inequality

$$|a_{11}^{(11)}|+|a_{22}^{(22)}|+\cdots+|a_{pp}^{(pp)}|\leqq\sqrt{p}. \tag{4.5}$$

Now, the number of the elements of $\boldsymbol{S}_m$ is $(2m)!/2^m m!$, and for each fixed pair j, k $(j<k)$ there are $(2m-2)!/2^{m-1}(m-1)!$ mappings $\sigma\in\boldsymbol{S}_m$ such that $\sigma(j)=k$. Therefore from the inequalities (4.4) and (4.5) we have

$$\frac{(2m-2)!}{2^{m-2}(m-1)!}(|a_{11}^{(11)}|+\cdots+|a_{1p}^{(1p)}|+|a_{22}^{(22)}|+\cdots+|a_{2p}^{(2p)}|+\cdots+|a_{pp}^{(pp)}|)$$
$$\leqq\frac{(2m)!}{2^m m!}\sqrt{p}+\frac{(2m-2)!}{2^{m-2}(m-1)!}\sqrt{p}=\frac{(2m-2)!\,2m(2m+1)}{2^m m!}\sqrt{p}$$

and so

$$|a_{11}^{(11)}|+\cdots+|a_{1p}^{(1p)}|+|a_{22}^{(22)}|+\cdots+|a_{2p}^{(2p)}|+\cdots+|a_{pp}^{(pp)}|\leqq\frac{p(p+1)}{2\sqrt{p}}. \tag{4.6}$$

By (4.1) and (4.6) we obtain

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right| \leqq p^{-p(p+1)/4}.$$

Next we consider the case that p is odd, i. e., $p=2m+1$. Denote by $\boldsymbol{T}_m$ the set of all one-to-one mappings τ from $\{1, \cdots, p\}$ onto itself such that $\tau(j_0)=j_0$ for a certain j_0 and $\tau(j)\neq j$, $\tau\circ\tau(j)=j$ for all other j. Let $\tau\in\boldsymbol{T}_m$ and let $j_0, j_1, \cdots, j_m$ be the natural numbers such that

$$1\leqq j_1<j_2<\cdots<j_m<p, \quad j_v<\tau(j_v) \ (v=1, \cdots, m), \quad \tau(j_0)=j_0.$$

We take a unitary matrix $V=(v_{\alpha\beta})$ of order p^2 such that $v_{1\beta}=e^{i\theta_j}/\sqrt{p}$ for $\beta=(j-1)p+\tau(j)$ $(j=1, \cdots, p)$ and $v_{1\beta}=0$ for the other β's, and denote by ϕ the automorphism of B_{p^2} given by V. Considering the mapping $\phi\circ F$ and the points $Z=(z_{jk})$ such that $z_{j_0j_0}=\alpha_0\zeta$, $z_{j_v\tau(j_v)}=z_{\tau(j_v)j_v}=\sqrt{2}\,\alpha_v\zeta$ $(v=1, \cdots, m)$ and $z_{jk}=0$ for the other j, k, where $|\alpha_v|=1$ $(v=0, 1, \cdots, m)$ and $|\zeta|<1$, we obtain the inequality

$$(4.7) \qquad |a_{j_0j_0}^{(j_0j_0)}|+2(|a_{j_1\tau(j_1)}^{(j_1\tau(j_1))}|+\cdots+|a_{j_m\tau(j_m)}^{(j_m\tau(j_m))}|)\leqq\sqrt{p}.$$

Furthermore we have

$$(4.8) \qquad |a_{11}^{(11)}|+|a_{22}^{(22)}|+\cdots|a_{pp}^{(pp)}|\leqq\sqrt{p}.$$

The number of the elements of $\boldsymbol{T}_m$ is $(2m+1)!/2^m m!$ and for each fixed pair j, k $(j<k)$ there are $(2m-1)!/2^{m-1}(m-1)!$ mappings $\tau\in\boldsymbol{T}_m$ such that $\tau(j)=k$, and further, for each fixed j there are $(2m)!/2^m m!$ mappings $\tau\in\boldsymbol{T}_m$ such that $\tau(j)=j$. Hence, using the inequalities (4.7) and (4.8) we obtain

$$\frac{(2m-1)!}{2^{m-2}(m-1)!}(|a_{11}^{(11)}|+\cdots+|a_{1p}^{(1p)}|+|a_{22}^{(22)}|+\cdots+|a_{2p}^{(2p)}|+\cdots+|a_{pp}^{(pp)}|)$$
$$\leqq\frac{(2m+1)!}{2^m m!}\sqrt{p}+\left(\frac{(2m-1)!}{2^{m-2}(m-1)!}-\frac{(2m)!}{2^m m!}\right)\sqrt{p}=\frac{(2m+2)(2m)!}{2^m m!}\sqrt{p}$$

i. e.,

$$|a_{11}^{(11)}|+\cdots+|a_{1p}^{(1p)}|+|a_{22}^{(22)}|+\cdots+|a_{2p}^{(2p)}|+\cdots+|a_{pp}^{(pp)}|\leqq\frac{p(p+1)}{2\sqrt{p}}.$$

Therefore we have

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right| \leqq p^{-p(p+1)/4}.$$

Since the mapping

$$w_{jk}=\frac{1}{\sqrt{p}}z_{jk} \qquad (j=1, \cdots, p;\ j\leqq k\leqq p)$$

belongs to $\mathcal{F}_0(\hat{R}_{\mathrm{II}}(p))$, we obtain

$$M(0, \hat{R}_{\text{II}}(p))=p^{-p(p+1)/4}.$$

By an analogous argument we can prove (1.3).

5. Finally we prove (1.4). Let f be a mapping in $\mathcal{F}_0(R_{\text{IV}}(n))$. We set

$$f=(f_1, \cdots, f_n),$$

$$f_j(z)=a_{j1}z_1+\cdots+a_{jn}z_n+(\text{higher powers}).$$

We may assume that $\left(\frac{\partial f}{\partial z}\right)_{z=0}$ is a triangular matrix of order n:

$$\left(\frac{\partial f}{\partial z}\right)_{z=0}=\begin{pmatrix} a_{11} & a_{12}\cdots a_{1n} \\ 0 & a_{22}\cdots a_{2n} \\ \cdots & \cdots\cdots\cdots \\ 0 & 0\ \cdots a_{nn} \end{pmatrix}.$$

Hence

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|=|a_{11}a_{22}\cdots a_{nn}|.$$

Let k be a natural number such that $1\leqq k\leqq n$. If $|\zeta|<1$, then the point $z=(z_1, \cdots, z_n)$ such that $z_k=\zeta$ and $z_j=0$ for $j\neq k$ belongs to $R_{\text{IV}}(n)$. Hence the function

$$\tilde{f}(\zeta)=f_k(z)=a_{kk}\zeta+(\text{higher powers})$$

is holomorphic in $|\zeta|<1$ and satisfies the conditions $|\tilde{f}(\zeta)|<1$, $\tilde{f}(0)=0$. Hence we have

$$|a_{kk}|\leqq 1$$

and (1.4) follows. This concludes the proof of the Theorem.

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