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THE STABLE GROUP OF SELF-HOMOTOPY EQUIVALENCES OF SPHERE BUNDLES OVER THE SPHERE

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Introduction.

The group of homotopy classes of homotopy equivalences of a CW-complex X with itself, denoted here as $G(X)$, has been studied by various authors. Specially a stabilization of $G(X)$ is defined by the suspension homomorphism $G(X)\to G(\Sigma X)$, denoted here as $G^s(X)=\lim_N G(\Sigma^N X)$, and has been studied by several authors [3, 4, 5, 6]. Nevertheless, our examples for the computation of stable groups are not abundant. The purpose of this paper is to give a computation of the stable group $G^s(X)$ for sphere bundles over the sphere. If X is a finite complex we can achieve $\lim_N G(\Sigma^N X)$ after a finite number of steps. Hence, it is sufficient for us to compute the group $G(\Sigma^N X)$ for a sufficiently large number N. The method is essentially based on Barcus-Barratt's theorem. However, we drive our main exact sequence from Puppe's sequence because our case is a stable one.

This is done in § 1. In § 2 some calculations which are needed in the later are done in the slightly more general situation. In § 3 our theorem, stated as follows, is proved. Let ξ be a S^p-bundle over S^q and we denote by K_ξ the total space of ξ. Let $\bar{\lambda}(\xi)$, $\bar{J}(\xi)$ be the stable classes of P_*-image and J-image of the characteristic class of ξ respectively, where $P_*:\pi_{q-1}(SO(p+1))\to\pi_{q-1}(S^p)$, $J:\pi_{q-1}(SO(p+1))\to\pi_{p+q}(S^{p+1})$ are usual ones.

THEOREM A. *Let L be the complex $S^p\cup e^q$ which is obtained from attaching q-cell to S^p by the P_*-image $\lambda(\xi)$. Then we have split extensions* $(q>p+1)$:

$$1\longrightarrow\theta_\xi\longrightarrow G^s(K_\xi)\longrightarrow G^s(L)\longrightarrow 1 \qquad \text{if } q\equiv 0 \bmod 4 \text{ and } 2\bar{J}(\xi)\neq 0,$$

$$1\longrightarrow\theta_\xi\longrightarrow G^s(K_\xi)\longrightarrow G^s(L)\times Z_2\longrightarrow 1 \qquad \text{if } q\not\equiv 0 \bmod 4,$$

where θ_ξ is isomorphic to $\pi^s_{p+p+1}(\Sigma L)$ if $q\not\equiv 0 \bmod 8$ and to $\pi^s_{p+q+1}(\Sigma L)/\bar{J}(\xi)\cdot\pi^s_1(S^0)$ if $q\equiv 0 \bmod 8$.

Further, about $G^s(L)$, we have

THEOREM B. *There exist split extensions* $(q>p+1)$:

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$$1 \longrightarrow \pi^s_{q-p}(S^0)/\bar{\lambda}(\xi)\cdot\pi^s_1(S^0) \longrightarrow G^s(L) \longrightarrow Z_2\times Z_2 \longrightarrow 1$$

if $2\xi=0$ *and* p *is even,*

$$1 \longrightarrow \pi^s_{q-p}(S^0)/\bar{\lambda}(\xi)\cdot\pi^s_1(S^0) \longrightarrow G^s(L) \longrightarrow Z_2 \longrightarrow 1$$

if $2\bar{\lambda}(\xi)\neq 0$ *and* p *is even.*

§1. Fundamental exact sequences

Let L be a 1-connected CW-complex and let K be a complex of the form $L\bigcup_\alpha e^n(\dim L+2\leqq n)$. Then as the Puppe's exact sequence we have the sequence

$$[\Sigma^2 L, \Sigma K] \underset{\Sigma^2\alpha}{\longrightarrow} [S^{n+1}, \Sigma K] \longrightarrow [\Sigma K, \Sigma K] \underset{R}{\longrightarrow} [\Sigma L, \Sigma L], \qquad \text{(I)}$$

where R denotes the natural ring homomorphism induced by the restriction of maps. Since $G(\Sigma K)$ is the set consisting of invertible elements of $[\Sigma K, \Sigma K]$ R induces the homomorphism

$$R\times Or:\ G(\Sigma K) \longrightarrow G(\Sigma L)\times Z_2,$$

where $Or(f)$ is defined as the degree of $f: \Sigma K\to\Sigma K$ on the $(n+1)$-cell of ΣK.

LEMMA 1.1. *The kernel of* $R\times Or$ *is isomorphic to*

$$\pi_{n+1}(\Sigma L)/[\Sigma^2 L, \Sigma L]\cdot\Sigma^2\alpha\cup\Sigma\alpha\cdot\pi_{n+1}(S^n).$$

Proof. Suppose that $R\times Or(f)=(1, 1)$ for f of $[\Sigma K, \Sigma K]$. Then, in the sequence (I), we have that $1-f$ is contained in the image of $[S^{n+1}, \Sigma K]$. Hence we have

$$f=1+\mu, \qquad \mu\in\pi_{n+1}(\Sigma K).$$

Since $Or(f)=Or(1)+Or(\mu)$ we obtain $Or(\mu)=0$ and this means $\mu\in\pi_{n+1}(\Sigma L)$. Converesely if $\mu\in\pi_{n+1}(\Sigma L)$ it is clear that $1+\mu$ is contained in $G(\Sigma K)$ and $R\times Or(1+\mu)=(1, 1)$. Thus the proof is completed from the diagram

$$\begin{array}{ccccc}
[\Sigma^2 L, \Sigma K] & \longrightarrow & [S^{n+1}, \Sigma K] & \longrightarrow & [\Sigma K, \Sigma K] \\
\uparrow & & \uparrow & & \\
[\Sigma^2 L, \Sigma L] & & [S^{n+1}, \Sigma L] & & \\
 & & \uparrow & & \\
 & & [S^{n+1}, S^n] & &
\end{array}$$

Example 1.2 (Theorem 3.13 of [8]). Let K be a CW-complex $S^{m-1}\bigcup_f e^{n-1}$ $(m+2\leqq n)$. Then we have two exact sequences:

$$1 \longrightarrow H \longrightarrow G(S^m \underset{\Sigma f}{\cup} e^n) \longrightarrow Z_2 \times Z_2 \longrightarrow 1 \qquad \text{if } 2\Sigma f = 0,$$

$$1 \longrightarrow H \longrightarrow G(S^m \underset{\Sigma f}{\cup} e^n) \longrightarrow Z_2 \longrightarrow 1 \qquad \text{if } 2\Sigma f \neq 0.$$

where H denotes $\pi_n(S^m)/\pi_{m+1}(S^m)\circ\Sigma^2 f \cup \Sigma f \circ \pi_n(S^{n-1})$.

Example 1.3. Let us consider the N-fold suspension

$$\Sigma^N : G(\Sigma K) \longrightarrow G(\Sigma^{N+1} K) \qquad (N \geqq 1)$$

in the case of $K = S^{m-1} \underset{f}{\cup} e^{n-1}$ $(m+2 \leqq n)$. From Example 1.2 we can obtain the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_n(S^m)/\pi_{m+1}(S^m)\circ\Sigma^2 f \cup \Sigma f\circ\pi_n(S^{n-1}) & \longrightarrow & G(\Sigma K) & \longrightarrow & Q & \longrightarrow & 1 \\ & & \Sigma^n \downarrow & & \Sigma^n \downarrow & & id \downarrow & & \\ 1 & \longrightarrow & W & \longrightarrow & G(\Sigma^{N+1}) & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

where Q is $Z_2 \times Z_2$ or Z_2 and W denotes the group

$$\pi_{n+N}(S^{m+N})/\pi_{m+1+N}(S^{m+N})\circ\Sigma^{N+2} f \cup \Sigma^{N+1} f \circ \pi_{n+N}(S^{n-1+N}).$$

Then, by using the standard argument, we have the exact sequence

$$1 \longrightarrow W_f \longrightarrow G(\Sigma K) \xrightarrow{\Sigma} G(\Sigma^{N+1} K) \longrightarrow \pi_{n+N}(S^{m+N})/\Sigma^N \pi_n(S^m) \longrightarrow 1,$$

where W_f is the group $\Sigma^{-N}(0)/\Sigma^{-N}(0) \cap (\pi_{m+1}(S^m)\circ\Sigma^2 f \cup \Sigma f \circ \pi_n(S^{n-1}))$.

§ 2. Some calculations

Let L_1 and L_2 be CW-complexes $S^{p+1} \underset{\alpha}{\cup} e^{q+1}$, $S^{p+1} \underset{\beta}{\cup} e^{p+q+1}$ $(q > p \geqq 2)$ respectively. Let K be the CW-complex obtained from identifying the common sphere of L_1 and L_2. Our purpose of this section is to calculate $G(\Sigma^N K)$ for $N \geqq 2$. Since $\Sigma^N K$ has a CW-decomposition

$$\Sigma^N K = S^{p+1+N} \underset{\Sigma^N \alpha}{\cup} e^{q+1+N} \underset{\Sigma^N \beta}{\cup} e^{p+q+1+N},$$

we have the following exact sequence by applying Lemma 1.1 to the pair $(\Sigma^N K, L = \Sigma^N L_1)$

$$1 \longrightarrow H_{\alpha,\beta} \longrightarrow G(\Sigma^N K) \longrightarrow G(L) \times Z_2,$$

where $H_{\alpha,\beta} = \pi_{N+p+q+1}(L)/[\Sigma L, L]\circ\Sigma^{N+1}\beta \cup \Sigma^N \beta \circ \pi_{N+p+q+1}(S^{N+p+q})$.

LEMMA 2.1. *If $N \geqq 2$ we have*

$$[\Sigma L, L]\circ\Sigma^{N+1}\beta \cup \Sigma^N \beta \circ \pi_{N+p+q+1}(S^{N+p+q}) = \Sigma^N \beta \circ \pi_{N+p+q+1}(S^{N+p+q}).$$

Proof. First we note a formula of [1] i.e.

$$\Sigma^{b+1}x\circ\Sigma^{m+1}y=(-1)^{*}\Sigma^{a+1}y\circ\Sigma^{n+1}x$$

for any $x\in\pi_{m+1}(S^{a+1})$, $y\in\pi_{n+1}(S^{b+1})$.

Let $\eta: S^3\to S^2$ be the Hopf map. By the above formula we have

$$\Sigma^{N+p-1}\eta\circ\Sigma^{N+1}\alpha=\Sigma^{N}\alpha\Sigma^{N+q-2}\eta \qquad (N\geqq 2).$$

Hence, $\Sigma^{N+p-1}\eta$ is extendable to a map $\Sigma L\to L$ and the sub-group $[\Sigma L, L]\circ\Sigma^{N+1}\beta$ is generated by the i-image of $\Sigma^{N+p-1}\eta\circ\Sigma^{N+1}\beta$ ($i: S^{N+p+1}\to L$ is the inclusion). Thus the proof is completed by the formula

$$\Sigma^{N+p-1}\eta\circ\Sigma^{N+1}\beta=\Sigma^{N}\beta\circ\Sigma^{N+p+q-2}\eta \qquad (N\geqq 2).$$

LEMMA 2.2. *The image of the homomorphism*

$$R\times Or: G(\Sigma^N K)\longrightarrow G(L)\times Z_2$$

is equal to $G(L)$ *if* $2\Sigma^N\beta\notin\Sigma^N\alpha\circ\pi_{N+p+q}(S^{N+q})$ *and to* $G(L)\times Z_2$ *if* $2\Sigma^N\beta\in\Sigma^N\alpha\pi_{N+p+q}(S^{N+q})$.

Proof. Since Hurewicz-image: $\pi_*(\Sigma^N K)\to H_*(\Sigma^N K)=Z$ ($*=N+p+q+1$) contains ± 2 if and only if $2\Sigma^N\beta=0$ in $\pi_{N+p+q}(L)$ the proof follows from the standard obstruction theory.

Next, consider the group $G(L)$. Then, from Example 1.2 and the formula $\Sigma^N\alpha\circ\Sigma^{N+q-2}\eta=\Sigma^{N+p-1}\eta\circ\Sigma^{N+1}\alpha$ ($N\geqq 2$) we can easily obtain

LEMMA 2.3. *The following sequences are exact*

$$1\longrightarrow\pi_{N+q+1}(S^{N+p+1})/\{\Sigma^N\alpha\circ\Sigma^{N+p-2}\eta\}\longrightarrow G(L)\longrightarrow Z_2\times Z_2\longrightarrow 1 \qquad \text{if } 2\Sigma^N\alpha=0$$

$$1\longrightarrow\pi_{N+q+1}(S^{N+p+1})/\{\Sigma^N\alpha\circ\Sigma^{N+p-2}\eta\}\longrightarrow G(L)\longrightarrow Z_2\longrightarrow 1 \qquad \text{if } 2\Sigma^N\alpha\neq 0.$$

Now, from lemmas we obtain

PROPOSITION 2.4. *If* $N\geqq 2$ *we have the exact sequence*

$$1\longrightarrow H\longrightarrow G(\Sigma^N K)\longrightarrow \mathrm{Aut}(\bar{H}_*(\Sigma^N K)),$$

where H *is a group-extension as follows* ($*=N+p+q+1$)

$$1\longrightarrow\pi_*(L)/\{\Sigma^N\beta\circ\Sigma^{*-3}\eta\}\longrightarrow H\longrightarrow\pi_{N+q+1}(S^{N+p+1})/\{\Sigma^N\alpha\circ\Sigma^{N+q-2}\eta\}\longrightarrow 1.$$

Moreover the image of $G(\Sigma^N K)\to\mathrm{Aut}(\bar{H}_*(\Sigma^N K))$ *is given as follows (we identify* $\mathrm{Aut}(\bar{H}_*(\Sigma^N K))$ *with* $Z_2\times Z_2\times Z_2$)

$$Z_2\times Z_2\times Z_2 \quad \text{if } 2\Sigma^N\alpha=0 \text{ and } 2\Sigma^N\beta\in\Sigma^N\alpha\circ\pi_{N+p+q}(S^{N+q}),$$

$$1\times Z_2\times Z_2 \quad \text{if } 2\Sigma^N\alpha=0 \text{ and } 2\Sigma^N\beta\notin\Sigma^N\alpha\circ\pi_{N+p+q}(S^{N+q}),$$

$$Z_2\times 1\times Z_2 \quad \text{if } 2\Sigma^N\alpha\neq0 \text{ and } 2\Sigma^N\beta\in\Sigma^N\alpha\circ\pi_{N+p+q}(S^{N+q}),$$

$$1\times 1\times Z_2 \quad \text{if } 2\Sigma^N\alpha\neq0 \text{ and } 2\Sigma^N\beta\notin\Sigma^N\alpha\circ\pi_{N+p+q}(S^{N+q}).$$

§ 3. The case of sphere bundles over the sphere

Let ξ be a S^p-bundle over S^q $(q>p+1\geqq3)$. It is known in [7] that the total space K_ξ has a CW-decomposition

$$K_\xi=S^p\cup e^q\cup e^{p+q}.$$

Let $T(\xi)$ be the Thom complex of the vector bundle associated with ξ. Since we may regard $T(\xi)$ as the mapping cone of the projection of ξ (up to homotopy) there exists the natural map of degree 1

$$C_\xi=T(\xi)\longrightarrow C_\xi/S^q=\Sigma(K_\xi).$$

Hence, by using the CW-decomposition of $T(\xi)$;

$$T(\xi)=S^{p+1}\bigcup_{J(\xi)}e^{p+q+1},$$

we have a CW-decomposition of ΣK_ξ as follow

$$\Sigma K_\xi=e^{p+q+1}\bigcup_{J(\xi)}S^{p+1}\bigcup_{\lambda(\xi)}e^{q+1},$$

where $J(\xi)$ is the usual notation and $\lambda(\xi)$ denotes the P_*-image $(P_*:\pi_{q-1}(SO(p+1))\to\pi_{q-1}(S^p))$.

Thus we can apply Prop. 2.4 to the case of $(K, L)=(K_\xi, L)$.

Let $\bar{J}(\xi)$ $(\in\pi^s_{q-1}(S^0))$, $\bar{\lambda}(\xi)$ $(\in\pi^s_{q-p-1}(S^0))$ be the stable classes of $J(\xi)$, $\lambda(\xi)$ respectively. Then it is well known that

$$\begin{aligned}\bar{J}(\xi)&=0 && \text{if } q\equiv3,5,6,7 \bmod 8,\\ &\in Z_2 && \text{if } q\equiv1,2 \bmod 8,\\ &\in Z_{m(q)} && \text{if } q\equiv0 \bmod 8\end{aligned}$$

and $\bar{J}(\xi)\circ\eta=0$ if $q\not\equiv0 \bmod 8$, $\neq0$ if $q\equiv0 \bmod 8$. Hence we have

PROPOSITION 3.1. *There exists following exact sequences*:

$$1\longrightarrow\theta_\xi\longrightarrow G^s(K_\xi)\longrightarrow \mathrm{Aut}(\bar{H}_*(K_\xi))$$

and for $q\not\equiv0 \bmod 8$

$$1\longrightarrow K_\xi\longrightarrow\theta_\xi\longrightarrow\pi^s_{q-p}(S^0)/\{\bar{\lambda}(\xi)\circ\eta\}\longrightarrow1,$$

$$1 \longrightarrow \pi_q^s(S^0)/\{\bar{\lambda}(\xi)\circ\pi_{p+1}^s(S^0)\} \longrightarrow K_\xi \longrightarrow \{x \mid x\in\pi_q^s(S^0),\ \bar{\lambda}(\xi)\circ x=0\} \longrightarrow 1\,.$$

Next, Let consider the splitting of group-extensions which are already obtained in § 2.

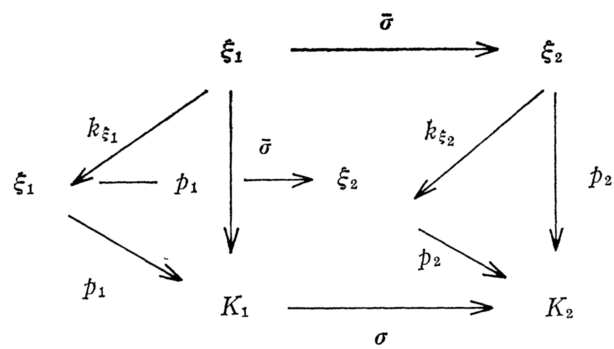
LEMMA 3.2. *There exists an involution* $h_\xi : \Sigma^N T(\xi)\to\Sigma^N T(\xi)$ *which is degree* -1 *on* S^{N+p+1} $(N\geqq 2)$.

Proof. Since we may consider K_ξ as $(S^p\times E_-^q)\cup(S^p\times E_+^q)$ we have the map $k_\xi : K_\xi\to K_\xi$ defined by

$$k_\xi(v,\ x_\varepsilon)=(-v,\ x_\varepsilon) \qquad (\varepsilon=\pm,\ x\in E_\varepsilon^q)\,.$$

Then k_ξ induces an involution $T(k_\xi) : T(\xi)\to T(\xi)$ of degree $(-1)^{p+1}$ on S^{N+p+1}. Hence, it is sufficient to take h_ξ as the suspension of $T(k_\xi)$ for even p and of $T(K_{\xi\oplus 1})$ for odd p.

Remark. We note that k_ξ is commutative with any bundle map, i. e. we have a commutative diagram:



Next, for any homotopy equivalence $f : L\to L$, define $\bar{f} : \Sigma^N K\to\Sigma^N K$ by

$$\bar{f}\,|\,L=f\,,\quad\text{and}\quad \bar{f}\,|\,\Sigma^N T(\xi)=h_\xi^{(3+\deg f)/2}\,,$$

where $\deg f$ is the degree on $S^{N+p+1}\subset L$.

LEMMA 3.3. *Define* $S : G(L)\to G(\Sigma^N K_\xi)$ *by* $S(f)=\bar{f}$. *Then,* S *is a homomorphism and the composite* $R\circ S$ *is the identity.*

Proof. For $f,\ g : L\to L$, we have

$$f\circ g\,|\,L=f\circ g\,|\,L=f\,|\,L\circ g\,|\,L$$

and

$$f\circ g\,|\,\Sigma^N T(K_\xi)=h_\xi^{(3+\deg f g)/2}=h_\xi^{(3+\deg f\,\circ\,\deg g)/2}$$
$$=h_\xi^{(3+\deg f)/2}\circ h_\xi^{(3+\deg g)/2}=\bar{f}\,|\,\Sigma^N T(\xi)\circ\bar{g}\,|\,\Sigma^N T(\xi)\,.$$

LEMMA 3.4. *If* $q\not\equiv 0 \mod 4$ *there exists and involution (up to homotopy)* l_ξ: $\Sigma^N T(\xi)\to\Sigma^N T(\xi)$ *(sufficiently large* N*) which is of degree* 1 *on* S^{N+p+1} *and of degree* -1 *on* $e^{N+p+q+1}$, *and further we have* $h_\xi\circ l_\xi=l_\xi\circ h_\xi$.

Proof. First we note the following sub-lemma

SUB-LEMMA. Let $X=S^N\cup_\sigma e^{N+p}$ be a CW-complex (N: sufficiently large). Let $f: X\to X$ be a homotopy equivalence with degree 1 on S^N and degree -1 on e^{N+p}. Then the composite $f\circ f$ is the identity in $[X, X]$.

For, from Puppe's sequence

$$[S^{N+p}, X]\longrightarrow[X, X]\longrightarrow[S^N, X]$$

there exists an element $\alpha\in\pi_{N+p}(X)$ such that $f=1_X+\alpha$. Hence, we have $f\circ f=(1_X+\alpha)(1_X+\alpha)=1_X+2\alpha+(\deg\alpha)\alpha$. On the other hand, since degree $f=1+$degree α, we have degree $\alpha=-2$. Thus the proof is complete.

Secondly we prove Lemma 3.4. Since $2\xi=0$ in the stable range the bundle induced by a map: $S^q\to S^q$ of degree -1 is equivalent to ξ. Hence, from sub-lemma, we have a bundle map: $K_\xi\to K_\xi$ which induces the map l_ξ, as desired. The commutativity follows from the above remark.

For $q\not\equiv 0 \mod 4$, we define an involution $\bar{l}_\xi: \Sigma^N K\to\Sigma^N K$ by

$$\bar{l}_\xi|L=l_L \quad\text{and}\quad \bar{l}_\xi|\Sigma^N T(\xi)=l_\xi.$$

Now, consider the correspondence $V: G^s(L)\times Z_2\to G^s(K)$ defined by

$$V(f, \varepsilon)=\bar{l}_\xi^*\circ S(f)=\bar{l}_\xi^*\circ\bar{f},$$

where $*$ denotes $(\varepsilon+3\deg f)/2$ and ε is 1 or -1.

LEMMA 3.5. *V is a homomorphism and the composite $R\times Or\cdot V$ is the identity.*

Proof. The former follows from the commutativity of Lemma 3.4 and the latter is easily obtained from the definitions.

Now the theorem stated in the introduction is an easy consequence from Lemmas 1.1, 2.2, 3.4 and 3.5.

Example 3.6. Let ξ be the trivial bundle, hence $K_\xi=S^p\times S^q$ ($q>p\geqq 2$). Then we have a split extension

$$1\longrightarrow\pi_q^s(S^0)\oplus\pi_p^s(S^0)\longrightarrow G^s(K_\xi)\longrightarrow G\times Z_2\longrightarrow 1,$$

where G denotes the group $\left\{\begin{pmatrix}\varepsilon_2 & \gamma\\ 0 & \varepsilon_1\end{pmatrix}, \gamma\in\pi_{q-p}^s(S^0), \varepsilon_i=1 \text{ or } -1\right\}$.

Further, $G^s(K_\xi)$ is isomorphid to the group

$$\left\{\begin{pmatrix} \varepsilon_3 & \alpha & \beta \\ 0 & \varepsilon_2 & \gamma \\ 0 & 0 & \varepsilon_1 \end{pmatrix}, \alpha\in\pi_p^s(S^0),\ \beta\in\pi_q^s(S^0),\ \gamma\in\pi_{q-p}^s(S^0),\ \varepsilon_i=1, -1\right\}.$$

Example 3.7. Let $W_{n,2}$ be the Stiefel manifold $U(n)/U(n-2)$ and let ξ_n be the sphere bundle $S^{2n-3}\rightarrow W_{n,2}\rightarrow S^{2n-1}$. Then it is well known that $\Sigma^2 J(\xi_n)=0$ (1.17 of [7]) and $\lambda(\xi_n)=0$ if n is even, $=\eta(\in\pi_1^s(S^0))$ if n is odd. Thus if n is even the case reduces to Example 3.6, namely K_{ξ_n} is stablly equivalent to $S^p\times S^q$ (up to homotopy). Next, let CP^2 be the complex projective plane. Then, for odd n, we have a split extension:

$$1 \longrightarrow \pi_{4n-4}^s(CP^2) \longrightarrow G^s(W_{n,2}) \longrightarrow Z_2\times Z_2\times Z_2 \longrightarrow 1.$$

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