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TOTALLY REAL SUBMANIFOLDS OF AN ALMOST HERMITIAN MANIFOLD I

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§ **1.—Introduction.**—Let $(\bar{M}, g, J)$ be an almost Hermitian manifold, that is, the tangent bundle of $\bar{M}$ has an almost complex structure J and a Riemannian metric, g, such that $g(JX, JY)=g(X, Y)$ for all $X, Y \in T\bar{M}$. Then $\dim \bar{M}=2m$ and $\bar{M}$ is orientable.

In (2), B. Y. Chen and K. Ogiue studied some fundamental properties of totally real submanifolds of a Kaehler manifold.

In (3), B. Y. Chen, C. S. Houh and H. S. Lue follow the study of totally real submanifolds in a Kaehler manifold.

In (10), L. Vanhecke studied some fundamental properties of totally real submanifolds of a generalized complex space forms.

In this paper we study some properties of totally real submanifolds of an almost hermitian manifold (In particular, a Nearly Kaehler manifold).

We obtain some generalizations for results of (3), (6), (7) and (12).

In the last section we study a Hermitian connection (4), (9), respect to a totally real submanifold in an almost hermitian manifold. In particular, we obtain some basic formulas for this connection (Formulas of Gauss and Weingarten, equations of Gauss, Codazzi, Ricci$\cdots$).

§ **2.—Basic formulas.**—Let $\bar{M}^{2m}$ be a $2m$-dimensional almost Hermitian manifold with almost complex structure J and metric tensor g. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$.

It is well-known that $\bar{M}$ is a Nearly Kaehler manifold if

$$(\bar{\nabla}_X J)X=0 \tag{2.1}$$

for all $X \in T\bar{M}$, where $T\bar{M}$ is the tangent bundle of M. For $X \in T\bar{M}$, we denote a section (tangent vector field) in this vector bundle.

Let M^n be an n-dimensional totally real submanifold of $\bar{M}$, that is, for $x \in M$, $J(T_xM)$ is perpendicular to T_xM. Then the second fundamental form σ is given by

$$\sigma(X, Y)=\bar{\nabla}_X Y-\nabla_X Y \tag{2.2}$$

for all $X, Y \in TM$, where TM is the tangent bundle to M and ∇ is the induced connection of M.

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The mean curvature vector is given by $H=\frac{1}{n}$ trace σ. For a normal vector field ξ, we write

$$\bar{\nabla}_X\xi=-A_\xi X+D_X\xi \tag{2.3}$$

where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. normal) component of $\bar{\nabla}_X\xi$. Then we have,

$$g(\sigma(X, Y), \xi)=g(A_\xi X, Y) \tag{2.4}$$

A normal vector field ξ is called a parallel section in the the normal bundle $T^\perp M$ if $D\xi=0$.

A subbundle S of the normal bundle $T^\perp M$ is holomorphic if S is invariant under J, i. e. $JS=S$.

A subbundle S of $T^\perp M$ is said to be parallel if S is invariant under parallel translation, i. e. for every local section ξ in S, $D_X\xi$ is also a section in S. It is clear that a unit normal vector field ξ is parallel if and only if the line bundle generated by ξ is parallel. For a subbundle S of $T^\perp M$, there exists a unique subbundle S^c of $T^\perp M$ such that S and S^c are orthogonal and $S\oplus S^c=T^\perp M$. We call S^c the complementary subbundle of S. It is clear that for a totally real submanifold M in $\bar{M}$, the complementary subbundle $(J(TM))^c$ of $J(TM)$ is always holomorphic. Moreover, S is parallel if and only if S^c is parallel.

We call the complementary subbundles of holomorphic subbundles of $T^\perp M$, the coholomorphic subbundles of $T^\perp M$. Then a subbundle S of $T^\perp M$ is coholomorphic if and only if S is the direct sum of $J(TM)$ and a holomorphic subbundle of $T^\perp M$.

§ **3.—Parallel subbundles.**—In this section we consider an almost Hermitian manifold which is a Nearly kaehlerian.

LEMMA 3.1.—*Let M^n be a totally real submanifold of a Nearly Kaehler manifold $\bar{M}^{2m}$. If S is a $2r$-dimensional parallel holomorphic subbundle of $T^\perp M$, then $\sigma/S=0$.*

Proof.—It is very easy to prove that

$$g(\sigma(X, Y), \xi)=-g((\bar{\nabla}_X J)Y, J\xi) \tag{3.1}$$

for all $X, Y\in TM$ and $\xi\in S$.

If we use that $\bar{M}$ is a Nearly Kaehler manifold and σ is symmetric, we have from (3.1) that $g(\sigma(X, Y), \xi)=0$ for all $\xi\in S$. Then $\sigma/_{S=0}$ (Q. E. D.)

LEMMA 3.2.—*Let M^n be a totally real submanifold of a Nearly Kaehler manifold $\bar{M}^{2m}$. If S is a parallel coholomorphic subbundle of $T^\perp M$, then* $\mathrm{Im}\,\sigma\subset S$, *where* $\mathrm{Im}\,\sigma=\{\sigma(X, Y)/X, Y\in TM\}$.

Proof.—It is easy to see that S is parallel if and only if S^c is parallel.

Then, the result follows from Lemma 3.1.

LEMMA 3.3.—*Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. We suppose that M is a totally real submanifold of a $2(n+s)$-dimensional totally geodesic complex submanifold $N^{2(n+s)}$ of $\bar{M}^{2m}$. Then, there exists an $(n+2s)$-dimensional parallel coholomorphic subbundle S of $T^{\perp}M$.*

Proof.—We define $S=T_N^{\perp}M$, that is, S is the normal bundle of M in N.

It is clear that $T^{\perp}M=S\oplus T^{\perp}N$, $\dim S=n+2s$ and that S is coholomorphic. Since N is totally geodesic in $\bar{M}$, we have

$$\bar{\nabla}_Y\eta=\bar{D}_Y\eta \tag{3.2}$$

for all $Y\in TN$ and $\eta\in T^{\perp}N$. Moreover

$$g(\bar{\nabla}_X\xi,\ \eta)+g(\xi,\ \bar{\nabla}_X\eta)=0 \tag{3.3}$$

for all $\xi\in S$, $\eta\in T^{\perp}N$ and $X\in TM$.

Substituting (2.3) and (3.2) in (3.3), we get $g(D_Y\xi,\ \eta)=0$. Hence $D_Y\xi\in S$.

(Q. E. D.)

Remark.—Lemma 3.1 and Lemma 3.2. has been proved by B. Y. Chen, C. S. Houh and H. S. Lue (3) for a Kaehler manifold.

§ 4.—f-structure in the normal bundle.

Let ξ be any normal vector field on M^n in $\bar{M}^{2m}$. We put

$$J\xi=P\xi+f\xi \tag{4.1}$$

where $P\xi$ and $f\xi$ denote respectively the tangential and the normal component of $J\xi$. Then P is a tangent bundle valued 1-form and f is an endomorphism of the normal bundle. Then,

$$-\xi=JP\xi+Jf\xi \tag{4.2}$$

and making use of (4.1)

$$-\xi=JP\xi+Pf\xi+f^2\xi \tag{4.3}$$

Comparing the tangential and normal parts in (4.3), we get

$$Pf\xi=0 \tag{4.4}$$

$$f^2\xi+JP\xi=-\xi \tag{4.5}$$

In particular, if $\xi=JX$ for $X\in TM$, we have $-X=PJX+fJX$ thus $fJX=0$ and $-X=PJX$. By applying f to (4.3), we get $f^3\xi=-f\xi$. Since ξ is an arbitrary normal vector field $f^3+f=0$. Therefore, if the endomorphism f doesn't vanish (i. e. if $n<m$) it defines an f-structure in $T^{\perp}M$.

We define the covariant derivative of f with respect to D by

$$(D_X f)\xi = D_X f\xi - fD_X\xi \tag{4.6}$$

for all $X \in TM$ and $\xi \in T^\perp M$.

Moreover, we define the covariant derivative of P with respect to the connection in $TM \oplus T^\perp M$ obtained by combining the connections ∇ in TM and D in $T^\perp M$

$$(\hat{\nabla}_X P)\xi = \nabla_X P\xi - PD_X\xi \tag{4.7}$$

for all $X \in TM$ and $\xi \in T^\perp M$.

If $D_X f = 0$ (respectively, $\hat{\nabla}_X P = 0$ for all tangent vector fields X, then the f-structure in the normal bundle (respectively, the tangent bundle valued 1-form P) is said to be parallel.

§ 5.—Parallel f-structure.

LEMMA 5.1.—*Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$, then, for all X, $Y \in TM$ and $\xi \in T^\perp M$, we have*

$$(\bar{\nabla}_X J)\xi = (\hat{\nabla}_X P)\xi - A_{f\xi}X + (D_X f)\xi + JA_\xi X + \sigma(X, P\xi) \tag{5.1}$$

and

$$(\bar{\nabla}_X J)Y = -A_{JY}X - P\sigma(X, Y) + D_X JY - J\nabla_X Y - f\sigma(X, Y) \tag{5.2}$$

Proof.—For $\xi \in T^\perp M$ and $X \in TM$, we have

$$\bar{\nabla}_X J\xi = \bar{\nabla}_X P\xi + \bar{\nabla}_X f\xi$$

Then,

$$(\bar{\nabla}_X J)\xi - JA_\xi X + JD_X\xi = \nabla_X P\xi + \sigma(X, P\xi) - A_{f\xi}X + D_X f\xi$$

From (4.6) and (4.7), we obtain (5.1). In the proof of (5.2), we use a similar reasoning.

COROLLARY 5.2.—*Let M^n be a totally real submanifold of a Nearly Kaehler manifold $\bar{M}^{2m}$. Then for all X, $Y \in TM$, we have*

$$P\sigma(X, Y) = -\frac{1}{2}(A_{JX}Y + A_{JY}X) \tag{5.3}$$

and

$$f\sigma(X, Y) = \frac{1}{2}(D_X JY + D_Y JX - JD_X Y - JD_Y X) \tag{5.4}$$

The proof is immediate.

If we consider that $\bar{M}^{2m}$ is a Kaehler manifold, then we have the following result.

THEOREM 5.3.—*Let M^n be a totally real submanifold of a Kaehler manifold $\bar{M}^{2m}$ then the following statements are equivalent*

a) *The f-structure in the normal bundle is parallel.*
b) M^n *is geodesic w. r. t.* $(J(TM))^c$.
c) *The tangent bundle valued 1-form P is parallel.*
d) $(J(TM))^c$ *is parallel.*
e) $J(TM)$ *is parallel.*

Proof.—a) implies b) is proved in (12) and b) implies a) is proved in (6).

It is very easy to prove that $\operatorname{Im} f=(J(TM))^c$ and $\operatorname{Ker} f=J(TM)$, then taking in account that $\bar{M}$ is a Kaehler manifold ($\bar{\nabla}J=0$) and from (5.1), we have the others implications.

PROPOSITION 5.4.—*Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. Then, the following statements are equivalent.*

a) *M is geodesic w. r. t.* $(J(TM))^c$;
b) $\operatorname{Im}\sigma\subset J(TM)$; c) $\sigma/_{(J(TM))^c}=0$

where $\operatorname{Im}\sigma=\{\sigma(X, Y)/X, Y\in TM\}$

The proof is immediate.

In the next theorem, we give a generalization of a result of K. Yano-M. Kon (12), for a Nearly-Kaehler manifold.

THEOREM 5.5.—*Let M^n be a totally real submanifold of a Nearly-Kaehler manifold $\bar{M}^{2m}$. If the f-structure in the normal bundle is parallel, then M^n is geodesic w. r. t. $(J(TM))^c$. Moreover, for all $X\in TM$ and $\xi\in(J(TM))^c$, $(\bar{\nabla}_X J)\xi=0$.*

Proof.—It is well-known that an almost Hermitian manifold N satisfies

$$g((\bar{\nabla}_X J)Y, Y)=0 \tag{5.5}$$

for all $X, Y\in TN$, where J is the almost Hermitian structure and $\bar{\nabla}$ the Riemannian connection.

By (5.1), for all $Y\in TM$, we have

$$g((\bar{\nabla}_X J)\xi, JY)=g((D_X f)\xi, JY)+g(\sigma(X, Y), \xi)+g(\sigma(X, P\xi), JY) \tag{5.6}$$

If f is parallel, for all $\xi\in(J(TM))^c$ from (4.5), we get,

$$g(\sigma(X, Y), \xi)=-g((\bar{\nabla}_X J)Y, J\xi) \tag{5.7}$$

Since σ is symmetric and $\bar{M}$ is a Nearly-Kaehler manifold, $g(\sigma(X, Y), \xi)=0$ for all $X, Y\in TM$ and $\xi\in(J(TM))^c$. Moreover, $(\bar{\nabla}_X J)\xi\in TM$ for all $X\in TM$ and $\xi\in(J(TM))^c$. Since $(J(TM))^c$ is a holomorphic subbundle of $T^\perp M$, it is clair that

$$(\bar{\nabla}_X J)\xi=0 \qquad \text{(Q. E. D.)}$$

THEOREM 5.6.—*Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. We suppose that*

a) *M is geodesic w.r.t.* $(J(TM))^c$

b) *For all* $X \in TM$ *and* $\xi \in (J(TM))^c$, $(\bar{\nabla}_X J)\xi = 0$

Then, the f-structure in the normal bundle is parallel.

Proof.—If $\xi \in (J(TM))^c$, then $A_\xi = 0$, $P\xi = 0$ and $(\bar{\nabla}_X J)\xi = 0$, from (5.1), $(D_X f)\xi = 0$. If $\xi = JY$ for $Y \in TM$, then we can consider two cases

i) Let η be a normal vector field in $(J(TM))^c$, then $A_\eta = 0$ and

$$g((D_X f)\xi, \eta) = -g((\bar{\nabla}_X J)\eta, \xi) + g(A_\xi X, J\eta) = 0$$

ii) Let η be a normal vector field in $J(TM)$, then $\eta = JZ$ for $Z \in TM$

$$\begin{aligned} g((D_X f)\xi, \eta) &= g((\bar{\nabla}_X J)JY, JZ) - g(A_{JY}X, Z) + g(\sigma(X, Y), JZ) \\ &= -g(\bar{\nabla}_X Y, JZ) - g(\bar{\nabla}_X JY, Z) - g(A_{JY}X, Z) + g(\sigma(X, Y), JZ) = 0 \end{aligned}$$

Thus, $g((D_X f)\xi, \eta) = 0$ for all $\eta \in T^\perp M$, then $(D_X f)\xi = 0$ for all $\xi \in T^\perp M$ and $X \in TM$. (Q.E.D.)

COROLLARY 5.7.—*Let* M^n *be a totally real submanifold of a Nearly Kaehler manifold* $\bar{M}^{2m}$. *Then the f-structure in the normal bundle is parallel if and only if the following statements hold*

a) *M is geodesic w.r.t.* $(J(TM))^c$

b) *For all* $X \in TM$ *and* $\xi \in (J(TM))^c$, $(\bar{\nabla}_X J)\xi = 0$.

THEOREM 5.8.—*Let* M^n *be a totally real submanifold of an almost Hermitian manifold* $\bar{M}^{2m}$. *We suppose that* $(\bar{\nabla}_X J)\xi = 0$ *for all* $X \in TM$ *and* $\xi \in (J(TM))^c$. *Then the following statements are equivalent*

1) *The f-structure in the normal bundle is parallel.*

2) *M is geodesic w.r.t.* $(J(TM))^c$.

The proof is immediate.

THEOREM 5.9.—*Let* M^n *be a totally real submanifold of a Nearly Kaehler manifold* $\bar{M}^{2m}$. *If the f-structure in the normal bundle is parallel, then the normal subbundle* $(J(TM))^c$ *is parallel.*

Proof.—For all $X \in TM$ and $\xi \in (J(TM))^c$, we have

$$0 = (\bar{\nabla}_X J)\xi = (-A_{J\xi}X - PD_X\xi) + (D_X J\xi - fD_X\xi + JA_\xi X)$$

Thus,

$$PD_X\xi = -A_{J\xi}X \quad \text{and} \quad fD_X\xi = D_X J\xi + JA_\xi X$$

since $\mathrm{Im}\,\sigma \subset J(TM)$, we have

$$PD_X\xi = 0\,, \qquad fD_X\xi = D_X J\xi$$

Then, $D_X J\xi \in (J(TM))^c$. Since $(J(TM))^c$ is holomorphic, we get that $(J(TM))^c$ is parallel.

COROLLARY 5.10.— *Let M^{m-1} be a totally real submanifold of a Nearly-Kaehler manifold M^{2m}. Then the following statements are equivalent*

1) *The f-structure in the normal bundle is parallel.*
2) *M is geodesic w. r. t. $(J(TM))^c$.*

$$(\bar{\nabla}_X J)\xi=0 \quad \textit{for all} \quad X\in TM \textit{ and } \quad \xi\in(J(TM))^c.$$

3) *The normal subbundle $(J(TM))^c$ is parallel.*
4) *The normal subbundle $J(TM)$ is parallel.*

§ 6.—Parallel 1-form P.

—In this section, we study, in which cases the tangent bundle valued 1-form P is parallel.

THEOREM 6.1.— *Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. We suppose that*

a) *M is geodesic w. r. t. the normal subbundle $(J(TM))^c$.*
b) *For all $X\in TM$ and $\xi\in T^{\perp}M$, $(\bar{\nabla}_X J)\xi\in T^{\perp}M$.*

Then, the tangent bundle valued 1-form P is parallel.

Proof.—From (5.1), we have

$$g((\bar{\nabla}_X J)\xi,\ Y)=g((\hat{\nabla}_X P)\xi,\ Y)-g(A_{f\xi}X,\ Y) \tag{6.1}$$

for all $Y\in TM$.

Since $f\xi\in(J(TM))^c$, from (6.1), we have $(\hat{\nabla}_X P)\xi=0$. (Q. E. D.)

THEOREM 6.2.—*Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. We suppose that $(\bar{\nabla}_X J)\xi\in T^{\perp}M$ for all $X\in TM$ and $\xi\in T^{\perp}M$; then, the following statements are equivalent*

1) *The tangent bundle valued 1-form P is parallel.*
2) *M is geodesic w. r. t. $(J(TM))^c$.*

Proof.—(2) implies (1) is proved in Theorem 5.1. If P is parallel, from (5.1), we have $g(A_{f\xi}X, Y)=0$ for all $X, Y\in TM$ and $\xi\in T^{\perp}M$. Since $f\xi\in(J(TM))^c$ and $\mathrm{Im} f=(J(TM))^c$ we get the other implication.

§ 7.—Totally umbilical submanifolds.

—In this section, we consider that M^n is totally umbilical, that is

$$\sigma(X,\ Y)=g(X,\ Y)H \tag{7.1}$$

for all $X, Y\in TM$, where H is the mean curvature vector.

THEOREM 7.1.—*Let M^n be a totally real submanifold of a Nearly-Kaehler manifold $\bar{M}^{2m}$. We suppose that:*

a) *M is totally umbilical.*

b) *The f-structure in the normal bundle is parallel.*

Then M is totally geodesic.

Proof.—If $\bar{M}$ is a Nearly-Kaehler manifold, that is, $(\bar{\nabla}_X J)X=0$ then

$$J\sigma(X, X)=-A_{JX}X+D_X JX-J\nabla_X X \tag{7.2}$$

for all $X\in TM$.

If X is any unit vector field perpendicular to Y, then from (7.1) and (7.2)

$$0=g(X, X)\cdot g(H, JX)=g(H, JY) \tag{7.3}$$

thus $H\in(J(TM))^c$. From Theorem 5.5 and Proposition 5.4 wet get $H=0$.

(Q. E. D.)

COROLLARY 7.2.— *Let M^n $(n>1)$ be a totally real submanifold of a Kaehler manifold $\bar{M}^{2m}$. We suppose that:*

a) *M is totally umbilical.*

b) *The f-structure in the normal bundle is parallel.*

Then, M is totally geodesic.

Remark. Corollary 7.2 has been proved by G. D. Ludden, M. Okumura and K. Yano (7), in the case $m=n$.

COROLLARY 7.3.— *Let M^m $(m>1)$ be a totally real submanifold of a Nearly Kaehler manifold $\bar{M}^{2m}$. If M is totally umbilical, then, M is totally geodesic.*

§ 8.—**On a Hermitian connection.**—Let $\bar{M}^{2m}$ be an almost Hermitian manifold with Riemannian connection $\bar{\nabla}$. Then we can define a new connection $\bar{\nabla}'$ on $\bar{M}$ by(4),

$$\bar{\nabla}'_X Y=\frac{1}{2}(\bar{\nabla}_X Y-J\bar{\nabla}_X JY) \tag{8.1}$$

for all X, $Y\in T\bar{M}$.

It is well-known that $(\bar{\nabla}'_X J)=0$ and so $\bar{\nabla}'$ is a Hermitian connection in the sense of (8).

Let M^n be a totally real submanifold on $\bar{M}^{2m}$. If X, $Y\in TM$, we can write

$$\bar{\nabla}'_X Y=\nabla'_X Y+\sigma'(X, Y) \tag{8.2}$$

where, $\nabla'_X Y$ (resp. $\sigma'(X, Y)$) denotes the tangential component (resp. the normal component) of $\bar{\nabla}'_X Y$.

PROPOSITION 8.1.— *If M^n is a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$, then*

a) *∇' is a connection on M.*

b) *The mapping $\sigma': TM\times TM\to T^{\perp}M$ is bilinear over $F(M)$.*

c) *We have the following relations*

$$\nabla'_X Y = \frac{1}{2}(\nabla_X Y - PD_X JY) \tag{8.4}$$

and

$$\sigma'(X, Y) = \frac{1}{2}(\sigma(X, Y) + JA_{JY}X - fD_X JY) \tag{8.5}$$

where, $F(M)$ is the algebra of C^∞ differentiable functions on M.

Next, if $X \in TM$ and $\xi \in T^\perp M$, we write

$$\nabla'_X \xi = -A'_\xi X + D'_X \xi \tag{8.6}$$

where, $-A'_\xi X$ and $D'_X\xi$ are symbols for the tangential and normal components.

PROPOSITION 8.2.— *Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. Then*

a) *D' is a connection in the normal bundle $T^\perp M$.*

b) $g(\sigma'(X, Y), \xi) = g(A'_\xi X, Y)$ (8.7)

for all $X, Y \in TM$ and $\xi \in T^\perp M$.

c) *The mapping $A' : (X, \xi) \in TM \times T^\perp M \to A'_\xi X \in TM$ is bilinear over $F(M)$.*

d) *We have the following relations*

$$A'_\xi X = \frac{1}{2}(A_\xi X + P\sigma(X, P\xi) + PD_X f\xi) \tag{8.8}$$

$$D'_X \xi = \frac{1}{2}(D_X\xi + JA_{f\xi}X - J\nabla_X P\xi - f\sigma(X, P\xi) - fD_X f\xi) \tag{8.9}$$

The proofs of Propositions 8.1. and 8.2. are immediate.

We call the formulas (8.2) and (8.6) the equations of Gauss and Weingarten for the Hermitian connection $\bar{\nabla}'$.

If $\bar{R}'(X, Y) = [\bar{\nabla}'_X, \bar{\nabla}'_Y] - \bar{\nabla}'_{[X,Y]}$ is the curvature operator determined by $\bar{\nabla}'$, (4), then we write $g(\bar{R}'(X, Y)Z, W) = \bar{R}'(X, Y, Z, W)$.

It is very easy to obtain the equation of Gauss for $\bar{\nabla}'$, that is,

$$\bar{R}'(X, Y, Z, W) = R'(X, Y, Z, W) + g(\sigma'(X, Z), \sigma'(Y, W)) - g(\sigma'(Y, Z), \sigma'(X, W)) \tag{8.10}$$

for all $X, Y, Z, W \in TM$, where $R'(X, Y, Z, W) = g(R'(X, Y)Z, W)$.

We define the covariant derivative of σ' with respect to the connection in $TM \oplus T^\perp M$ obtained by combining the connections ∇' in TM and D' in $T^\perp M$, that is,

$$(\bar{\nabla}'_X \sigma')(Y, Z) = D'_X \sigma'(Y, Z) - \sigma(\nabla'_X Y, Z) - \sigma'(Y, \nabla'_X Z) \tag{8.11}$$

for all $X, Y, Z \in TM$. Then it is very easy to prove the equation of Codazzi for $\bar{\nabla}'$.

PROPOSITION 8.3.—(*Equation of Codazzi*). *The normal component of* $\bar{R}'(X, Y)Z$ *is given by*

$$(\bar{R}'(X, Y)Z)^n = (\bar{\nabla}'_X \sigma')(Y, Z) - (\bar{\nabla}'_Y \sigma')(X, Z) + \sigma'(T_{\nabla'}(X, Y), Z) \quad (8.12)$$

for all $X, Y, Z \in TM$, *where* $T_{\nabla'}(X, Y) = \nabla'_X Y - \nabla'_Y X - [X, Y]$ *is the torsion of* ∇'.

Let $R^{D'}$ be the curvature tensor associated with D', i.e. $R^{D'}(X, Y) = [D'_X, D'_Y] - D'_{[X,Y]}$. Then we can obtain of a very easy manner the equation of Ricci for $\bar{\nabla}'$

$$\bar{R}(X, Y, \xi, \eta) = R^{D'}(X, Y, \xi, \eta) + g(A'_\eta Y, A'_\xi X) - g(A'_\eta X, A'_\xi Y) \quad (8.13)$$

for all $X, Y \in TM$ and $\xi, \eta \in T^\perp M$.

LEMMA 8.4.—*Let* M^n *be a totally real submanifold of an almost Hermitian manifold* $\bar{M}^{2m}$. *Then the following statements are equivalents*

a) σ' *is symmetric, i.e.* $\sigma'(X, Y) = \sigma'(Y, X)$

b) $A_{JX} Y = A_{JY} X$ *and* $f D_X JY = f D_Y JX$

c) $T_{\bar{\nabla}'}(X, Y) = -\frac{1}{2}(PD_X JY - PD_Y JX + [X, Y])$

for all $X, Y \in TM$. *Where* $T_{\bar{\nabla}'}$ *is the torsion of* $\bar{\nabla}'$.

Remark.—We observe that c) implies that $T_{\bar{\nabla}'}(X, Y) \in TM$ for all $X, Y \in TM$. We can say that if σ' is symmetric for a totally real submanifold M, then M is torsion-invariant with respect to $\bar{\nabla}'$, in the same sense that a submanifold is curvature-invariant.

If σ' is symmetric, we can define the mean curvature vector H' for $\bar{\nabla}'$

$$H' = \frac{1}{n} \text{trace}\, \sigma'$$

Then, it is easy to find the following relation

$$H' = \frac{1}{2}\{H + R_1 + R_2\}$$

where $H = \frac{1}{n} \text{trace}\, \sigma$ is the mean curvature vector and

$$R_1 = \frac{1}{n} \sum_{i=1}^n JA_{JX_i} X_i \in J(TM), \qquad R_2 = f\left(-\frac{1}{n} \sum_{i=1}^n D_{X_i} JX_i\right) \in (J(TM))^c$$

$\{X_1, \cdots, X_n\}$ is a local frame of vector fields in TM.

COROLLARY 8.5.— *Let M^n be a totally real submanifold of a Nearly Kaehler manifold $\bar{M}^{2m}$. Then σ' is symmetric if and only if $\sigma=\sigma'$.*

Proof.—If $\bar{M}$ is a Nearly-Kaehler manifold, we have

$$0=(\bar{\nabla}_X J)Y+(\bar{\nabla}_Y J)X$$

$$=-A_{JX}Y-A_{JY}X+D_X JY+D_Y JX-J\nabla_X Y-J\nabla_Y X-2J\sigma(X, Y)$$

then,

$$A_{JX}Y+A_{JY}X=-2P\sigma(X, Y) \tag{8.14}$$

$$D_X JY+D_Y JX=J\nabla_X Y+J\nabla_Y X+2f\sigma(X, Y) \tag{8.15}$$

From (8.15)

$$fD_X JY+fD_Y JX=2Jf\sigma(X, Y) \tag{8.16}$$

for all $X, Y\in TM$.

By Lemma 8.4, (8.14), (8.16) and (8.5), $\sigma=\sigma'$. The converse is obvious.

PROPOSITION 8.6.—*Let M^n be a totally real submanifold of a Nearly-Kaehler manifold $\bar{M}^{2m}$. Then $H=H'$.*

Proof.—If $\bar{M}$ is a Nearly-Kaehler manifold, then we have from (1.1) that $\bar{\nabla}'_X X=\bar{\nabla}_X X$. Thus $\sigma(X, X)=\sigma'(X, X)$ for all $X\in TM$. Hence $H=H'$.

COROLLARY 8.7.— *Let M^n be a totally real submanifold of a Nearly-Kaehler manifold $\bar{M}^{2m}$. Then M is minimal for the Hermitian connection $\bar{\nabla}'$ i.e. $H'=0$ if and only if is minimal for the Riemannian connection $\bar{\nabla}$ i.e. $H=0$.*

In the following, we study the relation between $\{f, D', J(TM)$ and $(J(TM))^c\}$

PROPOSITION 8.8.—*Let M^n be a totally real submanifold of a Nearly-Kaehler manifold $\bar{M}^{2m}$. We supose that $(J(TM))^c$ is parallel with respect to D. The $(J(TM))^c$ is parallel with respect to D'.*

Proof.—From (8.9), we have

$$2D'_X\xi=D_X\xi+JA_{J\xi}X-fD_X J\xi$$

for all $X\in TM$ and $\xi\in(J(TM))^c$. Since $(J(TM))^c$ is parallel with respect to D, $A_{J\xi}=0$ and $D_X\xi, fD_X J\xi\in(J(TM))^c$, thus $D'_X\xi\in(J(TM))^c$, then $(J(TM))^c$ is parallel with respect to D'.

PROPOSITION 8.9.—*Let M^n be a totally real submanifold of a Nearly Kaehler monifold $\bar{M}^{2m}$. If S is a subbundle of $T^{\perp}M$ such that S is parallel with respect to D', then*

1) *If S is holomorphic, then $\sigma'/_S=0$*

2) *If S is coholomorphic, then* $\operatorname{Im}\sigma'\subset S$.

Proof.—Since $(\bar{\nabla}'J)=0$, we have

$$g(\sigma'(X, Y), \xi)=g(\bar{\nabla}'_X JY, J\xi)=g(D'_X JY, J\xi)$$

If S is holomorphic and $\xi\in S$, we have $g(\sigma'(X, Y), \xi)=0$, thus $\sigma'/_S=0$.
In the proof of 2) we use a similar reasoning.

LEMMA 8.10.—*Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. Then,*

$$(\hat{\nabla}'_X P)\xi=A'_{f\xi}X \tag{8.17}$$

and

$$(D'_X f)\xi=-JA'_\xi X-\sigma'(X, P\xi) \tag{8.18}$$

for all $X\in TM$ and $\xi\in T^\perp M$, *where,*

$$(\hat{\nabla}'_X P)\xi=\nabla'_X P\xi-PD'_X\xi\,, \qquad (D'_X f)\xi=D'_X f\xi-fD'_X\xi\,.$$

Proof.—Since $(\nabla'J)=0$, (8.18) and (8.17) follow from (8.2) and (8.6).

THEOREM 8.11.—*Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. Then the following statements are equivalent*

1) *The f-structure in the normal bundle is parallel with respect to D'.*
2) $A'_\xi=0$ *for all* $\xi\in(J(TM))^c$.
3) $(J(TM))^c$ *is parallel with respect to D'.*
4) $J(TM)$ *is parallel with respect to D'.*
5) *The tangent bundle valued 1-form P is parallel with respect to D'.*

Proof.—
1)⇒2) is trivial.
2)⇒1) From (8.18), we have $(D'_X f)\xi=0$ for all $X\in TM$ and $\xi\in(J(TM))^c$.
Then, it is sufficient to prove that

$$g((D'_X f)\xi, \eta)=0 \tag{8.19}$$

for all $\xi\in J(TM)$ and $\eta\in T^\perp M$. We consider two cases:

i) If $\eta\in(J(TM))^c$. Using (8.18) and 1), it is very easy to prove (8.19).

ii) If $\eta\in J(TM)$, $\eta=JY$ and $\xi=JZ$ for $Y, Z\in TM$

$$g((D'_X f)JZ, JY)=-g(JA'_{JZ}X, JY)+g(\sigma'(X, Z), JY)$$
$$=-g(A'_{JZ}X+P\sigma'(X, Z), Y)$$

Since $(\bar{\nabla}'J)=0$, $\bar{\nabla}'_X JY=J\bar{\nabla}'_X Y$ and we obtain $g((D'_X f)JZ, JY)=0$.
2)⇒3)
Since $A'_\xi=0$ for all $\xi\in(J(TM))^c$ and $(J(TM))^c$ is a holomorphic subbundle of

$T^{\perp}M$, we have

$$0=g(\overline{\nabla}'_X J\xi, Z)=g(J\overline{\nabla}'_X\xi, Z)=-g(\overline{\nabla}'_X\xi, JZ)=-g(D'_X\xi, JZ)$$

for all $X, Z\in TM$ and $\xi\in(J(TM))^c$, thus we obtain 3).

3)⇒4) It is immediate.

4)⇒5)

If $\quad \xi\in J(TM), \quad f\xi=0 \quad \text{and} \quad (\hat{\nabla}'_X P)=0$

If $\quad \xi\in(J(TM))^c, \quad P\xi=0, \quad \text{and} \quad g((\hat{\nabla}'_X P)\xi, Y)=g(A'_{J\xi}X, Y)$

for all $X, Y\in TM$. Then

$$g((\hat{\nabla}'_X P)\xi, Y)=g(J\overline{\nabla}'_X\xi, Y)=-g(\overline{\nabla}'_X\xi, JY)=-g(D'_X\xi, JY)=0$$

5)⇒2) It follows from (8.18). (Q. E. D.)

Let M^n be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. We suppose that $\sigma'(X, Y)=\sigma'(Y, X)$ for all $X, Y\in TM$. Moreover, we can say that M is totally geodesic with respect to σ', if $\sigma'=0$. Then we have the following

THEOREM 8.12.—*Let M^n $(n>1)$ be a totally real submanifold of an almost Hermitian manifold $\bar{M}^{2m}$. Suppose that*

a) *M is totally umbilical* with *respect to σ'.*

b) *The f-structure is parallel respect to D'.*

Then M is totally geodesic respect to σ'.

Proof.—For all $X, Y\in TM$, $(\overline{\nabla}'_X J)Y=0$, then

$$J\sigma'(X, Y)=-A'_{JY}X+D'_X JY-J\nabla'_X Y \tag{8.20}$$

By (8.20), we have

$$g(\sigma'(X, Y), JZ)=g(\sigma'(X, Z), JY)$$

Using a), we have

$$g(X, Y)g(H', JZ)=g(X, Z)g(H', JY)$$

$$g(H', JZ)Y=g(H', JY)Z$$

If Y and Z are lineary independent, then $g(H', JZ)=0$ for all $Z\in TM$, thus $H'=0$. (Q. E. D.)

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