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CURVATURE INVARIANTS OF CR-MANIFOLDS

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§ 0. Introduction.

In [2], S. Bochner defined a certain curvature tensor as a complex analogy of Weyl conformal curvature tensor without geometrical interpretation. At present this tensor is called *Bochner curvature tensor*. Recently Webster [9], [11] gave this geometrical interpretation as a pseudoconformal invariant on a CR-manifold. Indeed Bochner curvature tensor is the 4th curvature invariant given in Chern-Moser's paper [4] (cf. Tanaka [7]). In this paper we shall also derive Bochner curvature tensor from our argument of CR-structure.

In [6] we studied almost contact structures standing on the viewpoint of pseudoconformal geometry and gave the change of canonical connections associated with almost contact structures belonging to the same CR-structure. The point under our discussion is the fact that almost contact structures belonging to a CR-structure play the same role as Riemannian structures belonging to a conformal structure and canonical connections correspond to Riemannian connections. Like the conformal change of Riemannian connections, a gradient vector appears in the change of canonical connections. Therefore we compute the difference of their curvature tensors and eliminate the gradient vector. Then we get a curvature invariant.

In § 1 we recall definitions and results given in [6]. § 2 is devoted to the study of curvatures of canonical connections. We in § 3 obtain the curvature invariant of the pseudo-conformal geometry.

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§ 1. Preliminaries.

Let $\mathcal{M}$ be a connected orientable C^∞-manifold of dimension $2n+1$ $(n\geqq 1)$ and $(\mathcal{D}, J)$ a pair of a hyperdistribution $\mathcal{D}$ and a complex structure J on $\mathcal{D}$. The pair $(\mathcal{D}, J)$ is called a *CR-structure* if the following two conditions hold:

$$\text{(C. 1)} \qquad [JX, JY]-[X, Y]\in\Gamma(\mathcal{D}),$$

$$\text{(C. 2)} \qquad [JX, JY]-[X, Y]-J([X, JY]+[JX, Y])=0$$

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for every $X, Y\in\Gamma(\mathcal{D})$ where $\Gamma(\mathcal{D})$ denotes the set of all vector fields contained in $\mathcal{D}$. Let θ be a local 1-form annihilating the hyperdistribution $\mathcal{D}$. If the restriction of the 2-form $d\theta$ to $\mathcal{D}$ is nondegenerate, then the CR-structure $(\mathcal{D}, J)$ is called to be *nondegenerate*. In the sequel $(\mathcal{D}, J)$ will be a nondegenerate CR-structure.

Now let the manifold $\mathcal{M}$ admit a CR-structure $(\mathcal{D}, J)$. An *almost contact structure* (ϕ, ξ, θ) is a triplet of (1, 1) tensor field ϕ, a vector field ξ and an 1-form θ satisfying

$$\theta(\xi)=1, \quad \phi^2=-I+\theta\otimes\xi\,, \tag{1.1}$$

which imply

$$\phi\xi=0, \quad \theta\circ\phi=0 \text{ and rank } \phi=2n.$$

If the 1-form θ annihilates $\mathcal{D}$ and the restriction of ϕ to $\mathcal{D}$ coincides with J, then we say that the almost contact structure (ϕ, ξ, θ) *belongs* to the CR-structure $(\mathcal{D}, J)$. Define ω by

$$\omega=-2d\theta\,. \tag{1.2}$$

Then ω satisfies

$$\omega(JX, JY)=\omega(X, Y) \tag{1.3}$$

for every $X, Y\in\Gamma(\mathcal{D})$ because of the condition (C. 1). Moreover define $g:\mathcal{D}\times\mathcal{D}\to R$ by

$$g(X, Y)=\omega(JX, Y)\,, \tag{1.4}$$

which is called *Levi metric* and satisfies the equations

$$g(X, Y)=g(Y, X)\,, \tag{1.5}$$

$$g(JX, JY)=g(X, Y)\,. \tag{1.6}$$

From a given almost contact structure belonging to $(\mathcal{D}, J)$ we can always make an almost contact structure belonging to the same $(\mathcal{D}, J)$ and satisfying the following condition

$$[\xi, \Gamma(\mathcal{D})]\subset\Gamma(\mathcal{D}) \tag{*}$$

(cf. [6]). This condition (*) is equivalent to

$$\mathcal{L}_\xi\theta=0 \tag{1.7}$$

or

$$\omega(X, \xi)=0\,,$$

where $\mathcal{L}_\xi$ denotes the Lie differentiation with respect to ξ. Such an almost contact structure will be denoted by $(\phi, \xi, \theta)^*$ and we shall restrict our attention to the family of almost contact structures with condition (*) which belong to the CR-structure $(\mathcal{D}, J)$. We proved in [6]

LEMMA 1.1. *If $(\phi, \xi, \theta)^*$ and $(\phi', \xi', \theta')^*$ belong to $(\mathcal{D}, J)$, then they are related by*

$$\theta'=\varepsilon e^{2\mu}\theta,\quad \xi'=\varepsilon e^{-2\mu}(\xi-2Q),\quad \phi'=\phi-2\theta\otimes P, \tag{1.8}$$

where $\varepsilon=\pm 1$, μ is a C^∞-function, $P\in\Gamma(\mathcal{D})$ is defined by $g(P, X)=d\mu(X)$ for $X\in\Gamma(\mathcal{D})$ and $Q=JP$.

Next we shall explain canonical connections associated to almost contact structures with condition (*) and their change. Before mentioning the existence of canonical connections, we prepare the notations. For $(\phi, \xi, \theta)^*$ belonging to $(\mathcal{D}, J)$ there always exists a linear connection ∇ such that $\nabla\phi=0$, $\nabla\xi=0$ and $\nabla\theta=0$. Let D denote the induced connection on the hyperdistribution $\mathcal{D}$. Then D satisfies $DJ=0$. Since the equation $\nabla\theta=0$ implies that the parallel displacement with respect to ∇ preserves $\mathcal{D}$, the torsion tensor field T of ∇ satisfies

$$T(X, Y)=T_{\mathcal{D}}(X, Y)-\omega(X, Y)\xi, \tag{1.9}$$

$$T_{\mathcal{D}}(X, Y)=\nabla_X Y-\nabla_Y X-[X, Y]_{\mathcal{D}} \tag{1.10}$$

for every $X, Y\in\Gamma(\mathcal{D})$ where $[X, Y]_{\mathcal{D}}$ denotes the $\mathcal{D}$-component of $[X, Y]$ and we note that $T_{\mathcal{D}}(X, Y)$ is the $\mathcal{D}$-component of $T(X, Y)$. Let F be a tensor field of type (1, 1) defined by

$$FX=T(\xi, X),\qquad X\in T\mathcal{M}. \tag{1.11}$$

Then from the definition of the torsion tensor field T we see that

$$\nabla_\xi X=FX+[\xi, X],\qquad X\in\Gamma(\mathcal{D}). \tag{1.12}$$

Tanaka [7] proved (cf. [6])

LEMMA 1.2. *Let $(\phi, \xi, \theta)^*$ be an almost contact structure satisfying the condition (*) and belonging to $(\mathcal{D}, J)$. Then there exists uniquely a linear connection ∇ such that $\nabla\phi=0$, $\nabla\xi=0$, $\nabla\theta=0$, $Dg=0$, $T_{\mathcal{D}}=0$ and $F=-1/2\,\phi\mathcal{L}_\xi\phi$.*

The linear connection stated in the above lemma is called a *canonical connection* associated with $(\phi, \xi, \theta)^*$. To conclude this section we give the following (cf. [6])

LEMMA 1.3. *Let $(\phi, \xi, \theta)^*$ and $(\phi', \xi', \theta')^*$ be two almost contact structures which belong to the CR-structure $(\mathcal{D}, J)$. Let ∇ and ∇' be canonical connections associated with $(\phi, \xi, \phi)^*$ and $(\phi', \xi', \theta')^*$ respectively. Define the difference H between ∇ and ∇' by*

$$H(X, Y)=\nabla_X{}'Y-\nabla_X Y, \qquad X, Y\in\Gamma(T\mathcal{M}).$$

Then we have

(1.13) $H(X, Y)=p(X)Y+p(Y)X-g(X, Y)P+q(X)JY+q(Y)JX-g(JX, Y)Q,$

(1.14) $H(\xi, X)=\nabla_{JX}P+\nabla_X Q-2q(X)P+2p(X)Q+2g(P, P)JX, \qquad X, Y\in\Gamma(\mathcal{D}),$

where $p=d\mu$ *and* $q=-p\circ\phi$.

Remark. We have $g(P, X)=p(X)$ and $g(Q, X)=q(X)$ for every $X\in\mathcal{D}$.

§ 2. Curvatures of canonical connections.

Let $(\phi, \xi, \theta)^*$ be an almost contact structure satisfying (*) and belonging to $(\mathcal{D}, J)$. Let ∇ be the canonical connection associated with $(\phi, \xi, \theta)^*$ and R be the curvature tensor field of ∇ defined by

$$R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

for $X, Y, Z\in\Gamma(T\mathcal{M})$. Since $\nabla\xi=0$, we have

$$R(X, Y)\xi=0, \qquad X, Y\in T\mathcal{M}. \tag{2.1}$$

The property $\nabla_X\Gamma(\mathcal{D})\subset\Gamma(\mathcal{D})$ implies that

$$R(X, Y)\mathcal{D}\subset\mathcal{D}, \qquad X, Y\in T\mathcal{M}. \tag{2.2}$$

Moreover from $\nabla\phi=0$ we have

$$R(X, Y)\phi=\phi R(X, Y), \qquad X, Y\in T\mathcal{M}. \tag{2.3}$$

If we put $R(X, Y, Z, W)=g(R(X, Y)Z, W)$ for $X, Y\in T\mathcal{M}$ and $Z, W\in\mathcal{D}$, then we have the equation

$$R(X, Y, Z, W)=-R(X, Y, W, Z). \tag{2.4}$$

Next we give first and second Bianchi identities. In general first Bianchi identity is the formula (cf. [5])

$$\mathfrak{S}\{R(X, Y)Z\}=\mathfrak{S}\{T(T(X, Y), Z)+(\nabla_X T)(Y, Z)\},$$

where $X, Y, Z\in T\mathcal{M}$ and $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z. If $X, Y, Z\in\Gamma(\mathcal{D})$, then we have

$$\mathfrak{S}\{R(X, Y)Z\}=\mathfrak{S}\{-T(\omega(X, Y)\xi, Z)-\nabla_X(\omega(Y, Z)\xi)+\omega(\nabla_X Y, Z)\xi+\omega(Y, \nabla_X Z)\xi\}$$

because of (1.9) and $T_{\mathcal{D}}=0$. Thus we obtain

$$\mathfrak{S}\{R(X, Y)Z\}=-\mathfrak{S}\{\omega(X, Y)FZ\} \tag{2.5}$$

for every $X, Y, Z \in \mathcal{D}$. Putting $X=\xi$ and letting $Y, Z \in \Gamma(\mathcal{D})$ in the general Bianchi identity, we have

$$
\begin{aligned}
& R(\xi, Y)Z + R(Y, Z)\xi + R(Z, \xi)Y \\
& = T(T(\xi, Y), Z) + T(T(Z, \xi), Y) + (\nabla_\xi T)(Y, Z) + (\nabla_Y T)(Z, \xi) + (\nabla_Z T)(\xi, Y) \\
& = -\omega(FY, Z)\xi + \omega(FZ, Y)\xi - \nabla_Y(FZ) + F\nabla_Y Z + \nabla_Z(FY) - F\nabla_Z Y,
\end{aligned}
$$

where we have used (1.11). Therefore we get

$$
R(\xi, Y)Z - R(\xi, Z)Y = -(\nabla_Y F)Z + (\nabla_Z F)Y \tag{2.6}
$$

for $Y, Z \in \mathcal{D}$. Since the general second Bianchi identity is

$$
\mathfrak{S}\{(\nabla_X R)(Y, Z)\} = -\mathfrak{S}\{R(T(X, Y), Z)\}, \qquad X, Y, Z \in T\mathcal{M},
$$

we have immediately

$$
\mathfrak{S}\{(\nabla_X R)(Y, Z)\} = \mathfrak{S}\{\omega(X, Y)R(\xi, Z)\}, \qquad X, Y, Z \in \mathcal{D}. \tag{2.7}
$$

Furthermore if we put $X=\xi$ and $Y, Z \in \mathcal{D}$ in the general second Bianchi identity, then

$$
(\nabla_\xi R)(Y, Z) - (\nabla_Y R)(\xi, Z) + (\nabla_Z R)(\xi, Y) = -R(FY, Z) + R(FZ, Y), \qquad Y, Z \in \mathcal{D}. \tag{2.8}
$$

We shall prove the following formula

$$
\begin{aligned}
R(X, Y, Z, W) - R(Z, W, X, Y) = {} & \omega(X, Z)g(FY, W) - \omega(Y, Z)g(FX, W) \\
& + \omega(Y, W)g(FX, Z) - \omega(X, W)g(FY, Z)
\end{aligned} \tag{2.9}
$$

where $X, Y, Z, W \in \mathcal{D}$. If we put

$$
\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W),
$$

then we have

$$
\begin{aligned}
& \tilde{R}(X, Y, Z, W) - \tilde{R}(Y, Z, W, X) - \tilde{R}(Z, W, X, Y) + \tilde{R}(W, X, Y, Z) \\
& = 2\{R(Y, Z, X, W) - R(X, W, Y, Z)\}
\end{aligned}
$$

because of (2.4). Since the equation (2.5) shows

$$
\tilde{R}(X, Y, Z, W) = -\{\omega(X, Y)g(FZ, W) + \omega(Y, Z)g(FX, W) + \omega(Z, X)g(FY, W)\},
$$

we obtain

$$
\begin{aligned}
R(Y, Z, X, W) - R(X, W, Y, Z) = {} & \omega(Y, X)g(FZ, W) - \omega(Z, X)g(FY, W) \\
& + \omega(Z, W)g(FY, X) - \omega(Y, W)g(FZ, X),
\end{aligned}
$$

where we have used the fact that F is symmetric with respect to the Levi metric g. Therefore (2.9) has been proved. From (2.6) we have

$$R(\xi, Y, Z, W)-R(\xi, Z, Y, W)=-g((\nabla_Y F)Z, W)+g((\nabla_Z F)Y, W),$$

in which we permute the letters Y, Z and W cyclically and subtract one from the sum of the other two. Noting that $\nabla_X F$ is symmetric with respect to g, we obtain

$$R(\xi, Y, Z, W)=g(Y, (\nabla_Z F)W-(\nabla_W F)Z) \tag{2.10}$$

for every $Y, Z, W \in \mathcal{D}$.

For later use we need the following formula

$$\begin{aligned} &R(JX, JY)Z-R(X, Y)Z \\ &=g(JX, Z)FY-g(JY, Z)FX+g(X, Z)FJY-g(Y, Z)FJX \\ &\quad +f(X, Z)JY-f(Y, Z)JX+f(JX, Z)Y-f(JY, Z)X \end{aligned} \tag{2.11}$$

where $X, Y, Z \in \mathcal{D}$ and we have defined f by

$$f(X, Y)=g(FX, Y), \qquad X, Y \in \mathcal{D}.$$

This formula can be proved by using equations (1.6), (2.3) and (2.9). In fact we see that

$$\begin{aligned} &R(JX, JY, Z, W) \\ &=R(Z, W, JX, JY)+\omega(JX, Z)g(FJY, W)-\omega(JY, Z)g(FJX, W) \\ &\quad +\omega(JY, W)g(FJX, Z)-\omega(JX, W)g(FJY, Z) \\ &=R(Z, W, X, Y)+g(X, Z)g(FJY, W)-g(Y, Z)g(FJX, W) \\ &\quad +g(Y, W)f(JX, Z)-g(X, W)f(JY, Z) \\ &=R(X, Y, Z, W)+\omega(Z, X)g(FW, Y)-\omega(W, X)g(FZ, Y) \\ &\quad +\omega(W, Y)g(FZ, X)-\omega(Z, Y)g(FW, X)+g(X, Z)g(FJY, W) \\ &\quad -g(Y, Z)g(FJX, W)+g(Y, W)f(JX, Z)-g(X, W)f(JY, Z) \\ &=R(X, Y, Z, W)+g(JX, Z)g(FY, W)-g(JY, Z)g(FX, W) \\ &\quad +g(X, Z)g(FJY, W)-g(Y, Z)g(FJX, W)+f(X, Z)g(JY, W) \\ &\quad -f(Y, Z)g(JX, W)+f(JX, Z)g(Y, W)-f(JY, Z)g(X, W). \end{aligned}$$

Remark. If $F=0$, then $R(JX, JY)=R(X, Y)$ holds for every $X, Y \in \mathcal{D}$. The condition $F=0$ is equivalent to that the almost contact structure $(\phi, \xi, \theta)^*$ is normal (for the definition, see [1]).

Next we turn to the study of the Ricci tensor field. We shall define two kinds of Ricci tensors. In general Ricci tensor field s is defined by

$$s(X, Y)=\text{trace of } (V\to R(V, X)Y), \tag{2.12}$$

where $X, Y\in T\mathcal{M}$. We define another Ricci tensor field k by

$$k(X, Y)=\frac{1}{2}\text{ trace } (\phi R(X, \phi Y)), \qquad X, Y\in T\mathcal{M}. \tag{2.13}$$

The tensor field s is symmetric when restricted to $\mathcal{D}$, because we have for $X, Y\in\mathcal{D}$

$$s(X, Y)-s(Y, X)=\text{trace } (V\to\mathfrak{S}\{R(V, X)Y\}),$$

where we may take $\mathcal{D}$ as range of V and hence from (2.5)

$$\begin{aligned} s(X, Y)-s(Y, X)&=-\text{trace } (V\to\mathfrak{S}\{\omega(V, X)FY\}) \\ &=-\omega(X, Y)\text{ trace } F=0, \end{aligned}$$

which proves

$$s(X, Y)=s(Y, X), \qquad X, Y\in\mathcal{D}. \tag{2.14}$$

The tensor field k does not always coincide with s. Indeed we have

$$k(X, Y)=s(X, Y)-(n-1)f(JX, Y), \qquad X, Y\in\mathcal{D}. \tag{2.15}$$

Thus we see that when the almost contact structure $(\phi, \xi, \theta)^*$ is normal k coincides with s on $\mathcal{D}$. The equation (2.15) can be shown as following;

$$\begin{aligned} s(X, Y)&=\text{trace } (V\to -JR(V, X)JY) \qquad (V\in\mathcal{D}) \\ &=\text{trace } (V\to JR(X, JY)V+JR(JY, V)X \\ &\quad +\omega(V, X)FY+\omega(X, JY)JFV+\omega(JY, V)JFX) \\ &=2k(X, Y)+\text{trace } (V\to JR(JY, V)X), \end{aligned}$$

where we have used the equation (2.5) and the fact that F anticommutes with J, and using (2.11) we have

$$\begin{aligned} &\text{trace } (V\to JR(JY, V)X) \\ &=\text{trace } (JV\to JR(JY, JV)X) \\ &=\text{trace } (V\to R(JY, JV)X) \\ &=\text{trace } (V\to R(Y, V)X+g(JY, X)FV-g(JV, X)FY+g(Y, X)FJV \\ &\quad -g(V, X)FJY+f(Y, X)JV-f(V, X)JY+f(JY, X)V-f(JV, X)Y) \\ &=-s(Y, X)+2(n-1)f(JX, Y), \end{aligned}$$

which implies (2.15). From equations (2.14) and (2.15) we obtain

(2.16) $$k(X, Y)=k(Y, X), \qquad X, Y\in\mathcal{D}.$$

The defining equation (2.13) of k shows the following property

(2.17) $$k(JX, JY)=k(X, Y), \qquad X, Y\in\mathcal{D}.$$

It follows that we have

(2.18) $$s(JX, JY)=s(X, Y)-2(n-1)f(JX, Y), \qquad X, Y\in\mathcal{D}.$$

It is easy to show

(2.19) $$s(X, \xi)=0, \qquad X\in\mathcal{D}.$$

Furthermore by making use of (2.6) we obtain

(2.20) $$s(\xi, X)=\text{trace }(V\to(\nabla_V F)X), \qquad X\in\mathcal{D},$$

where the range of V is $\mathcal{D}$.

We introduce two notations for use in the subsequent section. First define $S\in\Gamma(\mathcal{D}^*\otimes\mathcal{D})$ by

(2.21) $$g(SX, Y)=s(X, Y), \qquad X, Y\in\mathcal{D}.$$

Secondly define ρ by

(2.22) $$\rho=\text{trace } S,$$

which is a smooth function on $\mathcal{M}$ and will be called *scalar curvature*.

§3. An invariant of pseudo-conformal geometry.

Let $(\phi, \xi, \theta)^*$ and $(\phi', \xi', \theta')^*$ be two almost contact structures belonging to the same CR-structure $(\mathcal{D}, J)$. Let ∇ and ∇' be canonical connections associated with $(\phi, \xi, \theta)^*$ and $(\phi', \xi', \theta')^*$ respectively. Then the difference tensor H between ∇ and ∇' is given in Lemma 1.3. Thus we may calculate the difference $R'(X, Y)Z-R(X, Y)Z$ $(X, Y, Z\in\mathcal{D})$. We shall introduce suitable 2-forms into the resulting long equation and rewrite it comfortably. This is the first lemma in the present section. Next using this lemma we shall calculate $k'-k$ and $\varepsilon e^{2\mu}\rho'-\rho$, from which the above introduced 2-forms will be solved. In this way, the difference $R'(X, Y)Z-R(X, Y)Z$ will be an equation consisting of only k, k', ρ and ρ'. Finally by transposing the terms with dashes and without dashes to right and left sides respectively, we shall find an invariant of the change (1.8), that is an invariant of the pseudoconformal geometry. This argument is analogous to that of conformal geometry (cf. [12], [13]).

To begin with we prove

(3.1) $$R'(X, Y)Z-R(X, Y)Z$$
$$=(\nabla_X H)(Y, Z)-(\nabla_Y H)(X, Z)$$
$$+H(X, H(Y, Z))-H(Y, H(X, Z))-\omega(X, Y)H(\xi, Z)$$

for every $X, Y, Z\in\Gamma(\mathcal{D})$. Since

$$\nabla'_X\nabla'_Y Z=\nabla'_X(\nabla_Y Z+H(Y, Z))$$
$$=\nabla_X\nabla_Y Z+H(X, \nabla_Y Z)+(\nabla_X H)(Y, Z)+H(\nabla_X Y, Z)$$
$$+H(Y, \nabla_X Z)+H(X, H(Y, Z)),$$

we have

$$R'(X, Y)Z=R(X, Y)Z+(\nabla_X H)(Y, Z)-(\nabla_Y H)(X, Z)+H(\nabla_X Y-\nabla_Y X, Z)$$
$$-H([X, Y]_{\mathcal{D}}+\omega(X, Y)\xi, Z)+H(X, H(Y, Z))-H(Y, H(X, Z)).$$

Taking (1.10) and $T_{\mathcal{D}}=0$ into account, we get (3.1). The first two terms of the equation (3.1) is given by

(3.2) $$(\nabla_X H)(Y, Z)-(\nabla_Y H)(X, Z)$$
$$=-(\nabla_Y p)(Z)X+(\nabla_X p)(Z)Y-(\nabla_Y q)(Z)JX+(\nabla_X q)(Z)JY$$
$$-g(Y, Z)\nabla_X P+g(X, Z)\nabla_Y P-g(JY, Z)\nabla_X Q+g(JX, Z)\nabla_Y Q$$
$$+\{(\nabla_X q)(Y)-(\nabla_Y q)(X)\}JZ+p(\xi)\omega(X, Y)Z,$$

which is calculated as follows: Using (1.13) we immediately have

$$(\nabla_X H)(Y, Z)=(\nabla_X p)(Y)Z+(\nabla_X p)(Z)Y-g(Y, Z)\nabla_X P$$
$$+(\nabla_X q)(Y)JZ+(\nabla_X q)(Z)JY-g(JY, Z)\nabla_X Q.$$

Note that

(3.3) $$(\nabla_X p)(Y)-(\nabla_Y p)(X)=p(\xi)\omega(X, Y)$$

holds for every $X, Y\in\Gamma(\mathcal{D})$. Then we may easily show (3.2). The second two tirms of the equation (3.1) is given by

(3.4) $$H(X, H(Y, Z))-H(Y, H(X, Z))$$
$$=\{p(Y)p(Z)-q(Y)q(Z)-p(P)g(Y, Z)\}X$$
$$-\{p(X)p(Z)-q(X)q(Z)-p(P)g(X, Z)\}Y$$
$$+\{q(Y)p(Z)+p(Y)q(Z)-p(P)g(JY, Z)\}JX$$
$$-\{q(X)p(Z)+p(X)q(Z)-p(P)q(JX, Z)\}JY$$

$$+\{p(X)g(Y, Z)-p(Y)g(X, Z)+q(X)g(JY, Z)-q(Y)g(JX, Z)+2q(Z)g(JX, Y)\}P$$
$$-\{q(X)g(Y, Z)-q(Y)g(X, Z)+p(Y)g(JX, Z)-p(X)g(JY, Z)+2p(Z)g(JX, Y)\}Q$$

for every $X, Y, Z\in\Gamma(\mathcal{D})$. We have only to substitute (1.13) into the left hand side of (3.4). The calculation is long but routine and hence we omit the proof. Therefore we see that the equation (3.1) becomes

$$\begin{aligned}(3.5)\quad & R'(X, Y)Z-R(X, Y)Z\\ &=-\{(\nabla_Y p)(Z)-p(Y)p(Z)+q(Y)q(Z)+p(P)g(Y, Z)\}X\\ &\quad+\{(\nabla_X p)(Z)-p(X)p(Z)+q(X)q(Z)+p(P)g(X, Z)\}Y\\ &\quad-\{(\nabla_Y q)(Z)-q(Y)p(Z)-p(Y)q(Z)+p(P)g(JY, Z)\}JX\\ &\quad+\{(\nabla_X q)(Z)-q(X)p(Z)-p(X)q(Z)+p(P)g(JX, Z)\}JY\\ &\quad-g(Y, Z)\{\nabla_X P-p(X)P+q(X)Q\}+g(X, Z)\{\nabla_Y P-p(Y)P+q(Y)Q\}\\ &\quad-g(JY, Z)\{\nabla_X Q-q(X)P-p(X)Q\}+g(JX, Z)\{\nabla_Y Q-q(Y)P-p(Y)Q\}\\ &\quad+\{(\nabla_X q)(Y)-(\nabla_Y q)(X)\}JZ+g(JX, Y)\{\nabla_{JZ}P+\nabla_Z Q+2p(P)JZ-p(\xi)Z\},\end{aligned}$$

where we have used (1.14).

Now we put

$$(3.6)\quad \alpha(Y, Z)=(\nabla_Y p)(Z)-p(Y)p(Z)+q(Y)q(Z)+\frac{1}{2}p(P)g(Y, Z)+\frac{1}{2}p(\xi)g(JY, Z)$$

for $Y, Z\in\mathcal{D}$ and so we see that $\alpha\in\Gamma(\mathcal{D}^*\otimes\mathcal{D}^*)$. Also we define $\gamma\in\Gamma(\mathcal{D}^*\otimes\mathcal{D}^*)$ by

$$(3.7)\quad \gamma(Y, Z)=(\nabla_Y q)(Z)-q(Y)p(Z)-p(Y)q(Z)+\frac{1}{2}p(P)g(JY, Z)-\frac{1}{2}p(\xi)g(Y, Z).$$

Then they are related as

$$(3.8)\quad \alpha(Y, Z)=\gamma(Y, JZ).$$

The 1-form p is a differential of the function μ, but the canonical connection ∇ has a torsion and so we have the equation (3.3). To require that the bilinear form α is symmetric, we need the last term in the definition. Thus

$$(3.9)\quad \alpha(Y, Z)=\alpha(Z, Y).$$

Furthermore define $A, C\in\Gamma(\mathcal{D}^*\otimes\mathcal{D})$ by

$$(3.10)\quad AY=\nabla_Y P-p(Y)P+q(Y)Q+\frac{1}{2}p(P)Y+\frac{1}{2}p(\xi)JY$$

$$CY=\nabla_Y Q-q(Y)P-p(Y)Q+\frac{1}{2}p(P)JY-\frac{1}{2}p(\xi)Y\,. \tag{3.11}$$

Then we have

$$g(AY, Z)=\alpha(Y, Z)\,, \qquad g(CY, Z)=\gamma(Y, Z)\,. \tag{3.12}$$

and from (3.8)

$$JA=C\,. \tag{3.13}$$

Substituting equations (3.6), (3.7), (3.10) and (3.11) into (3.5), we obtain

LEMMA 3.1. *Let ${}^tC$ denote the transpose of the linear transformation C of $\mathcal{D}$, i.e., $g({}^tCY, Z)=g(Y, CZ)$. Then*

$$\begin{aligned}(3.14)\quad & R'(X, Y)Z-R(X, Y)Z\\ &=-\alpha(Y, Z)X+\alpha(X, Z)Y-\gamma(Y, Z)JX+\gamma(X, Z)JY-g(Y, Z)AX+g(X, Z)AY\\ &\quad -g(JY, Z)CX+g(JX, Z)CY+\{\gamma(X, Y)-\gamma(Y, X)\}JZ+g(JX, Y)(CZ-{}^tCZ).\end{aligned}$$

Proof. The straightforward computation shows

$$\begin{aligned}& R'(X, Y)Z-R(X, Y)Z\\ &=(\text{the right hand side of (3.14) except the last term})\\ &\quad +g(JX, Y)\{\nabla_{JZ}P+\nabla_Z Q+p(P)JZ-p(\xi)Z\}\,.\end{aligned}$$

Thus it suffices to prove

$$g(\nabla_{JZ}P+\nabla_Z Q+p(P)JZ-p(\xi)Z, W)=g(CZ-{}^tCZ, W)$$

for every $W\in\Gamma(\mathcal{D})$. In virtue of (3.3) and (3.11) we have

$$\begin{aligned}&(\text{L. H. S. of the above equation})\\ &=(\nabla_{JZ}p)(W)+g(CZ, W)+q(Z)p(W)+p(Z)q(W)\\ &\quad +\frac{1}{2}p(P)g(JZ, W)-\frac{1}{2}p(\xi)g(Z, W)\\ &=g(CZ, W)-\gamma(W, Z)=g(CZ, W)-g({}^tCZ, W)\,.\end{aligned}$$

Q. E. D

Next we shall compute $k'(Y, Z)-k(Y, Z)$ ($Y, Z\in\mathcal{D}$). Before contracting the equation (3.14), we consider the symmetric part of γ. The bilinear form α is symmetric. However we can not expect γ to be skewsymmetric. This is caused by the following equation;

$$\gamma(Y, Z)+\gamma(Z, Y)=-f'(Y, Z)+f(Y, Z)\,. \tag{3.15}$$

Since we have (cf. [6])

$$f'(X, Y)-f(X, Y)=(\nabla_{JX}p)(Y)-(\nabla_{X}q)(Y)+2p(X)q(Y)+2q(X)p(Y),$$

the bilinear form α satisfies

$$\alpha(JY, Z)+\alpha(Y, JZ)=f'(Y, Z)-f(Y, Z),$$

which implies (3.15). It is clear that (3.15) is equivalent with

$$\alpha(JY, JZ)-\alpha(Y, Z)=f'(JY, Z)-f(JY, Z). \tag{3.16}$$

Well we compute $s'(Y, Z)-s(Y, Z)$. From (3.14) we see that

$$\begin{aligned} &s'(Y, Z)-s(Y, Z)\\ &=\mathrm{trace}(X\rightarrow-\alpha(Y, Z)X+\alpha(X, Z)Y-\gamma(Y, Z)JX+\gamma(X, Z)JY\\ &\qquad -g(Y, Z)AX+g(X, Z)AY-g(JY, Z)CX+g(JX, Z)CY\\ &\qquad +\{\gamma(X, Y)-\gamma(Y, X)\}JZ+g(JX, Y)(CZ-{}^{t}CZ))\\ &=-(2n+1)\alpha(Y, Z)+2\gamma(JY, Z)+\gamma(JZ, Y)-g(Y, Z)\,\mathrm{trace}\,A-g(JY, Z)\,\mathrm{trace}\,C. \end{aligned}$$

Noting the fact that trace $F=$trace $F'=0$, we obtain trace $C=0$ by virtue of the equation (3.15), and moreover substituting

$$\gamma(JY, Z)=-\alpha(Z, Y)-f'(JY, Z)+f(JY, Z)$$

into R. H. S. of the above equation, we have

$$\begin{aligned} s'(Y, Z)-s(Y, Z)=&-2(n+2)\alpha(Y, Z)-3f'(JY, Z)\\ &+3f(JY, Z)-g(Y, Z)\,\mathrm{trace}\,A. \end{aligned} \tag{3.17}$$

Therefore by the equation (2.15) we get

LEMMA 3.2. *The difference $k'(Y, Z)-k(Y, Z)$ is given by*

$$k'(Y, Z)-k(Y, Z)=-(n+2)\{\alpha(Y, Z)+\alpha(JY, JZ)\}-g(Y, Z)\,\mathrm{trace}\,A \tag{3.18}$$

for arbitrary $Y, Z\in\mathcal{D}$.

Using the equation (3.17) we have

$$\varepsilon e^{2\mu}S'-S=-2(n+2)A-3\varepsilon e^{2\mu}F'J+3FJ-(\mathrm{trace}\,A)I_{\mathcal{D}},$$

where $I_{\mathcal{D}}$ denotes the identity transformation of $\mathcal{D}$. Taking trace of both sides we obtain

LEMMA 3.3. *The difference $\varepsilon e^{2\mu}\rho'-\rho$ is given as*

$$\varepsilon e^{2\mu}\rho'-\rho=-4(n+1)\operatorname{trace} A. \tag{3.19}$$

Let's define l and m by

$$l(X, Y)=-\frac{1}{2(n+2)}k(X, Y)+\frac{1}{8(n+1)(n+2)}\rho g(X, Y) \tag{3.20}$$

and

$$m(X, Y)=-\frac{1}{2(n+2)}k(JX, Y)+\frac{1}{8(n+1)(n+2)}\rho g(JX, Y) \tag{3.21}$$

respectively, where $X, Y\in\mathcal{D}$ and we note that l (resp. m) is symmetric (resp. skewsymmetric). Clearly l and m satisfy

$$l(JX, JY)=l(X, Y), \qquad m(JX, JY)=m(X, Y) \tag{3.22}$$

and

$$m(X, Y)=l(JX, Y). \tag{3.23}$$

Moreover we need the following definitions. Define L and M by $g(LX, Y)=l(X, Y)$ and $g(MX, Y)=m(X, Y)$ for every $X, Y\in\mathcal{D}$ respectively. Then they satisfy ${}^tL=L$, ${}^tM=-M$, $LJ=JL=M$ and $MJ=JM$.

Under these notations we solve α, γ, A and C.

LEMMA 3.4. *The bilinear form α on $\mathcal{D}$ is given by*

$$\alpha(Y, Z)=l'(Y, Z)-l(Y, Z)-\frac{1}{2}\{f'(JY, Z)-f(JY, Z)\}, \tag{3.24}$$

so that we have

$$A=\varepsilon e^{2\mu}L'-L-\frac{1}{2}(\varepsilon e^{2\mu}F'J-FJ), \tag{3.25}$$

and the bilinear form γ is given by

$$\gamma(Y, Z)=m'(Y, Z)-m(Y, Z)-\frac{1}{2}\{f'(Y, Z)-f(Y, Z)\}, \tag{3.26}$$

so that we have

$$C=\varepsilon e^{2\mu}M'-M-\frac{1}{2}(\varepsilon e^{2\mu}F'-F). \tag{3.27}$$

Proof. It suffices to prove the equation (3.24) from which the others are trivially derived. From the defining equation (3.20), we have

$$l'(Y, Z)-l(Y, Z)=-\frac{1}{2(n+2)}\{k'(Y, Z)-k(Y, Z)\}$$

$$+\frac{1}{8(n+1)(n+2)}\{\rho' g'(Y, Z)-\rho g(Y, Z)\}$$

in which we substitute equations (3.18) and (3.19). Then we have

$$l'(Y, Z)-l(Y, Z)=\frac{1}{2}\{\alpha(Y, Z)+\alpha(JY, JZ)\}.$$

Combining this equation with (3.16), we obtain (3.24). Q. E. D.

Finally we state

THEOREM 3.5. *Let define* $B_0, B_1 \in \Gamma(\mathcal{D}^{*3}\otimes\mathcal{D})$ *by*

$$B_0(X, Y)Z=R(X, Y)Z+l(Y, Z)X-l(X, Z)Y+m(Y, Z)JX-m(X, Z)JY \tag{3.28}$$
$$+g(Y, Z)LX-g(X, Z)LY+g(JY, Z)MX-g(JX, Z)MY$$
$$-2\{m(X, Y)JZ+g(JX, Y)MZ\},$$

$$B_1(X, Y)Z=\frac{1}{2}\{R(JX, JY)Z-R(X, Y)Z\}. \tag{3.29}$$

Then $B=B_0+B_1$ *is invariant under the change* (1.8), *i.e.*, $B=B'$ *holds.*

Proof. Substitute the equations (3.24)~(3.27) into (3.14). Then it is easy to see that $B=B'$. Q. E. D.

Remark. When the almost contact structure $(\phi, \xi, \theta)^*$ is normal, B_0 is Bochner curvature tensor and B_1 vanishes. We also note that if $n=1$, then B_0, B_1 and hence B vanish identically.

We summarize the identities satisfied by B. Let $B(X, Y)Z=B_0(X, Y)Z+B_1(X, Y)Z$ and so $B(X, Y)\in\Gamma(\mathcal{D}^*\otimes\mathcal{D})$ for every $X, Y\in\Gamma(\mathcal{D})$. The followings are clear:

$$B(X, Y)=-B(Y, X) \quad \text{and} \tag{3.30}$$

$$g(B(X, Y)Z, W)=-g(B(X, Y)W, Z). \tag{3.31}$$

The straightforward calculation shows $\mathfrak{S}B_0(X, Y)Z=\mathfrak{S}R(X, Y)Z$ and $\mathfrak{S}B_1(X, Y)Z=\mathfrak{S}g(X, JY)FZ$. It follows from (2.5) that

$$\mathfrak{S}B(X, Y)Z=0. \tag{3.32}$$

In general a tensor satisfying the equation (3.30)~(3.32) always satisfies (cf. the proof of (2.9))

$$g(B(X, Y)Z, W)=g(B(Z, W)X, Y). \tag{3.33}$$

It is easy to show

(3.34) $$B(JX, JY)=B(X, Y) \quad \text{and}$$

(3.25) $$B(X, Y)J=JB(X, Y).$$

Since we have

$$\operatorname{trace}(X\to B_0(X, Y)Z)$$
$$=s(Y, Z)+2(n+2)l(Y, Z)+g(Y, Z)\operatorname{trace} L$$
$$=s(Y, Z)-k(Y, Z)$$

and

$$\operatorname{trace}(X\to B_1(X, Y)Z)=-(n-1)f(JY, Z)$$

the equation (2.15) implies

(3.26) $$\operatorname{trace}(X\to B(X, Y)Z)=0.$$

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