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UNIFORM ASYMPTOTIC PROPERTIES OF THE WKB METHOD

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§ 1. Introduction.

Suppose that the wave function $\psi(x, t)$ depends on time in oscillatory manner such that

$$\psi(x, t)=y(x)e^{-i\omega t},$$

and the function $y(x)$ satisfies the second order ordinary differential equation of the form

$$\frac{d^2y}{dx^2}+\lambda^2k^2(x, \omega)y=0, \tag{1.1}$$

where λ is a large parameter and $k^2(x, \omega)$ is the local wave number depending on a parameter ω. Usually the equation (1.1) is associated with certain boundary conditions at infinity such that, for example,

(i) $y(x)$ decays zero at $x=\pm\infty$
(ii) $y(x)$ represents an outgoing wave at infinity, or
(iii) $y(x)$ decays zero at $x=-\infty$ and represents an outgoing wave at $x=\infty$.

In general, these boundary conditions does not set up self adjoint boundary value problems, and accordingly we can not apply the Hilbert space technique. Alternatively to solve these problems, the so-called WKB approximation of solution is effectively used. The existence of turning points physically plays significant roles such as reflection, transmission or tunneling effects.

As we shall see from the examples given below, turning points may depend on the parameter or eigenvalues so that it is impossible to know a priori the location of these points. Since the boundary condition are complicated, so we can expect neither real eigenvalues nor that turning points are on the real axis even if $k^2(x, w)$ is real for x, w real.

Physicist may be possible to foresee the location of these points from the physical consideration, but how can we mathematically overcome this difficulty?. One of the answers to this question is to consider a dependence of the WKB approximations of solutions on a parameter ω, and to construct uniformly valid approximations with respect to ω. Not only in the physical applications, but also in the asymptotic theory of ordinary differential equations, as pointed out by

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Olver [6], these uniform asymptotic expansions of solutions of the equations having movable turning points have significant meanings and many challenging problems.

The purpose of this note is to give a few theorem about the uniformity of WKB type asymptotic expansions of solutions and their connection formulas.

There are many fields of physics in which the above sorts of mathematical problems are encountered. We give here three examples of such problems.

(1) It is familiar in the inelastic scattering theory that the one-dim. Schrödinger equation

$$\frac{d^2\psi}{dx^2}+\frac{2m}{h}(\omega-U(x))\psi=0\,,$$

and the outgoing wave condition at infinity may give us a set of complex eigenvalues ω, the so-called quasi-stationary state of the wave functions. From the physical reasonings, eigenvalues have to be of the form $E-i\Gamma$ where E and Γ are positive, and Γ small. The value Γ^{-1} represents the life time of the system, Landau and Lifshitz [3], P440 ff.

(2) The anharmonic oscillator in quantum mechanics defined by the differential equation

$$-\frac{d^2\phi}{dx^2}+\left(\frac{1}{4}x^2+\frac{1}{4}\beta x^4\right)\phi=E\phi\,,$$

and the boundary condition

$$\lim_{x\to\pm\infty}\phi(x)=0$$

or the generalized boundary condition for complex x

$$\lim_{|x|\to\infty}\phi(x)=0\,,\quad \text{when}\quad \left|\arg(\pm x)+\frac{1}{6}\arg\beta\right|<\frac{\pi}{6}$$

is of particular interest because it is a model of $\beta\phi^4$ field theory in one dim. space time. There are deep investigations about the analytic and asymptotic structure of the energy levels E as a function of complex β, Bender and Wu [1] and Simon [8].

(3) In the linear dynamical theory of the density waves of spiral galaxies, the problem is reduced in some circumstances to the investigation of the differential equation

$$\frac{d^2y}{dx^2}+\lambda^2x(x-\omega)^2y=0$$

with the boundary condition that $y(x)$ exponentially decays zero as x goes to $-\infty$ and represents an outgoing wave as x tends to $+\infty$. It is expected and proved that there exist complex eigenvalues ω with negative imaginary part of small absolute values and these correspond to unstable nomal modes of the density waves, Lin and Lau [4] and Nishimoto [5].

In section 2, we outline the so-called Fedoryuk theory of the WKB method, Evgrafov and Fedoryuk [2]. In section 3, the continuous dependence of asymptotic formulas on ω are considered, and a set of singular point and singular curves in the complex ω-plane are introduced. A few theorems on the uniform asymptotic expansions of solutions and uniform connection formulas with respect to ω are presented. It is proved that the region of ω where the uniform asymptotic formulas of WKB type are valid are open and their boundaries are singular curves. In the last section, we consider relations between the singular curves and asymptotic distributions of eigenvalues of certain boundary value problems.

§ 2. WKB method.

In this section, we give a summary of the Fedoryuk theory of the WKB method. Throught this note, we assume that the function $k^2(x, \omega)$ is a polynomial of x and ω. Let ω be fixed. In the complex x-plane we plot turning points $a_j (j=1, 2, \cdots, l)$ where $k^2(a_j, \omega)=0$. From each turning point a_j, we describe the Stokes curves:

$$\operatorname{Re} \xi(x, a_j, \omega)=0, \quad \xi(x, a_j, \omega)=\int_{a_j}^{x} ik(x, \omega)dx .$$

Here Ref means the real part of f and Imf the imaginary part of f. Then the complex x-plane is divided by these Stokes curves into a finite number of simply connected unbounded regions which we call Stokes regions. There are two types of Stokes regions, one is the half-plane type and the other is the strip type. The canonical domain introduced by Evegrafov and Fedoryuk [2] is a union of an appropriate number of adjacent Stokes regions, bounded by Stokes curves, containing no turning point in its interior, and is mapped by

$$\xi(x, x_0, \omega)=i\int_{x_0}^{x} k(x, \omega)dx$$

onto the whole ξ-plane cut by a finite number of unbounded verticals. For each canonical domain D, we can assign a turning point a which is on the boundary of D, and a Stokes curve S issuing from a and contained in the interior of D. Then there exists a fundamental system of solutions of (1.1) $Y\{a, S, D\}\equiv\{y_1, y_2\}$ such that for every compact subregion of D y_1 and y_2 have asymptotic expansions as $\lambda\to\infty$ of the form

$$\text{(2.1)}\qquad \begin{aligned} y_1(x, \lambda, w)&\cong ck(x, \omega)^{-(1/2)}\exp\{-\lambda\xi(x, a, \omega)\}, \\ y_2(x, \lambda, \omega)&\cong ck(x, \omega)^{-(1/2)}\exp\{\lambda\xi(x, a, \omega)\}, \end{aligned}$$

where

$$c=e^{-i\alpha}, \quad \alpha=\lim_{\substack{x\to a \\ x\in S}} \arg\{k(x, \omega)^{-(1/2)}\},$$

and the branch of $\xi(x, a, \omega)$ is determined by $\operatorname{Re}\xi(x, a, \omega)>0$ on S. The solutions $y_1(x, \lambda, \omega)$ and $y_2(x, \lambda, w)$ are entire functions of x, ω and λ. (Sibuya [7], P. 15, ff.). The elementary connection matrix is the connection matrix between two fundamental systems of solutions defined at neighboring canonical domains. Then the connection matrix between any two thus defined fundamental systems can be obtained by successive multiplications of the elementary connection matrices.

§ 3. Parameter dependence.

For each parameter ω fixed, there corresponds a Stokes curve configuration in the complex x-plane which we denote by $SC(\omega)$. The position of turning points and Stokes curve configuration continuously change as the parameter ω moves in the complex ω-plane.

DEFINITION 1. Two Stokes curve configurations $SC(\omega)$ and $SC(\omega')$ are topologically equivalent if there exists a topological mapping, which transforms turning points to turning points, between two Stokes curve configurations.

DEFINITION 2. A parameter ω_0 is regular if there exists a neighborhood $U(\omega_0)$ of ω_0 such that for all ω in $U(\omega_0)$, $SC(\omega_0)$ and $SC(\omega)$ are topologically equivalent. If ω_0 is not regular, we call that ω_0 is singular, and denote by I_ω the set of all singular points.

DEFINITION 3. A continuous rectifiable curve C in the ω-plane connecting ω_1 and ω_2 is a equivalent connection between ω_1 and ω_2 if all $\omega \in C$ are regular. Then the Stokes curve configuration $SC(\omega_1)$ and $SC(\omega_2)$ are topologically equivalent. We say that ω_1 and ω_2 are equivalent if there exists an equivalent connection between ω_1 and ω_2.

Let us assume that the function $k^2(x, \omega)$ has a form

$$k^2(x, \omega)=a_m(\omega)x^m+a_{m-1}(\omega)x^{m+1}+\cdots a_0(\omega) \tag{3.1}$$

with polynominal coefficients of ω and has a factorization

$$k^2(x, \omega)=a_m(\omega)(x-b_1(\omega))^{\gamma_1}\cdots(x-b_l(\omega))^{\gamma_l}, \quad \gamma_1+\gamma_2+\cdots+\gamma_l=m,$$

where $b_j(\omega)$ are algebraic functions of ω and then $b_j(\omega)$ are finitely many valued analytic functions. Then, the set of singular points I_ω consists of $I_\omega^{(i)}(i=1, 2, 3)$, where $I_\omega^{(1)}$ is the set of zero of $a_m(\omega)$, $I_\omega^{(2)}$ is the set of points ω for which at least two $b_k(\omega)$ and $b_j(\omega)(k\neq j)$ coincide and $I_\omega^{(3)}$ is the set of point ω such that

$$I_\omega^{(3)}=\bigcup_{\substack{i<j \\ i,j=1,2,\cdots l}}\left\{\omega:\ I_m\int_{b_j(\omega)}^{b_i(\omega)}k(x, \omega)dx=0\right\}.$$

If $\omega_0\in I_\omega^{(3)}$, this means that the Stokes curve configuration of $k^2(x, \omega_0)$ has at least one Stokes curve which connects two turning points, and the algebraic singularities of $b_i(\omega)$ are contained in $I_\omega^{(2)}$. By the difinition, I_ω is a closed set.

$I_\omega^{(1)}$ and $I_\omega^{(2)}$ consist of at most finite number of points and $I_\omega^{(3)}$ defines curves in the ω-plane, which we call singular curves.

PROPOSITION 3.1. *The set of singular points I_ω does not depend on the choice of branches of $b_j(\omega)(j=1, 2, \cdots, l)$ and $k(x, \omega)$.*

Proof. This comes from the facts that the change of the choice of branch of $k(x, \omega)$ induces a permutation of Stokes curves and does not change the Stokes curve configuration in the whole x-plane, and the change of branch of algebraic functions $b_j(\omega)$ give us a permutation between some of $b_k(\omega)$'s.

Here we give a few properties of the set of singular points. The number of points of $I_\omega^{(1)}$ and $I_\omega^{(2)}$ is finite. If ω_0 does not belong to $I_\omega^{(1)}$ or $I_\omega^{(2)}$, the function $F_{ij}(\omega)$ defined by

$$F_{ij}(\omega)=\int_{b_i(\omega)}^{b_j(\omega)} k(x, \omega)dx ,$$

where the integral path in the x-plane is to be taken such that it does not pass through any point $b_k(\omega)(k=1, 2, \cdots, l\ k\neq i, j)$ for all ω in a neighborhood of ω_0, is an analytic function of ω in a neighborhood of ω_0. Then if ω_0 is in $I_\omega^{(3)}$, the set $I_{\omega ij}^{(3)}\equiv\{\omega: \operatorname{Im} F_{ij}(\omega)=0\}$ define curves passing through ω_0. Next, let ω_0 be in $I_\omega^{(1)}$ or $I_\omega^{(2)}$. ω_0 may be infinity. When ω approaches to ω_0, each of the $b_k(\omega)$'s either goes to infinity or tends to some finite value.

Let i and j be fixed $(i<j)$. Suppose at first that when ω tends ω_0, there is no $b_k(\omega)$ which goes to infinity, and then we can assume that

$$b_k(\omega)-b_i(\omega)=b_{ki}(\omega-\omega_0)^{\alpha_{ki}}(1+0[(\omega-\omega_0)^{\delta_{ki}}]) ,\quad b_{ki}\neq 0 ,$$
$$(k=1, 2, \cdots, l,\ k\neq i)$$

Here b_{ki} are constant, and α_{ki}, δ_{ki} are nonnegative rational numbers. Then

$$F_{ij}(\omega)=\sqrt{a_m(\omega)}\int_{b_i(\omega)}^{b_j(\omega)}\sqrt{(x-b_1(\omega))^{\gamma_1}\cdots(x-b_l(\omega))^{\gamma_l}}\,dx$$
$$=\sqrt{a_m(\omega)}\int_0^{b_j(\omega)-b_i(\omega)}\sqrt{(x+b_i(\omega)-b_1(\omega))^{\gamma_1}\cdots(x+b_i(\omega)-b_l(\omega))^{\gamma_l}}\,dx .$$

Here we assume without loss of generality that the exponents $\alpha_{ki}(k\neq i)$ satisfy

$$0\leqq\alpha_{1i}\leqq\alpha_{2i}\leqq\cdots\leqq\alpha_{s-1i}<\alpha_{si}=\cdots=\alpha_{ji}=\cdots=\alpha_{ti}<\alpha_{t+1i}\leqq\cdots\leqq\alpha_{li} .$$

Then we have for $k\leqq s-1$

$$x+b_i(\omega)-b_k(\omega)=x-b_{ki}(\omega-\omega_0)^{\alpha_{ki}}\{1+0[(\omega-\omega_0)^{\delta_{ki}}]\}$$
$$=(\omega-\omega_0)^{\alpha_{ki}}\{(\omega-\omega_0)^{\alpha_{ji}-\alpha_{ki}}\tau-b_{ki}(1+0[(\omega-\omega_0)^{\delta_{ki}}])\},$$

and for $s\leqq k\leqq t$,

$$x+b_i(\omega)-b_k(\omega)=(\omega-\omega_0)^{\alpha_{ji}}\{\tau-b_{ki}(1+0[(\omega-\omega_0)^{\delta_{ki}}])\},$$

and for $t+1\leqq k\leqq l$

$$x+b_i(\omega)-b_k(\omega)=(\omega-\omega_0)^{\alpha_{ji}}\{\tau-b_{ki}(\omega-\omega_0)^{\alpha_{ki}-\alpha_{ji}}(1+0[(\omega-\omega_0)^{\delta_{ki}}])\}.$$

Here we used a substitution $x=(\omega-\omega_0)^{\alpha_{ji}}\tau$.

Then $F_{ij}(\omega)$ becomes

$$F_{ij}(\omega)=G(\omega)\int_0^{b_{ji}(1+0[(\omega-\omega_0)^{\delta_{ji}}])}H(\tau,\ \omega)d\tau\ ,$$

where

$$G(\omega)=a_m(\omega)^{1/2}(\omega-\omega_0)^{\rho},\ \rho=\{\alpha_{1i}\gamma_1+\cdots+\alpha_{s-1,i}\gamma_{s-1}+\alpha_{ji}(\gamma_s+\cdots+\gamma_l+\gamma_i+2)\}\times\frac{1}{2}$$

$$H(\tau,\ \omega)=\prod_{k=1}^{s-1}\{(\omega-\omega_0)^{\alpha_{ji}-\alpha_{ki}}\tau-b_{ki}(1+0[(\omega-\omega_0)^{\delta_{ki}}])\}^{\gamma_k/2}$$

$$\times\prod_{k=s}^{t}\{\tau-b_{ki}(1+0[(\omega-\omega_0)^{\delta_{ki}}])\}^{\gamma_k/2}$$

$$\times\prod_{t+1}^{l}\{\tau-b_{ki}(\omega-\omega_0)^{\alpha_{ki}-\alpha_{ji}}(1+0[(\omega-\omega_0)^{\delta_{ki}}])\}\times\tau^{\gamma_i/2}.$$

Then we have

$$\lim_{\omega\to\omega_0}G(\omega)^{-1}F_{ij}(\omega)=\prod_{k=1}^{s-1}(-b_{1i})^{\gamma_k/2}\int_0^{b_{ji}}\prod_{k=s}^{t}(\tau-b_{ki})^{\gamma_k/2}\prod_{k=t+1}^{l}\tau^{\gamma_k/2}\times\tau^{\gamma_i/2}d\tau$$

$$=c\ (\text{const.})$$

If $c\neq 0$, then this means that when ω tends to ω_0

$$F_{ij}(\omega)\sim C\ a_m(\omega)^{-(1/2)}(\omega-\omega_0)^{\rho},\ \rho\quad \text{rational}\,,$$

and thus only a finite number of singular curves can start from ω_0(or enter to ω_0). The cases that when $\omega\to\omega_0$ some of $b_k(\omega)$'s go to the infinity and that ω_0 is infinity can be treated analogously. After all, from points in $I_\omega^{(1)}$, or $I_\omega^{(2)}$ a number of singular curves start, and tend to the singular points or extend to the infinity. The whole complex ω-plane is divided by the singular curves into a finite or an infinite number of components of regular points.

Before going into the study of the roles of these singular points, we give a few examples

EXAMPLE 1. $k^2(x,\ \omega)=\omega-x^2$ (Harmonic oscillator. Fig. 1 (i))

$$I_\omega=I_\omega^{(2)}\cup I_\omega^{(3)},\ I_\omega^{(2)}=\{0\},\ I_\omega^{(3)}=\{\omega:\ \mathrm{Im}\,\omega=0\}.$$

EXAMPLE 2. $k^2(x,\ \omega)=x(x-\omega)^2$ (Density wave theory of spiral galaxy [5], Fig. I, (ii)).

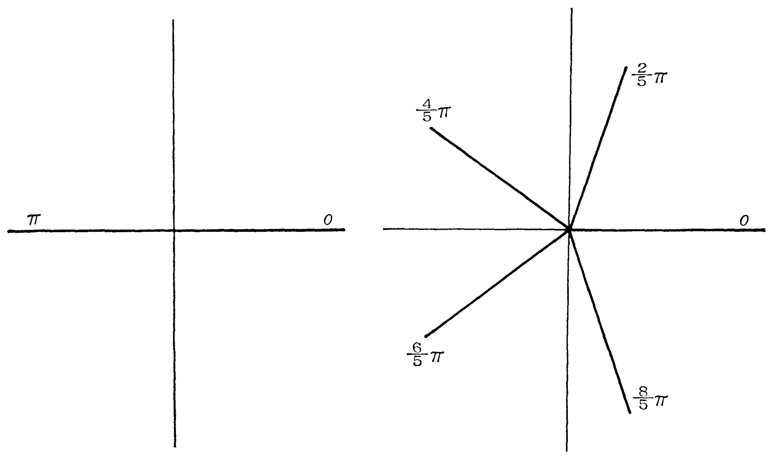
$$I_\omega=I_\omega^{(2)}\cup I_\omega^{(3)},\ I_\omega^{(2)}=\{0\},\ I_\omega^{(3)}=\{\omega:\ \mathrm{Im}\,\omega^{5/2}=0\}.$$

EXAMPLE 3. $k^2(x, \omega)=a(\omega)(x-b_1(\omega))^{\gamma_1}(x-b_2(\omega))^{\gamma_2}$ (Two turning point problem).

$$I_\omega=I_\omega^{(1)}\cup I_\omega^{(2)}\cup I_\omega^{(3)},$$

$$I_\omega^{(1)}=\{\omega:\ a(\omega)=0\},\ I_\omega^{(2)}=\{\omega:\ b_1(\omega)=b_2(\omega)\},$$

$$I_\omega^{(3)}=\left\{\omega:\ \mathrm{Im}\frac{\Gamma\left(\frac{\gamma_1}{2}+1\right)\Gamma\left(\frac{\gamma_2}{2}+1\right)}{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}+2\right)}e^{\pi i\gamma_2/2}a(\omega)^{1/2}[b_2(\omega)-b_1(\omega)]^{(\gamma_1+\gamma_2+2)/2}=0\right\}.$$



(i) $k^2(x, \omega)=\omega-x^2$ (ii) $k^2(x, \omega)=x(x-\omega)^2$

Fig. 1. Singular curves

Now let us consider the parameter dependence of the fundamental systems of solutions of the equation (1.1). As stated in the § 2, we can define several canonical domains for each parameter and the union of these cannonical domains cover the whole complex x-plane. For each canonical domain $D(\omega)$, associated with a turning point $a(\omega)$ and a Stokes curve $S(\omega)$, there exists a fundamental system of solutions of (1.1) that have asymptotic expansions of the form (2.1). Then we denote thus characterized fundamental system of solutions by $Y\{a, S, D: \omega\}$ and its asymptotic expansion (2.1) by $Y_{WKB}\{a, S, D: \omega\}$, that is

$$Y\{a, S, D: \omega\}\cong Y_{WKB}\{a, S, D: \omega\},\ (\lambda\to\infty). \tag{3.2}$$

THEOREM 3.1. *Let C be an equivalent connection between ω_0 and ω. If*

$$Y\{a, S, D: \omega_0\}\cong Y_{WKB}\{a, S, D: \omega_0\},$$

Then the analytic continuation of $Y\{a, S, D: \omega_0\}$ with respect to ω along C has

$$Y\{a, S, D: \omega\}\cong Y_{WKB}\{a, S, D: \omega\}.$$

Proof. It is sufficient for us to prove this theorem under the condition that ω is sufficiently close to ω_0. Then $D(\omega_0)$ and $D(\omega)$ have a sufficiently large common part. From the existence theorem, there exists a fundamental system of solutions $\bar{Y}\{a, S, D: \omega\}$ whose asymptotic expansion in $D(\omega)$ has a form $Y_{WKB}\{a, S, D: \omega\}$. Let B be any compact set contained in every $D(\omega')$ for $\omega' \in C$. Now we put $\bar{Y}\{a, S, D: \omega\} = \{\bar{y}_1(x, \omega), \bar{y}_2(x, \omega)\}$, and $Y\{a, S, D: \omega\} = \{y_1(x, \omega), y_2(x, \omega)\}$ that is obtained from $Y\{a, S, D; \omega_0\}$ by the analytic continuation with respect to ω. The asymptotic expression (2.1) for $\omega=\omega_0$ means that for $x \in B$,

$$|k(x, \omega_0)\exp\{\lambda\xi(x, a, \omega_0)\}[y_1(x, \omega_0)-ck(x, \omega_0)^{-(1/2)}\exp\{-\lambda\xi(x, a, \omega_0)\}]| < K\lambda^{-1},$$

$$|k(x, \omega_0)\exp\{-\lambda\xi(x, a, \omega_0)\}[y_2(x, \omega_0)-ck(x, \omega_0)^{-(1/2)}\exp\{\lambda\xi(x, a, \omega_0)\}]| < K\lambda^{-1}$$

for some positive constant K. But from the continuity of all the functions appearing in the above inequalities, the above inequalities are also true if ω_0 is replaced by ω in B. Since both of $y_i(x, \omega)$ and $\bar{y}_i(x, \omega)$ $(i=1, 2)$ are solutions of the same equation (1.1) and $D(\omega_0)$ is sufficiently close to $D(\omega)$, then $y_i(x, \omega)$ must have the same asymptotic expansion as $\bar{y}_i(x, \omega)$ in $D(\omega)$. Therefore we have proved

$$Y\{a, S, D; \omega\} \cong Y_{WKB}\{a, S, D; \omega\}.$$

THEOREM 3.2. *Suppose that ω_0 and ω are equivalent, and there exist two fundamental systems of solutions such that*

$$Y\{a_1, S_1, D_1; \omega_0\} \cong Y_{WKB}\{a_1, S_1, D_1, \omega_0\},$$

$$Y\{a_2, S_2, D_2; \omega_0\} \cong Y_{WKB}\{a_2, S_2, D_2, \omega_0\}.$$

Let $R(\omega_0)$ be the asymptotic expression of connection matrix between $Y\{a_1, S_1, D_1; \omega_0\}$ and $Y\{a_2, S_2, D_2; \omega_0\}$:

$$Y_{WKB}\{a_2, S_2, D_2; \omega_0\} \cong R(\omega_0) Y_{WKB}\{a_1, S_1, D_1; \omega_0\}.$$

Then we have

$$Y_{WKB}\{a_2, S_2, D_2; \omega\} \cong R(\omega) Y_{WKB}\{a_1, S_1, D_1; \omega\}. \tag{3.3}$$

Here we consider the fundamental system Y as a column vector.

Proof. This is obvious from the theorem 3.1.

From the above two theorem, we get the followings.

COROLLARY 3.1. *Let Ω be a component of regular points in the ω-plane. Then the asymptotic expansion* (3.2) *and the connection formula* (3.3) *uniformly hold with respect to ω in every compact subregion of Ω.*

We can extend the uniform validity of the above corollary across the boundary of Ω in some cases. To do so, we introduce the term "preservation of the canonical domain". In general, a canonical domain $D(\omega)$ for a regular point ω can not always become a canonical domain when ω tends to a singular point ω_0. This can be seen from a simple example in Fig. 2, a canonical domain $D(\omega)$ containing $S_1 \cup S_5 \cup S_3$ is splitted when ω tends to a positive real ω_0. But we have the followings. Suppose that the regular point ω tends to a singular point ω_0 in such a way that a canonical domain $D(\omega)$, associated turning point $a(\omega)$ and Stokes curve $S(\omega)$, become a canonical domain $D(\omega_0)$ associated turning point $a(\omega_0)$ and Stokes curve $S(\omega_0)$, then we say that the canonical domain $D(\omega)$ preserves as ω tends to ω_0. In this case, it is easy to see that the asymptotic expansion (3.2) is valid for $\omega = \omega_0$.

Let l be a singular curve both sides of which are components of regular points Ω and Ω'. Let ω_0 be on l, but not a point of $I_\omega^{(1)}$ or $I_\omega^{(2)}$. If $D(\omega_0)$, $a(\omega_0)$, $S(\omega_0)$ are a canonical domain, associated turning point and Stokes curve respectively, then there exists a triplet $D(\omega)$, $a(\omega)$ and $S(\omega)$ that becomes $D(\omega_0)$, $a(\omega_0)$ and $S(\omega_0)$ as ω tends to ω_0 from both sides of l. Thus we have a following theorem.

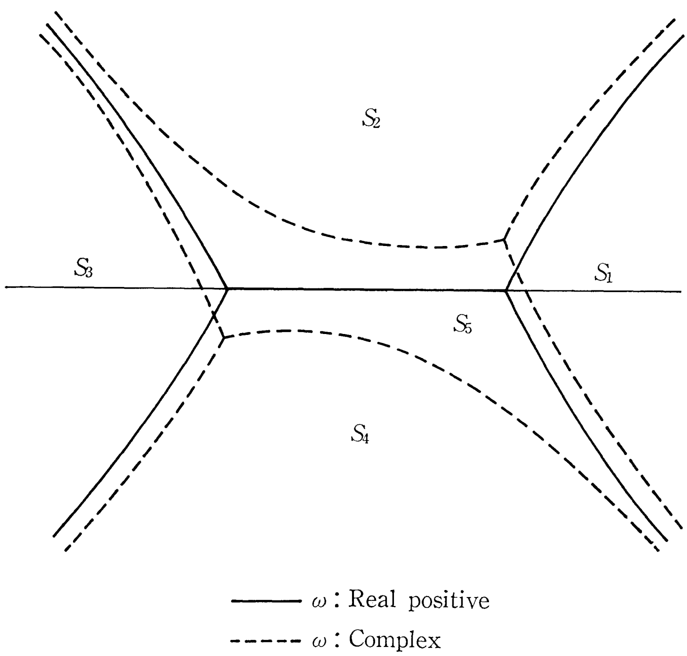


Fig. 2. Canonical domain preservation

THEOREM 3.3. *Let l be a singular curve which does not contain any points of $I_\omega^{(1)}$ or $I_\omega^{(2)}$ in its interior. If we have for $\omega_0 \in l$*

$$Y\{a, S, D ; \omega_0\} \cong Y_{WKB}\{a, S, D ; \omega_0\},$$

then this asymptotic expansion is uniformly valid in every compact svbset of $\Omega \cup \Omega' \cup l$. *Similarily if* $R(\omega_0)$ *is the asymptotic expansion of the connection matrix between two fundamental systems of solutions, that is*

$$Y\{a_2, S_2, D_2 ; \omega_0\} \cong R(\omega_0) Y\{a_1, S_1, D_1 ; \omega_0\},$$

then this connection formula is uniformly valid in every compact subregion of $\Omega \cup \Omega' \cup l$.

Now it remains problems of constructing uniform asymptotic expansions with respect to ω when ω moves in a neighborhood of ω_0 which is in $I_\omega^{(1)}$ or $I_\omega^{(2)}$. But this is in general very difficult and have to find completely different forms of asymptotic expansions from the WKB type expansions, [6].

§4. Distribution of eigenvalues.

In the previous section, we saw that the singular curves can be boundaries of the domain of ω where the asymptotic expansions (2.1) are uniformly valid with respect to ω. But these singular curves seem to play another special role in certain boundary value problems, that is, the example of the eigenvalue problems of harmonic oscillator and of the density wave theory [5] strongly suggest that the distributions of infinite sets of eigenvalues are asymptotic to some of the singular curves. More precisely, we can conjecture that if there exists an infinite set of discrete eigenvalues for a certain boundary value problem of the equation (1.1), then these eigenvalues are on one of the singular curves or asymptotic to it, and conversely, for each singular curve there corresponds a boundary vaule problem whose eigenvalues are on or asymptotic to the singular curve. This conjecture is partially true from a result of Evgrafov and Fedoryuk [2], page 44.

THEOREM 4.1 (Evgrofov and Fedoryuk). *Suppose that* ω *is not in the set of singular points and that* $k^2(x, \omega)$ *has all its zeros simple. Then if the solution* $y_1(x, \lambda, \omega)$ *has the asymptotic form* (2.1) *in some Stokes region* S_0 *of the half plane type where* $y_1(x, \lambda, \omega)$ *exponentially decreases as* $x \to \infty$ *in* S_0, *the asymptotic formula* (2.1) *of* $y_1(x, \lambda, w)$ *is valid in the whole complex x-plane except for arbitrarily small neighborhoods of certain Stokes curves that are uniquely defined by* S_0. *Further this solution exponentially increases in all canonical domains other than* S_0.

Therefore this is rewritten by using the term of singular points as follows:

COROLLARY 4.1 (Evgrafov and Fedoryuk). *Let the boundary value problem be to find the solution* $y(x, \lambda, \omega)$ *of* (1.1) *with the eigenvalue parameter* ω *such that* $\lim_{x \to \infty} y(x, \lambda, \omega) = 0$ *along two rays that extend to infinity in different Stokes*

regions of half-plane type.
Then if ω dose not belong to the set of singular points, it can not be an eigenvalue of the problem for sufficiently large λ. And if there exists an infinite number of discrete eigenvalues for this boundary value problem, they are on the singular curves.

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