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UNICITY THEOREMS FOR MEROMORPHIC OR ENTIRE FUNCTIONS

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1. Let f and g be meromorphic functions. We denote the order of f by ρ_f. In what follows we use the notation $f=a \to g=a$ in the following sense: Z_n is a zero of $g-a$ of order at least $\nu(n)$ whenever Z_n is a zero of $f-a$ of order $\nu(n)$. If k is a positive integer or ∞, let $E(a, k, f)=\{z \in \boldsymbol{C}: z$ is a zero of $f-a$ of order $\leqq k.\}$, where $\boldsymbol{C}$ is the complex plane. If a belongs to $\bar{\boldsymbol{C}}=\boldsymbol{C} \cup \{\infty\}$, we denote by $\bar{n}_k(r, a, f)$ the number of distinct zeros of order $\leqq k$ of $f-a$ in $|z| \leqq r$ (each zero of order $\leqq k$ is counted only once irrespective of its multiplicity). And we set

$$\bar{N}_k(r, a, f)=\int_0^r \frac{\bar{n}_k(t, a, f)-\bar{n}_k(0, a, f)}{t} dt+\bar{n}_k(0, a, f) \log r .$$

Further we denote by $n_0^{(k)}(r, a; f, g)$ the number of common zeros of order $\leqq k$ of $f-a$ and $g-a$ in $|z| \leqq r$, and we set

$$N_0^{(k)}(r, a; f, g)=\int_0^r \frac{n_0^{(k)}(t, a; f, g)-n_0^{(k)}(0, a; f, g)}{t} dt+n_0^{(k)}(0, a; f, g) \log r .$$

In this paper we shall prove some unicity theorems for meromorphic or entire functions.

2. Gopalakrishna and Bhoosnurmath have proved the following theorem in [1].

THEOREM A. *Let f and g be transcendental meromorphic functions. Assume that there exist distinct elements $a_1, \cdots, a_m$ in $\bar{\boldsymbol{C}}$ such that $E(a_i, k_i, f)=E(a_i, k_i, g)$ for $i=1, \cdots, m$; where each k_i is a positive integer or ∞ with $k_1 \geqq \cdots \geqq k_m$, and $\{k_i\}_1^m$ satisfies*

$$\sum_{i=2}^{m} \frac{k_i}{k_i+1}-\frac{k_1}{k_1+1}>2 .$$

Then $f \equiv g$.

From Theorem A several consequences including a theorem of Nevanlinna [4] are deduced.

THEOREM A$_1$. *Let f and g be transcendental meromorphic functions. If there*

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exist distinct seven elements $a_1, \cdots, a_7$ *in* $\bar{C}$ *such that* $E(a_i, 1, f)=E(a_i, 1, g)$, *then* $f\equiv g$.

THEOREM A_2. *Let* f *and* g *be transcendental meromorphic functions. If there exist distinct six elements* $a_1, \cdots, a_6$ *in* $\bar{C}$ *such that* $E(a_i, 2, f)=E(a_i, 2, g)$, *then* $f\equiv g$.

THEOREM A_3. *Let* f *and* g *be transcendental meromorphic functions. If there exist distinct five elements* $a_1, \cdots, a_5$ *in* $\bar{C}$ *such that* $E(a_i, p, f)=E(a_i, p, g)$, *where* p *is a positive integer* $\geqq 3$ *or* ∞, *then* $f\equiv g$.

We first remark that the assumption on the number of distinct elements $a_1, \cdots, a_m$ in $\bar{C}$ satisfying $E(a_i, k_i, f)=E(a_i, k_i, g)$ cannot be improved in the above each theorem A_i ($i=1, 2, 3$). This fact is clear in the case of Theorem A_3. In order to see this in the cases of Theorems A_1 and A_2 we may consider the following examples. Set

$$z=\phi_n(w)=\int_0^w (1-t^n)^{-2/n}dt \qquad (n=3, 4).$$

Then $\phi_3(w)$ maps the unit disc onto an equilateral triangle $z_1z_2z_3$, where $z_1=\phi_3(1)$, $z_2=\phi_3(\omega)$, and $z_3=\phi_3(\omega^2)$, where ω is a cubic root of 1. And $\phi_4(w)$ maps the unit disc onto a square $z_1z_2z_3z_4$, where $z_1=\phi_4(1)$, $z_2=\phi_4(i)$, $z_3=\phi_4(-1)$, and $z_4=\phi_4(-i)$. The inverse function of $z=\phi_n(w)$ ($n=3, 4$) can be analytically continued over the whole plane as a one-valued meromorphic function by Schwarz's reflection principle and the resulting function $w=f_n(z)$ is doubly periodic. In the case of $n=3$, we put $a_1=1$, $a_2=\omega$, $a_3=\omega^2$, $a_4=0$ and $a_5=\infty$. Then all the zeros of $f-a_i$ ($i=1, 2, 3$) are taken with multiplicity 3. In the case of $n=4$, we put $a_1=1$, $a_2=i$, $a_3=-1$, $a_4=-i$, $a_5=0$ and $a_6=\infty$. Then all the zeros of $f-a_i$ ($i=1, 2, 3, 4$) are taken with multiplicity 2. Therefore if we set $g_3=\omega f_3$ and $g_4=i\cdot f_4$, we have $E(a_i, 2, f_3)=E(a_i, 2, g_3)$ ($i=1, \cdots, 5$) and $E(a_i, 1, f_4)=E(a_i, 1, g_4)$ ($i=1, \cdots 6$). And it is clear that $\sum_{i=1}^{5}\delta(a_i, f_3)=\sum_{i=1}^{5}\delta(a_i, g_3)=0$ and $\sum_{i=1}^{6}\delta(a_i, f_4)=\sum_{i=1}^{6}\delta(a_i, g_4)=0$.

Secondly we note that in Theorem A, the Nevanlinna deficiencies of f and g are not taken into consideration. With respect to this point we shall prove

THEOREM 1. *Let* f *and* g *be transcendental meromorphic functions. Assume that there exist distinct elements* $a_1, \cdots, a_m$ *in* $\bar{C}$ *such that* $E(a_i, k_i, f)=E(a_i, k_i, g)$ *for* $i=1, \cdots m$; *where each* k_i *is a positive integer or* ∞ *with* $k_1\geqq \cdots \geqq k_m$, *and the sequence* $\{k_i\}_1^m$ *satisfies*

$$\sum_{i=2}^{m}\frac{k_i}{k_i+1}-\frac{k_1}{k_1+1}-2\leqq 0.$$

Then the inequality:

$$(*) \qquad \sum_{i=1}^{m}\min(\delta(a_i, f), \delta(a_i, g))>\left\{2+\frac{k_1}{k_1+1}-\sum_{i=2}^{m}\frac{k_i}{k_i+1}\right\}(k_1+1)$$

implies $f \equiv g$. *Especially, if the right hand side of* (*) *is equal to zero,* "min" *in the condition* (*) *can be replaced by* "max". *Further if both* ρ_f *and* ρ_g *are finite,* "δ" *in the condition* (*) *can be replaced by* "Δ" (*Valiron deficiency*).

From this, we deduce several consequences which include the following three results.

1° Let f and g be transcendental meromorphic functions. Assume that there exist distinct six elements $a_1, \cdots, a_6$ in $\bar{C}$ such that $E(a_i, 1, f)=E(a_i, 1, g)$ and

$$\sum_{i=1}^{6} \max(\delta(a_i, f), \delta(a_i, g))>0 .$$

Then $f \equiv g$.

2° Let f and g be transcendental meromorphic functions. Assume that there exist distinct five elements $a_1, \cdots, a_5$ in $\bar{C}$ such that $E(a_i, 2, f)=E(a_i, 2, g)$ and

$$\sum_{i=1}^{5} \max(\delta(a_i, f), \delta(a_i, g))>0 .$$

Then $f \equiv g$.

3° Let f and g be transcendental meromorphic functions. Assume that there exist distinct five elements $a_1, \cdots, a_5$ in $\bar{C}$ such that $E(a_i, 1, f)=E(a_i, 1, g)$ and

$$\sum_{i=1}^{5} \min(\delta(a_i, f), \delta(a_i, g))>1 .$$

Then $f \equiv g$.

COROLLARY 1 (*of* 3°) *Let f and g be transcendental entire functions. Assume that there exist distinct four complex numbers* $a_1, \cdots, a_4$ *such that* $E(a_i, 1, f)= E(a_i, 1, g)$ *and*

$$\sum_{i=1}^{4} \max(\delta(a_i, f), \delta(a_i, g))>0 .$$

Then $f \equiv g$. *In the case of* $\rho_f, \rho_g<\infty$, "δ" *in the assumption can be replaced by* "Δ".

COROLLARY 2 (*of* 3°) *Let f and g be transcendental entire functions of order*$>1/2$. *Assume that there exist distinct four complex numbers* $a_1, \cdots a_4$ *such that* $E(a_i, 1, f)=E(a_i, 1, g)$. *Further assume that all the zeros of* $f-a_1$ *lie on the negative real axis and that they have a finite exponent of convergence. Then* $f \equiv g$.

Proof of Theorem 1. First suppose that $a_1, \cdots, a_m$ are all finite. By the second fundamental theorem, we have

$$(2.1) \quad (m-2)\{T(r, f)+T(r, g)\} \leqq \sum_{i=1}^{m} \{\bar{N}(r, a_i, f)+\bar{N}(r, a_i, g)\}+S(r, f)+S(r, g),$$

where $S(r, f)=o(T(r, f))$, $S(r, g)=o(T(r, g))$ as $r \rightarrow \infty$ outside a set E of finite linear measure. For brevity, we put $S(r)=S(r, f)+S(r, g)$. Here we use the following obvious inequalities:

$$\bar{N}(r, a_i, f) \leqq \frac{k_i \bar{N}_{k_i}(r, a_i, f) + N(r, a_i, f)}{k_i+1} \qquad (i=1, \cdots, m).$$

These inequalities hold also when we replace f by g. Substituting these into (2.1), we have

$$(2.2) \quad (m-2)\{T(r, f)+T(r, g)\} \leqq \sum_{i=1}^{m}\left[\frac{k_i}{k_i+1}\{\bar{N}_{k_i}(r, a_i, f)+\bar{N}_{k_i}(r, a_i, g)\} + \frac{1}{k_i+1}\{N(r, a_i, f)+N(r, a_i, g)\}\right]+S(r)$$

$$\leqq \frac{2k_1}{k_1+1}\sum_{i=1}^{m} N_0^{(k_i)}(r, a_i ; f, g) + \sum_{i=1}^{m}\frac{1}{k_i+1}\{N(r, a_i, f)+N(r, a_i, g)\}+S(r),$$

where we used the fact that $k_1/(k_1+1) \geqq \cdots \geqq k_m/(k_m+1)$ and $N_{k_i}(r, a_i, f)= N_{k_i}(r, a_i, g)=N_0^{(k_i)}(r, a ; f, g)$.

Now suppose that $f \not\equiv g$. For $a \in \boldsymbol{C}$, each common zero of $f-a$ and $g-a$ is a zero of $f-g$. Since $a_1, \cdots, a_m$ are all distinct, we have

$$(2.3) \quad \sum_{i=1}^{m} N_0^{(k_i)}(r, a_i ; f, g) \leqq N(r, 0, f-g) \leqq T(r, f-g)+0(1) \leqq T(r, f)+T(r, g)+0(1).$$

From (2.2) and (2.3), we obtain

$$(2.4) \quad \left(m-2-\frac{2k_1}{k_1+1}\right)\{T(r, f)+T(r, g)\} \leqq \sum_{i=1}^{m}\frac{1}{k_i+1}\{N(r, a_i, f)+N(r, a_i, g)\}+S(r).$$

By the definition of the Nevanlinna deficiency, for any $\varepsilon>0$, there exists $r_0\ (>0)$ such that $r \geqq r_0$ implies $N(r, a_i, f)+N(r, a_i, g)<(1-\delta(a_i, f)+\varepsilon)T(r, f)+(1-\delta(a_i, g)+\varepsilon)T(r, g)$ $(i=1, \cdots, m)$. Hence we have from (2.4)

$$(2.5) \quad \left(m-2-\frac{2k_1}{k_1+1}-\sum_{i=1}\frac{1}{k_i+1}\right)\{T(r, f)+T(r, g)\} + \sum_{i=1}^{m}\left\{\frac{(\delta(a_i, f)-\varepsilon)T(r, f)+(\delta(a_i, g)-\varepsilon)T(r, g)}{k_i+1}\right\} \leqq S(r).$$

Here we note that

$$m-2-\frac{2k_1}{k_1+1}-\sum_{i=2}^{m}\frac{1}{k_i+1}=\left(\sum_{i=2}^{m}\frac{k_i}{k_i+1}\right)-\frac{k_1}{k_1+1}-2 \leqq 0,$$

and the second term of the left hand side of (2.5)$\geqq$

$$\left\{\frac{1}{k_1+1}\sum_{i=1}^{m}\min(\delta(a_i, f), \delta(a_i, g))-m\varepsilon\right\}(T(r, f)+T(r, g)).$$

Thus as $r(\notin E)\to\infty$ we have

$$\left(\sum_{i=2}^{m}\frac{k_i}{k_i+1}-\frac{k_1}{k_1+1}-2\right)+\frac{1}{k_1+1}\sum_{i=1}^{m}\min(\delta(a_i, f), \delta(a_i, g))-m\varepsilon\leqq 0.$$

This inequality contradicts the condition (*). Hence we have $f\equiv g$.

Suppose now that some a_i is ∞. Then let a be a complex number different from $a_1, \cdots, a_m$. Then $(a_1-a)^{-1}, \cdots, (a_m-a)^{-1}$ are all distinct and finite. If we put $F=1/(f-a)$ and $G=1/(g-a)$, we have $E(a_i, k_i, f)=E(a_i, k_i, g)\Leftrightarrow E((a_i-a)^{-1}, k_i, F)=E((a_i-a)^{-1}, k_i, G)$, and $\delta(a_i, f)=\delta((a_i-a)^{-1}, F)$, $\delta(a_i, g)=\delta((a_i-a)^{-1}, G)$ $(i=1, \cdots, m)$. Hence f and g satisfy the assumptions in Theorem 1. $\Leftrightarrow$ F and G satisfy the assumptions in Theorem 1 when we replace a by $(a_i-a)^{-1}$. Thus by what we have proved above, $F\equiv G$, that is, $f\equiv g$.

Next, we consider the case of

$$\sum_{i=2}^{m}\frac{k_i}{k_i+1}-\frac{k_1}{k_1+1}-2=0.$$

In this case we have

$$\frac{1}{K}\leqq\frac{T(r, f)}{T(r, g)}\leqq K \qquad (r\notin E,\ r: \text{large enough}),$$

where K is a positive constant (>1) depending only on m. This is clear if $f\equiv g$. If $f\not\equiv g$, we note that the following inequality holds: For any positive number $\tau<1$, there exists r_0 (>0) such that $r\geqq r_0$ implies

$$\sum_{i=1}^{m}N_0^{(k_i)}(r, a_i; f, g)>\tau\{T(r, f)+T(r, g)\} \qquad (r\notin E,\ r: \text{large enough}). \tag{2.6}$$

If this were not the case, we would have a positive constant $\tau_0<1$ and a monotone increasing sequence $\{r_n\}$ tending to ∞ as $n\to\infty$ such that $\{r_n\}\cap E=\phi$ and

$$\sum_{i=1}^{m}N_0^{(k_i)}(r_n, a_i; f, g)\leqq\tau_0\{T(r_n, f)+T(r_n, g)\}.$$

Substituting this into (2.2) we would have instead of (2.5),

$$\left(m-2-\frac{2\tau_0 k_1}{k_1+1}-\sum_{i=1}^{m}\frac{1}{k_i+1}\right)\{T(r_n, f)+T(r_n, g)\}$$
$$+\sum_{i=1}^{m}\frac{(\delta(a_i, f)-\varepsilon)T(r_n, f)+(\delta(a_i, g)-\varepsilon)T(r_n, g)}{k_i+1}\leqq S(r_n).$$

However, since $m-2-2\tau_0 k_1/(k_1+1)-\sum_{i=1}^{m}(k_i+1)^{-1}>0$ in this case, the above inequality is absurd. Hence (2.6) holds. From this we obtain

$$mT(r, f),\ mT(r, g) \geqq \sum_{i=1}^{m} N_0^{(k_i)}(r, a_i; f, g) > \tau \cdot \{T(r, f)+T(r, g)\} \qquad (r \notin E).$$

This implies

$$\frac{\tau}{m-\tau} < \frac{T(r, f)}{T(r, g)} < \frac{m-\tau}{\tau} \qquad (r \notin E,\ r: \text{large enough}).$$

Now, in order to see that "min" in (*) can be replaced by "max" in this case, we may prove that

$$\sum_{i=1}^{m} \min(\delta(a_i, f), \delta(a_i, g))=0 \Rightarrow \sum_{i=1}^{m} \max(\delta(a_i, f), \delta(a_i, g))=0.$$

Assume that there occur both

$$\sum_{i=1}^{m} \min(\delta(a_i, f), \delta(a_i, g))=0$$

and

$$\sum_{i=1}^{m} \max(\delta(a_i, f), \delta(a_i, g))>0.$$

For example, we may assume that $\delta(a_1, f)>0=\delta(a_1, g)$. In this case, we have by (2.5),

$$\left\{\frac{\delta(a_1, f)}{k_1+1} - \varepsilon \sum_{i=1}^{m} \frac{1}{k_i+1}\right\} T(r, f) \leqq S(r) + \varepsilon\left(\sum_{i=1}^{m} \frac{1}{k_i+1}\right) T(r, g)$$

$$(r \notin E,\ r: \text{large enough}).$$

Taking r $(\notin E)$ large enough, $\{\ \}>0$ holds. Dividing the both hand sides by $T(r, g)$, we have

$$\varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{T(r, f)}{T(r, g)} \leqq \frac{\varepsilon \sum_{i=1}^{m} \frac{1}{k_i+1}}{\frac{\delta(a_1, f)}{k_1+1} - \varepsilon \sum_{i=1}^{m} \frac{1}{k_i+1}}.$$

Since ε is arbitrary, this implies

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{T(r, f)}{T(r, g)} = 0,$$

which is a contradiction.

Finally we consider the case of

$$\sum_{i=2}^{m} \frac{k_i}{k_i+1} - \frac{k_1}{k_1+1} - 2 = 0$$

and $\rho_f,\ \rho_g<\infty$. In this case, we have $E=\phi$, and

$$m-2-\frac{2k_1}{k_1+1} = \sum_{i=1}^{m} \frac{k_i}{k_i+1}.$$

Hence, if $f\not\equiv g$, we have by (2.4),

$$\left(m-2-\frac{2k_1}{k_1+1}\right)\{T(r, f)+T(r, g)\}\leqq\sum_{i=1}^{m}\frac{1}{k_i+1}\{N(r, a_i, f)+N(r, a_i, g)\}+S(r)$$

$$\leqq\sum_{i=1}^{m}\frac{1}{k_i+1}\{T(r, f)+T(r, g)\}+S(r)=\left(m-2-\frac{2k_1}{k_1+1}\right)\{T(r, f)+T(r, g)\}+S(r).$$

Thus $N(r, a_i, f)\sim T(r, f)$ and $N(r, a_i, g)\sim T(r, g)$ $(r\to\infty)$ $(i=1, \cdots, m)$. These imply $\Delta(a_i, f)=\Delta(a_i, g)=0$ $(i=1, \cdots, m)$. This completes the proof.

Proof of Corollaries 1 *and* 2. From 3°, in this case,

$$\sum_{i=1}^{4}\min(\delta(a_i, f), \delta(a_i, g))>0$$

implies $f\equiv g$. However, as in the proof of Theorem 1, we can replace "min" by "max". And further if $\rho_f, \rho_g<\infty$, we can replace "δ" by "Δ" as in the proof of Theorem 1. The details will be ommitted.

Now, we shall prove Corollary 2. If $\rho_f=\infty$, we have by the assumption $\delta(a_1, f)=1$. Hence from Corollary 1 we have $f\equiv g$. Let p be the genus of the canonical product formed with the sequence of zeros of $f-a_1$. If $1<\rho_f<\infty$ and $p\geqq 1$, a well known theorem due to Edrei and Fuchs [2] implies $\delta(a_1, f)>0$. Hence from Corollary 1 we have $f\equiv g$. If $1<\rho_f<\infty$ and $p=0$, we have $\delta(a_1, f)=1$. Hence we again have $f\equiv g$. If $1/2<\rho_f\leqq 1$, a theorem of Shea [8] yields $\Delta(a_1, f)=1-\sin\pi\rho_f>0$. So if $\rho_g<\infty$, Corollary 1 gives this proof. However if $\rho_g=\infty$, then $\delta(a_1, g)=1$. This also yields $f\equiv g$ by Corollary 1. This completes the proof.

3. Nevanlinna [4] proved the following result.

THEOREM B. *Suppose that f and g are transcendental meromorphic in the plane and let $\{a_i\}_1^4$ be four distinct elements in $\bar{C}$. Then if $f=a_i\rightleftarrows g=a_i$ $(i=1, 2, 3, 4)$, $f\equiv g$, or $g=S(f)$, where $S(z)$ is one of the linear transformations which fix two elements in $\{a_i\}_1^4$ and permute the other two elements in $\{a_i\}_1^4$.*

In this section, we shall improve the above theorem in the case that f and g are entire functions. Our results may be stated as follows.

THEOREM 2. *Let f and g be non-constant entire functions such that $f=0\rightleftarrows g=0$ and $f=1\rightleftarrows g=1$. Further assume that there exists a complex number a $(\neq 0, 1)$ satisfying $E(a, k, f)=E(a, k, g)$, where k is a positive integer $(\geqq 2)$ or ∞. Then f and g must satisfy one of the following four relations.*

(i) $f\equiv g$ (ii) $\left(f-\frac{1}{2}\right)\left(g-\frac{1}{2}\right)\equiv\frac{1}{4}$ (*This occurs only for* $a=1/2$.)

(iii) $fg\equiv 1$ (*This occurs only for* $a=-1$.)

(iv) $(f-1)(g-1)\equiv 1$ (*This occurs only for* $a=2$.)

THEOREM 3. *Let f and g be non-constant entire functions such that $f=0\rightleftarrows g=0$ and $f=1\rightleftarrows g=1$. Further suppose that there exists a complex number a ($\neq 0, 1$) satisfying $f=a\rightarrow g=a$. Then f and g must satisfy one of the following four relations.*

(i) $f\equiv g$

(ii) $f=e^{\alpha}+a,\ g=(1-a)\{1+ae^{-\alpha}\}$ (*α is a non-constant entire function.*)

(iii) $fg\equiv 1$ (*This occurs only for $a=-1$.*)

(iv) $(f-1)(g-1)\equiv 1$ (*This occurs only for $a=2$.*)

We shall prove Theorems 2 and 3 from the following result.

LEMMA. *Define f as (3.1) with two non-constant entire functions β and γ.*

$$f=\frac{1-e^{\beta}}{1-e^{\gamma}}. \tag{3.1}$$

Then if f is a non-constant entire function, for any complex number a ($\neq 0, 1$),

$$\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{\bar{N}(r, a, f)}{m(r, f)}>0, \tag{3.2}$$

where E is the set of finite linear measure depending only on f.

Proof. The proof proceeds in two stages. In the first stage, we prove under the assumptions of the lemma,

$$\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{N(r, a, f)}{m(r, f)}>0 \qquad (a\in \boldsymbol{C}-\{0, 1\}). \tag{3.3}$$

We assume (3.3) to be false for some a ($\neq 0, 1$) and seek a contradiction. This assumption implies that f has many zeros. And we note that the following inequalities hold:

$$2\leqq\varliminf_{\substack{r\to\infty\\ r\notin E}}\frac{m(r, e^{\beta})}{m(r, e^{\gamma})}\leqq\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{m(r, e^{\beta})}{m(r, e^{\gamma})}<\infty. \tag{3.4}$$

To prove the first inequality, we make use of the argument of the impossibility of Borel's identity. The detail is as follows. In view of (3.1) we have $e^{\beta}-fe^{\gamma}+f\equiv 1$. Put $\varphi_1\equiv e^{\beta}$, $\varphi_2\equiv -fe^{\gamma}$ and $\varphi_3\equiv f$. Then $\varphi_1+\varphi_2+\varphi_3\equiv 1$, and $\varphi_1^{(n)}+\varphi_2^{(n)}+\varphi_3^{(n)}\equiv 0$ ($n=1, 2$). Further put

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ \varphi_1'/\varphi_1 & \varphi_2'/\varphi_2 & \varphi_3'/\varphi_3\\ \varphi_1''/\varphi_1 & \varphi_2''/\varphi_2 & \varphi_3''/\varphi_3\end{vmatrix}, \qquad \Delta'=\begin{vmatrix}\varphi_2'/\varphi_2 & \varphi_3'/\varphi_3\\ \varphi_2''/\varphi_2 & \varphi_3''/\varphi_3\end{vmatrix}. \tag{3.5}$$

Assume that $\Delta\equiv 0$. In this case, we have $\Delta'\equiv 0$. This implies $\varphi_2=C\varphi_3+D$ (C, D: constants), i.e., $-fe^{\gamma}=Cf+D$. Since f is entire, C must vanish. Hence $f=-De^{\gamma}$ ($D\neq 0$). This contradicts our assumption. So, we deduce $\Delta\not\equiv 0$. In this case, we have from (3.5)

$$(3.6)\qquad \varphi_1=e^{\beta}=\Delta'/\Delta\,.$$

Thus

$$(3.7)\qquad m(r,\ e^{\beta})\leqq m(r,\ \Delta')+m(r,\ \Delta^{-1})+O(1)\leqq m(r,\ \Delta')+m(r,\ \Delta)+N(r,\ \infty,\ \Delta)\,.$$

Here we estimate $m(r, \Delta')$ and $m(r, \Delta)$. From (3.1) we have $N(r, 0, f)=N(r, 1, e^{\beta})-N(r, 1, e^{\gamma})\geqq 0$. This yields $m(r, e^{\beta})\geqq(1-o(1))m(r, e^{\gamma})$ $(r\notin E, r\rightarrow\infty)$. Hence $m(r, \varphi_3)\leqq m(r, e^{\beta})+m(r, e^{\gamma})+O(1)\leqq(2+o(1))m(r, e^{\beta})$ $(r\notin E, r\rightarrow\infty)$, and $m(r, \varphi_2)\leqq m(r, f)+m(r, e^{\gamma})+O(1)\leqq(3+o(1))m(r, e^{\beta})$ $(r\notin E, r\rightarrow\infty)$. Thus $m(r, \Delta)=O(\log rm(r, e^{\beta}))$ $(r\notin E, r\rightarrow\infty)$ and $m(r, \Delta')=O(\log rm(r, e^{\beta}))$ $(r\notin E, r\rightarrow\infty)$. Substituting these into (3.7) we deduce

$$(3.8)\qquad N(r,\ \infty,\ \Delta)\geqq(1-o(1))m(r,\ e^{\beta})\qquad (r\notin E,\ r\rightarrow\infty)\,.$$

However direct computation of Δ shows that $N(r, \infty, \Delta)\leqq 2N(r, 0, f)$. It follows from this and (3.8) that $2N(r, 1, e^{\beta})\geqq(1-o(1))m(r, e^{\beta})+2N(r, 1, e^{\gamma})$. From this we easily obtain

$$\varlimsup_{\substack{r\rightarrow\infty\\ r\notin E}}\frac{m(r,\ e^{\gamma})}{m(r,\ e^{\beta})}\leqq\frac{1}{2}\,.$$

This proves the first inequality of (3.4). Next, to prove the last inequality of (3.4), assume that this is not the case. Then there exists a sequence $\{r_n\}_1^{\infty}\subset(0, \infty)-E$ tending to ∞ such that

$$\lim_{n\rightarrow\infty}\frac{m(r_n,\ e^{\beta})}{m(r_n,\ e^{\gamma})}=\infty\,.$$

In view of (3.1) we have

$$(3.9)\qquad f-a=\frac{1-a-e^{\beta}+ae^{\gamma}}{1-e^{\gamma}}\equiv\frac{F(z)}{1-e^{\gamma}}\,.$$

Then using a result of Nevanlinna [3, p 47], we have

$$\begin{aligned}m(r_n,\ e^{\beta})&\leqq\bar{N}(r_n,\ 0,\ e^{\beta})+\bar{N}(r_n,\ \infty,\ e^{\beta})+\bar{N}(r_n,\ 0,\ F)+S(r_n,\ e^{\beta})\\&=\bar{N}(r_n,\ 0,\ F)+S(r_n,\ e^{\beta})\leqq(1+o(1))m(r_n,\ e^{\beta})\,.\qquad(n\rightarrow\infty)\end{aligned}$$

This shows that $N(r_n, 0, F)=(1+o(1))m(r_n, e^{\beta})$ $(n\rightarrow\infty)$. Hence $N(r_n, a, f)=N(r_n, 0, F)-N(r_n, 1, e^{\gamma})=(1+o(1))m(r_n, e^{\beta})$ $(n\rightarrow\infty)$. On the other hand, we easily obtain $m(r_n, f)=(1+o(1))m(r_n, e^{\beta})$ $(n\rightarrow\infty)$ from (3.1). Thus

$$\lim_{n\rightarrow\infty}\frac{N(r_n,\ a,\ f)}{m(r_n,\ f)}=1\,,$$

a contradiction. This proves the last inequality of (3.4).

Next, we observe the following equality:

(3.10) $$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{m(r, e^{\beta})}{m(r, e^{\beta-\gamma})}=1 .$$

By the second fundamental theorem, we have

$$m(r, f)\leqq\begin{Bmatrix} N(r, 0, f) \\ N(r, 1, f)\end{Bmatrix}+N(r, a, f)+N(r, \infty, f)+S(r, f) .$$

Hence by our assumption we deduce

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{N(r, 0, f)}{m(r, f)}=\lim_{\substack{r\to\infty \\ r\notin E}} \frac{N(r, 1, f)}{m(r, f)}=1 .$$

Thus

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{N(r, 0, f)-N(r, 1, f)}{m(r, f)}=\lim_{\substack{r\to\infty \\ r\notin E}} \frac{N(r, 1, e^{\beta})-N(r, 1, e^{\beta-\gamma})}{m(r, f)}=0 .$$

However, since $m(r, e^{\beta})+m(r, e^{\gamma})+O(1)\leqq(3/2+o(1))m(r, e^{\beta})$ $(r\notin E, r\to\infty)$, we have

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{N(r, 1, e^{\beta})-N(r, 1, e^{\beta-\gamma})}{m(r, e^{\beta})}=0 .$$

Therefore (3.10) follows.

Now, from our assumption, (3.9) and (3.4) we deduce $(1-o(1))m(r, e^{\gamma})\leqq N(r, 0, F)$ $\leqq(1+o(1))m(r, e^{\gamma})+o(m(r, f))=(1+o(1))m(r, e^{\gamma})+o(m(r, e^{\beta}))=(1+o(1))m(r, e^{\gamma})$ $(r\notin E, r\to\infty)$. This implies

(3.11) $$N(r, 0, F)=(1+o(1))m(r, e^{\gamma}) \qquad (r\notin E, r\to\infty) .$$

Then, in view of (3.9), (3.4), (3.10) and (3.11), we have

$$\begin{aligned} m(r, F)&\leqq N(r, 0, F)+N(r, \infty, F)+N(r, 1-a, F)-N(r, 0, F')+S(r, F) \\ &=(1+o(1)(m(r, e^{\gamma})+N(r, a, e^{\beta-\gamma})-N\left(r, 0, e^{\beta-\gamma}-\frac{a\gamma'}{\beta'}\right)+S(r, F) \\ &=(1+o(1))m(r, e^{\gamma})+(1+o(1))m(r, e^{\beta-\gamma})-(1+o(1))m(r, e^{\beta-\gamma}) \\ &\quad +o(m(r, e^{\beta})+m(r, e^{\gamma})) \\ &=(1+o(1))m(r, e^{\gamma})+o(m(r, e^{\beta-\gamma}))+o(m(r, e^{\beta})+m(r, e^{\gamma})) \\ &=(1+o(1))m(r, e^{\gamma}) \qquad (r\notin E, r\to\infty) . \end{aligned}$$

It follows from this and (3.11) that

(3.12) $$m(r, F)=(1+o(1))m(r, e^{\gamma}) .$$

On the other hand, we deduce from (3.10) and (3.4)

(3.13) $$m(r, F)\geqq(1-o(1))m(r, ae^{\gamma}-e^{\beta})\geqq(1-o(1))N(r, 0, ae^{\gamma}-e^{\beta})$$

$$=(1-o(1))N(r, a, e^{\beta-\gamma})=(1-o(1))m(r, e^{\beta-\gamma})$$

$$=(1-o(1))m(r, e^{\beta})\geqq(2-o(1))m(r, e^{\gamma}) \qquad (r\notin E,\ r\to\infty).$$

(3.12) and (3.13) lead to a contradiction. This proves (3.3). In the second stage, we prove (3.2). Assume first

$$\overline{\lim_{\substack{r\to\infty\\ r\notin E}}}\frac{m(r, e^{\beta})}{m(r, e^{\gamma})}=\lim_{n\to\infty}\frac{m(r_n, e^{\beta})}{m(r_n, e^{\gamma})}=\infty.$$

In this case, we have $m(r_n, f)=(1+o(1))m(r_n, e^{\beta})$ $(n\to\infty)$, and $N(r_n, a, f)=(1+o(1))m(r_n, e^{\beta})$ $(n\to\infty)$. This implies that

$$\overline{\lim_{\substack{r\to\infty\\ r\notin E}}}\frac{\bar{N}(r, a, f)}{m(r, f)}=1.$$

Next, assume that

$$\overline{\lim_{\substack{r\to\infty\\ r\notin E}}}\frac{m(r, e^{\beta})}{m(r, e^{\gamma})}<\infty.$$

Let $\{z_n\}$ be all the roots of $f=a$ with multiplicity$\geqq 3$. Then $\{z_n\}$ are the roots of $F'(z)=e^{\gamma}\{a\gamma'-\beta' e^{\beta-\gamma}\}=0$ with multiplicity$\geqq 2$. Here note that we may assume

$$\underline{\lim_{\substack{r\to\infty\\ r\notin E}}}\frac{m(r, e^{\beta})}{m(r, e^{\gamma})}\geqq 2.$$

If not, the above argument shows that $\Delta\equiv 0$. This implies $f=-De^{\gamma}$ $(D\neq 0)$. In this case, it is clear that (3.2) holds. Therefore we assume (3.4). This yields $m(r, e^{\beta-\gamma})\leqq m(r, e^{\beta})+m(r, e^{\gamma})+O(1)\leqq(3/2+o(1))m(r, e^{\beta})$ $(r\notin E,\ r\to\infty)$ and $m(r, e^{\beta-\gamma})\geqq m(r, e^{\beta})-m(r, e^{\gamma})+O(1)\geqq(1/2-o(1))m(r, e^{\beta})$ $(r\notin E,\ r\to\infty)$. Hence we have

$$m(r, \gamma')+m(r, \beta')=o(m(r, e^{\gamma})+m(r, e^{\beta}))=o(m(r, e^{\beta-\gamma})) \qquad (r\notin E,\ r\to\infty). \tag{3.14}$$

Noting (3.14), and applying the second fundamental theorem to $G=a\gamma'-\beta' e^{\beta-\gamma}$, we have

$$(1+o(1))m(r, G)\leqq\bar{N}(r, 0, G)+\bar{N}(r, \infty, G)+\bar{N}(r, 0, \beta' e^{\beta-\gamma})+S(r, G),$$

which implies $m(r, G)=(1+o(1))N(r, 0, G)=(1+o(1))\bar{N}(r, 0, G)$ $(r\notin E,\ r\to\infty)$. Hence

$$\lim_{\substack{r\to\infty\\ r\notin E}}\frac{N_1(r, 0, G)}{m(r, G)}=\lim_{\substack{r\to\infty\\ r\notin E}}\frac{N_1(r, 0, G)}{m(r, e^{\beta-\gamma})}=\lim_{\substack{r\to\infty\\ r\notin E}}\frac{N_1(r, 0, G)}{m(r, e^{\beta})}=0.$$

Since $m(r, f)\geqq m(r, e^{\beta})-m(r, e^{\gamma})+O(1)\geqq(1/2-o(1))m(r, e^{\beta})$, we deduce

$$\lim_{\substack{r\to\infty\\ r\notin E}}\frac{N_1(r, 0, G)}{m(r, f)}=0.$$

Thus we easily obtain

$$\bar{N}(r, a, f)\geqq\frac{1}{2}\Big\{N(r, a, f)-N_1(r, 0, G)\Big\}=\frac{1}{2}N(r, a, f)-o(m(r, f)) \tag{3.15}$$

$$(r \notin E,\ r\to\infty).$$

Substituting (3.15) into (3.3), (3.2) follows. This completes the proof of Lemma.

Proof of Theorem 2. By assumption, we have with two entire functions α and β

$$f=e^{\alpha}g\,,\qquad f-1=e^{\beta}(g-1)\,. \tag{3.16}$$

(A) Suppose that $e^{\beta}\equiv c\ (\neq 0)$. If f has a zero, $c=1$. Hence $f\equiv g$. If f has no zeros and $c\neq 1$, we have $f-cg=1-c\ (\neq 0)$. Using a result of Niino and Ozawa [5], we obtain $2=\delta(0, f)+\delta(0, g)\leqq 1$, a contradiction.

(B) Suppose that $e^{\beta-\alpha}\equiv c\ (\neq 0)$. If $c=1$, we have $f\equiv g$. If $c\neq 1$, we have

$$g=\frac{f}{(1-c)f+c}\,;\qquad f=\frac{e^{\gamma}-c}{1-c}\,,\qquad g=\frac{1-ce^{-\gamma}}{1-c} \tag{3.17}$$

(γ: a non-constant entire function).

Assume first that $a=-c/(1-c)$. Then $E(a, k, g)=E(a, k, f)=\phi\,(k\geqq 2)$. Hence by (3.17) we deduce $a=1/(1-c)$. So, we obtain $-c/(1-c)=1/(1-c)$, i.e., $c=-1$. Substituting this into (3.17), we deduce

$$\left(f-\frac{1}{2}\right)\left(g-\frac{1}{2}\right)\equiv\frac{1}{4}\,,\qquad a=\frac{1}{2}\,.$$

Assume next that $a\neq -c/(1-c)$. In this case, $f=a$ has infinitely many simple roots. Hence we have $a=a/\{a(1-c)+c\}$ i.e., $a=1$, a contradiction.

(C) Suppose neither e^{β} nor $e^{\beta-\alpha}$ are constants. In this case, we have by (3.16)

$$f=\frac{1-e^{\beta}}{1-e^{\beta-\alpha}}\,,\qquad g=\frac{1-e^{\beta}}{1-e^{\beta-\alpha}}e^{-\alpha}\,. \tag{3.18}$$

Using our lemma to f, we have

$$\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{\bar{N}(r, a, f)}{m(r, f)}>0\,. \tag{3.19}$$

Now, let $\{w_n\}$ be all the common roots of $f=a$ and $g=a$. From (3.18) we have $e^{\alpha(w_n)}=e^{\beta(w_n)}=1$. Hence

$$f=\frac{-\beta'(w_n)(z-w_n)+\cdots}{-(\beta'(w_n)-\alpha'(w_n))(z-w_n)+\cdots}\,.$$

This shows that $\{w_n\}$ are the roots of

$$a\alpha'(z)+(1-a)\beta'(z)=0\,. \tag{3.20}$$

Also let $\{z_n\}$ be all the roots of $f=a$ with multiplicity $\geqq 3$. The argument in the above lemma shows

(3.21) $$N(r,\ \{z_n\})=o(m(r,\ f)) \qquad (r\notin E,\ r\to\infty)\,.$$

If f has a zero, we have

$$\underline{\lim}_{\substack{r\to\infty\\ r\notin E}}\frac{m(r,\ e^{\beta})}{m(r,\ e^{\beta-\alpha})}\geqq 2\,.$$

This implies $m(r,\ e^{\beta})=O(m(r,\ f))$ and $m(r,\ \alpha')+m(r,\ \beta')=o(m(r,\ e^{\beta}))$ $(r\notin E,\ r\to\infty)$. Hence unless (3.20) is an identity, we combine (3.20) with (3.21) to deduce $\bar{N}(r,\ a,\ f)$ $o(m(r,\ f))$ $(r\notin E,\ r\to\infty)$, which contradicts (3.19). If (3.20) is an identity, we have

(3.22) $$\beta(z)=a(\beta(z)-\alpha(z))+C \qquad (C\text{: a constant})\,.$$

Since $\beta-\alpha$ is a non-constant entire function, it is easy to see that a is an integer, and so, C is an integral multiple of $2\pi i$. Further we note that $a\neq 0,\ 1,\ -1$ by our assumptions.

If $a=2$, we have from (3.1) and (3.22) $f=1+e^{\beta/2}=1+e^{\beta-\alpha}$. On the other hand, $g=fe^{-\alpha}=e^{-\beta/2}(1+e^{\beta/2})=1+e^{-\beta/2}$. Thus $(f-1)(g-1)\equiv 1$. If $a\geqq 3$,

$$\begin{cases} f=1+e^{\beta-\alpha}+\cdots+e^{(a-1)(\beta-\alpha)} \\ g=1+e^{-(\beta-\alpha)}+\cdots+e^{-(a-1)(\beta-\alpha)}\,. \end{cases}$$

If $a\leqq -2$,

$$\begin{cases} f=-e^{-|a|(\beta-\alpha)}\{1+e^{\beta-\alpha}+\cdots+e^{(|a|-1)(\beta-\alpha)}\} \\ g=-e^{\beta-\alpha}\{1+e^{\beta-\alpha}+\cdots+e^{(|a|-1)(\beta-\alpha)}\}\,. \end{cases}$$

In these cases, f and g do not satisfy $E(a,\ k,\ f)=E(a,\ k,\ g)$. Finally we consider the case that f has no zeros. It follows from (3.1) that

$$\frac{1-e^{\beta}}{1-e^{\beta-\alpha}}=e^{\delta} \qquad (\delta\text{: a non-constant entire function}),\ \text{i.e.,}$$

(3.23) $$e^{\delta}-e^{\beta-\alpha+\delta}+e^{\beta}\equiv 1\,.$$

Using again the result due to Niino and Ozawa, we have $e^{\beta-\alpha+\delta}\equiv -1$. Hence $f=e^{\delta}=-e^{\alpha-\beta}=-e^{\beta}$. On the other hand, $g=fe^{-\alpha}=-e^{\beta-\alpha}=-e^{-\beta}$. Thus $fg\equiv 1$. This can be occurred in the case of $a=-1$. This completes the proof of Theorem 2.

Remark 1. The proof of Theorem 3 are essentially contained by the above proof. So, we omit the proof of Theorem 3.

Remark 2. Theorem 2 does not hold in the case of $k=1$. For instance, we may put $f=1+e^{\gamma}+e^{2\gamma}$ and $g=1+e^{-\gamma}+e^{-2\gamma}$, where γ is a non-constant entire function. Then if $a=3/4$, $f-a=(e^{\gamma}+1/2)^2$ and $g-a=(e^{-\gamma}+1/2)^2$. This shows that $f=0\rightleftarrows g=0$, $f=1\rightleftarrows g=1$ and $E(a,\ 1,\ f)=E(a,\ 1,\ g)$ $(=\phi)$, but it is clear that f and g do not satisfy any relations (i)-(iv) in Theorem 2.

Remark 3. Combining Corollary 1 with the proof of Theorem 2, we have the following result.

THEOREM 2'. *Let f and g be non-constant entire functions satisfying the following conditions* (i)-(iii);

(i) $\sum_{c\neq\infty}\left\{1-\varliminf_{\substack{r\to\infty\\ r\notin E}}\frac{N(r, c, f)}{m(r, f)}\right\}>0$,

(ii) $f=0\rightleftarrows g=0$, $f=1\rightleftarrows g=1$,

(iii) *There exist two distinct complex numbers* a_1, a_2 $(\neq 0, 1)$ *such that* $E(a_i, 1, f)=E(a_i, 1, g)$ $(i=1, 2)$.

Then $f\equiv g$.

4. Ozawa [7] has proved the following result.

THEOREM C. *Let f and g be entire functions of finite non-integral order such that* $f=0\rightleftarrows g=0$ *and* $f=1\rightarrow g=1$. *Then* $f\equiv g$.

In this section, we shall prove the following results.

THEOREM 4. *Let f and g be entire functions of non-integral order such that* $f=0\rightleftarrows g=0$. *Further assume that there exist two distinct complex numbers* a_1, a_2 $(\neq 0)$ *satisfying* $E(a_i, k, f)=E(a_i, k, g)$ $(i=1, 2)$, *where k is a positive integer* $(\geqq 2)$ *or* ∞. *Then* $f\equiv g$.

THEOREM 5. *Let f and g be entire functions of finite non-integral order satisfying the following conditions* (i)-(iii);

(i) $f=0\rightleftarrows g=0$,

(ii) *There exist two distinct complex numbers* a_1, a_2 $(\neq 0)$ *such that* $E(a_i, 1, f)=E(a_i, 1, g)$,

(iii) $\delta(0, f)+\delta(a_1, f)+\delta(a_2, f)>0$.

Then $f\equiv g$.

First we remark that the condition (iii) of Theorem 5 cannot be dropped. Example: $f=\cos(z)^{n/2}$, $g=-f$, $a_1=1$, $a_2=-1$. (n: an odd integer $\geqq 3$) Next we remark that the assumption of non-integrity of ρ_f cannot also be dropped. For example, we may put $f=e^z$ and $g=e^{-z}$ $(a_1=1, a_2=-1)$.

The method of proof of the above two theorems is essentially the same, so we shall prove only Theorem 5.

Proof of Theorem 5. By the assumption (i), we have with a polynomial α

$$f=e^{\alpha}g. \tag{4.1}$$

Non-integrity of ρ_f, ρ_g and (4.1) imply

(4.2) $$\rho_f=\rho_g>\deg\alpha \quad (\equiv p).$$

(A) Assume that $p\geqq 1$. Let $\{w_n\}$ be all the simple roots of $f=a_1$. From the condition (ii) we have $a_1=e^{\alpha(w_n)}a_1$, i.e., $e^{\alpha(w_n)}=1$. Hence $N(r, \{w_n\}) \leqq N(r, 1, e^\alpha)\leqq(1+o(1))m(r, e^\alpha)=O(r^p)$. On the other hand, a well known theorem of Borel yields $\rho_{N(r, a_1, f)}=\rho_f>p$. Thus $\Theta(a_1, f)\geqq 1/2$. In the same way we have $\Theta(a_2, f)\geqq 1/2$. Here we use the condition (iii). If $\delta(0, f)>0$, then $\Theta(0, f)>0$. Hence $\Theta(a_1, f)+\Theta(a_2, f)+\Theta(0, f)>1$. This is impossible. If $\delta(a_1, f)>0$, the above argument implies

$$\Theta(a_1, f)=1-\overline{\lim_{r\to\infty}}\frac{\bar{N}(r, a_1, f)}{m(r, f)}\geqq 1-\frac{1}{2}\overline{\lim_{r\to\infty}}\frac{N(r, a_1, f)}{m(r, f)}$$

$$=1-\frac{1}{2}(1-\delta(a_1, f))>\frac{1}{2}.$$

Hence $\Theta(a_1, f)+\Theta(a_2, f)>1$. This is impossible.

(B) Assume that $p=0$. If we put $e^\alpha\equiv c\ (\neq 0)$, $f\equiv cg$. Suppose first that $E(a_i, 1, f)=E(a_i, 1, g)\neq\phi$ for $i=1$ or 2. In this case we have $c=1$. Hence $f\equiv g$. Suppose next $E(a_i, 1, f)=E(a_i, 1, g)=\phi$ for $i=1, 2$. In this case, the same argument as (A) derives a contradiction. This completes the proof of Theorem 5.

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