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QUASIHARMONIC L^p FUNCTIONS ON THE POINCARÉ N-BALL

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By definition, a function q on a Riemannian manifold R is quasiharmonic if $\Delta q=1$, with $\Delta=d\delta+\delta d$ the Laplace-Beltrami operator. There is fairly extensive literature on the existence of quasiharmonic functions in the classes P, B, D or C of functions which are positive, bounded, Dirichlet finite, or bounded Dirichlet finite, respectively. In contrast, very little is known about the existence of quasiharmonic functions in L^p.

Let Q be the class of quasiharmonic functions and set $QX=Q\cap X$ for $X=P$, B, D, C, L^p. For any function class F, denote by O_F the class of Riemannian manifolds which do not carry any nonconstant functions in F, and $\tilde{O}_F$ its complement. The purpose of the present study is to give a criterion for $R\in O_{QL^p}$ and to relate the class O_{QL^p} to some harmonic and quasiharmonic null classes. We also discuss interrelations between the classes O_{QL^p} for various p. For explicit results, we consider these problems on the Poincaré N-ball B_α^N, that is, the unit ball of N-space, $N\geqq 2$, endowed with the metric $ds_\alpha=(1-r^2)^\alpha ds_0$, $r=|x|$, with $\alpha\in\boldsymbol{R}$ and ds_0 the Euclidean metric.

We start by stating, with or without proofs, some auxiliary results, mostly known, to be called Propositions. The new results will be given in Lemmas 1-6 and Theorems 1-3.

§ 1. Preliminaries

Propositions 1 and 2 on the general behavior of quasiharmonic functions will greatly simplify earlier proofs on characterizing quasiharmonic null classes. Let $(r, \theta)=(r, \theta^1, \cdots, \theta^{N-1})$ be the polar coordinates in $\boldsymbol{R}^N$.

PROPOSITION 1. *Every quasiharmonic function $q(r, \theta)$ on B_α^N can be represented as*

$$q(r, \theta)=q(r)+h(r, \theta)=q_0(r)+c+h(r, \theta),$$

where $h(r, \theta)$ is a harmonic function, c a constant, and

$$q_0(r)=-\int_0^r(1-t^2)^{-(N-2)\alpha}t^{-(N-1)}\int_0^t(1-s^2)^{N\alpha}s^{N-1}ds\,dt.$$

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Proof. By direct computation, $\Delta q_0=1$. Thus, $q_0(r)+c+h(r, \theta)$ is a quasiharmonic function. For any quasiharmonic function $q_1(r, \theta)$, the difference $q_1(r, \theta)-q_0(r)$ satisfies $\Delta(q_1-q_0)=\Delta q_1-\Delta q_0=0$ and is, therefore, harmonic.

PROPOSITION 2. *Let* $N>2$.
(a) *For* $\alpha>0$, *as* $r\to 1$,

$$q_0(r)\sim c\int_0^r (1-t)^{-(N-2)\alpha}ds\,.$$

(b) *For* $-1/N<\alpha<0$, *if* c_0 *is a constant such that* $q_0(r)+c_0\to 0$ *as* $r\to 1$, *then*

$$q_0(r)+c_0\sim c(1-r)^{[-(N-2)\alpha+1]}\,.$$

(c) *For* $-1<\alpha<-1/N$, *and* c_0 *as in* (b),

$$q_0(r)+c_0\sim c(1-r)^{2(\alpha+1)}\,.$$

(d) *For* $\alpha\leqq -1$, $|q_0(r)|\to\infty$.
(e) *For* $\alpha=-1/N$,

$$q_0(r)=O\Big(\int_0^r (1-t^2)^{-(N-2)\alpha}t\log(1-t^2)dt\Big)\,.$$

Proof. As an illustration, we compute (c):

$$q_0(r)+c_0=-\int_0^r (1-t^2)^{-(N-2)\alpha}t^{-(N-1)}\Big(\int_0^t (1-s^2)^{N\alpha}s^{N-1}ds\Big)dt+c_0$$

$$\sim\int_r^1 (1-t^2)^{-(N-2)\alpha}t^{-(N-1)}\Big(\int_0^t (1-s^2)^{N\alpha}s^{N-1}ds\Big)dt$$

$$\sim c\int_r^1 (1-t^2)^{-(N-2)\alpha}(1-t^2)^{N\alpha+1}dt$$

$$\sim c\int_r^1 (1-t)^{1+2\alpha}=c(1-r)^{2(1+\alpha)}\,.$$

We will also make use of the following well-known results in the classification theory for quasiharmonic and harmonic functions.

PROPOSITION 3 (Nakai-Sario [2], Sario [3]). *The quasiharmonic null classes satisfy the strict inclusion relations*

$$O_G<O_{QP}\begin{matrix} & O_{QB} & \\ < & & < \\ & & \\ < & & < \\ & O_{QD} & \end{matrix}O_{QC}\,,$$

whereas there is no inclusion between O_{QB} *and* O_{QD}.

PROPOSITION 4 (Sario-Wang [4]). *For* $X=P, C$,

$$B_\alpha^N \in \tilde{O}_{QX} \quad \text{if and only if } \alpha\in(-1, 1/(N-2)).$$

For $Y=D, C$,

$$B_\alpha^N \in \tilde{O}_{QY} \quad \text{if and only if } \alpha\in(-3/(N+2), 1/(N-2)).$$

PROPOSITION 5 (Hada-Sario-Wang [1]). $B_\alpha^N \in \tilde{O}_{HD}$ *if and only if* $|\alpha|<1/(N-2)$.

§ 2. Characterization of quasiharmonic L^p null classes

In this section, we will give a criterion for $B_\alpha^N \in O_{QL^p}$. Then we briefly consider applications to harmonic and quasiharmonic classifications. On occasion, we make the assumption $N>2$, usually necessitated by the factor $(N-2)^{-1}$.

LEMMA 1. *For* $1\leqq p<\infty$, $B_\alpha^N \in O_{QL^p}$ *if and only if for each constant* c, $q_0(r)+c\notin L^p$, *with* $q_0(r)$ *as in Proposition* 1.

Proof. The necessity is immediate. Since $B_\alpha^N \in O_{QL^p}$, it is clear that $q_0(r)+c$, as a quasiharmonic function, cannot belong to L^p.

To show the sufficiency, assume $q_0(r)+c$ is not in L^p for any constant c. By Proposition 1, a general quasiharmonic function $q(r, \theta)$ can be represented as $q_0(r)+c+h(r, \theta)$ where $h(r, \theta)$ is a harmonic function of the form $\sum f_n S_n$ with the S_n spherical harmonics of degree n. We can absorb the constant term of $h(r, \theta)$ into c and thus assume that $h(r, \theta)$ contains no constant term. Since by assumption $q_0(r)+c$ is not in L^p, we choose $g(r)\in L^q$ such that $p^{-1}+q^{-1}=1$ and $\left|\int g(r)(q_0(r)+c)dV\right|=\infty$. We have

$$\left|\int g(r)[q_0(r)+c+h(r, \theta)]\,dV\right|=\left|\int g(r)[q_0(r)+c]\,dV\right|=\infty,$$

since $h(r, \theta)$ is assumed to be a sum of terms $f_n(r)S_n(\theta)$, $n>0$, and since $\int gf_nS_n dV=0$ for $n>0$ by a well-known property of spherical harmonics. Thus $q(r, \theta)\notin L^p$ for any quasiharmonic function, as claimed.

Next we characterize the B_α^N which belong to O_{QL^p}. We separate the cases $N>2$ and $N=2$.

LEMMA 2. *Suppose* $N>2$.

(a) *A necessary and sufficient condition for* $B_\alpha^N \in O_{QL^p}$ *is one of the following*:
 (i) $1/(N-2)$ *and* $\alpha[(N-2)p-N]\geqq p+1$; *or*
 (ii) $-1<\alpha<-1/N$ *and* $2p(\alpha+1)+N\alpha+1\leqq 0$; *or*
 (iii) $\alpha\leqq -1$.

(b) *For* $-1/N\leqq\alpha\leqq 1/(N-2)$, $B_\alpha^N \in \tilde{O}_{QL^p}$.

Proof. First we prove (b). The case $\alpha=0$ is immediate, because B_0^N is the Euclidean ball.

Suppose $-1/N<\alpha<0$. By Proposition 2(b), there is a c_0 such that as $r\to 1$, $q_0(r)+c_0\to 0$ and $|q_0(r)+c_0|^p \sim c(1-r)^{[-(N-2)\alpha+1]p}$.

In order to prove $B_\alpha^N \in \tilde{O}_{QL^p}$, it suffices to show that $(1-r)^{[-(N-2)\alpha+1]p}$ is integrable. We have

$$\int (1-r)^{[-(N-2)\alpha+1]p} dV = c\int_0^1 (1-r)^{[-(N-2)\alpha+1]p}(1-r)^{N\alpha}(1+r)^{N\alpha} r^{N-1} dr$$

$$\leqq 2c\int_0^1 (1-r)^{[-(N-2)\alpha+1]p+N\alpha} dr\,.$$

The last integral is finite if and only if $[-(N-2)\alpha+1]p+N\alpha>1$. Since $\alpha>-1/N$ is negative, we obtain $[-(N-2)\alpha+1]p+N\alpha>p+N\alpha>p-1\geqq 0>-1$. Thus $B_\alpha^N \in \tilde{O}_{QL^p}$.

Suppose $0<\alpha\leqq 1/(N-2)$. By Proposition 2(a),

$$|q_0(r)|^p \sim c\left|\int_0^r (1-r)^{-(N-2)\alpha} dr\right|^p \leqq |\log(1-r)|^p \qquad \text{as } r\to 1\,.$$

Thus,

$$\int |q_0(r)|^p dV \sim \int |(1-r)^{-(N-2)\alpha} dr|^p dV \leqq c\int |\log(1-r)|^p dV$$

$$= c\int |\log(1-r)|^p (1-r)^{N\alpha}(1+r)^{N\alpha} r^{N-1} dr < \infty$$

and $q_0(r)\in B_\alpha^N$.

Suppose $\alpha=-1/N$. By Proposition 2(e),

$$q_0(r) = O\left(\int_0^r (1-t^2)^{-(N-2)\alpha} t\log(1-t^2)\,dt\right),$$

which is bounded. There is a constant c such that

$$q_0(r)-c = O\left(\int_r^1 (1-t^2)^{-(N-2)\alpha} t\log(1-t^2)\,dt\right),$$

which goes to 0 faster than $(1-r)^\varepsilon$ as $r\to 1$ for some $\varepsilon>0$. Thus $q_0(r)-c$ is a QL^p function.

To prove (a), we consider three cases:

Case 1. $\alpha>1/(N-2)$. By Lemma 1, $B_\alpha^N \in O_{QL^p}$ if and only if $q_0(r)+c \notin L^p$ for any constant c. By Proposition 2(a),

$$\int |q_0(r)+c|^p dV \sim c\int \left|\int_0^r (1-r)^{-(N-2)\alpha} dr\right|^p dV$$

$$= c\int (1-r)^{[-(N-2)\alpha+1]p}(1-r)^{N\alpha}(1+r)^{N\alpha} r^{N-1} dr\,,$$

which is ∞ if and only if $[-(N-2)\alpha+1]p+N\alpha\leqq -1$, that is, $\alpha[(N-2)p-N]\geqq p+1$.

Case 2. $-1<\alpha<-1/N$. By Proposition 2(c), there is a constant c_0 such that, as $r\to 1$, $q_0(r)+c_0\to 0$ and $|q_0(r)+c_0|^p\sim c(1-r)^{2(\alpha+1)p}$. Thus if $B_\alpha^N\in O_{QL^p}$, then

$$\infty=\|q_0(r)+c_0\|_p^p\sim c\int(1-r)^{2(\alpha+1)p}dV$$

$$\sim c\int(1-r)^{2(\alpha+1)p}(1-r)^{N\alpha}(1+r)^{N\alpha}r^{N-1}dr\,,$$

or equivalently, $2p(\alpha+1)+N\alpha+1\leqq 0$. Conversely, since $q_0(r)+x_0\notin L^p$ implies $q_0(r)+c\notin L^p$ for any c, we obtain (ii).

Case 3. $\alpha\leqq -1$.
By Proposition 2(d), $|q_0(r)|^p\to\infty$. Hence, also $|q_0(r)+c|^p\to\infty$ for any c. Thus

$$\int|q_0(r)+c|^pdV\geqq\int dV=c\int(1-r)^{N\alpha}(1+r)^{N\alpha}r^{N-1}dr=\infty\,,$$

as $\alpha\leqq -1$. This completes the proof of the lemma.
It remains to consider the case $N=2$. We omit the proof.

LEMMA 3. (a) *$B_\alpha^2\in O_{QL^p}$ if and only if*
(i) *$-1<\alpha<-1/2$ and $2p(\alpha+1)+2\alpha+1\leqq 0$; or*
(ii) *$\alpha\leqq -1$.*
(b) *For $\alpha\geqq -1/2$, $B_\alpha^2\in\tilde{O}_{QL^p}$.*

§ 3. Application to harmonic classification

We will briefly indicate an application of the characterization of $B_\alpha^N\in O_{QL^p}$ to the harmonic classification theory. The classes to be considered are $O_{QL^p}\cap O_{HX}$, $O_{QL^p}\cap\tilde{O}_{HX}$, $O_{QL^p}\cap O_{HX}$, and $\tilde{O}_{QL^p}\cap\tilde{O}_{HX}$, where $X=G$, P, B, or D, with G the class of Green's functions.

We start with $O_{QL^p}\cap\tilde{O}_{HX}$. By Proposition 5, $|\alpha|<1/(N-2)$ assures that $B_\alpha^N\in\tilde{O}_{HD}$. Lemma 2(a) entails that $-1/(N-2)<\alpha<-1/N$ and $2p(\alpha+1)+N\alpha+1\leqq 0$ are necessary and sufficient for B_α^N to belong to $O_{QL^p}\cap\tilde{O}_{HD}$. Since $2p(\alpha+1)+N\alpha+1\leqq 0$ is equivalent to $\alpha\leqq-(2p+1)/(2p+N)=1-(N-1)/(2p+n)$, we obtain $\alpha\leqq 1-(N-1)/(2+N)$. In view of $-1/(N-2)<\alpha$, we must have $-1/(N-2)<1-(N-2)/(2+N)$, which gives $N<4$. Thus $N=3$ is the only possible candidate.

To see that for $N=3$, the inequality has a solution with $\alpha\in(-1/(N-2),\,-1/N)$ for each $p\geqq 1$, let p be given. Then

$$2p(\alpha+1)+1\leqq -3\alpha\,.$$

Note that the range of α is $(-1/(N-2),\,-1/N)=(-1,\,-1/3)$. Choose α so close to -1 that $\alpha+1$ is very small, hence $2p(\alpha+1)$ also very small, and -3α very close to 3. Clearly the condition is satisfied. *Thus we have the rather unexpected result that the dimension of a manifold plays here a crucial role.*

Since harmonic classification serves only illustrative purposes in this work, we will not consider other harmonic classes, but discuss only O_{HD} in this section.

In view of the above phenomenon, one might expect its analogues concerning the other three classes as well. However, all other classes are well behaved:

THEOREM 1. *For each $p\geqq 1$ and each $N>2$, the family $\mathscr{B}$ of the Poincaré N-balls decomposes into the following disjoint nonempty sets:*

For $N=3$, $\mathscr{B}=O_{QL^p}\cap O_{HD}\oplus O_{QL^p}\cap\tilde{O}_{HD}\oplus\tilde{O}_{QL^p}\cap O_{HD}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{HD}$.

For $N>3$, $\mathscr{B}=O_{QL^p}\cap O_{HD}\oplus\tilde{O}_{QL^p}\cap O_{HD}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{HD}$.

Proof. If suffices to show the nonemptiness of the other classes. For $O_{QL^p}\cap O_{HD}$, choose $\alpha\leqq -1$. Then $B_\alpha^N\in O_{QL^p}\cap O_{HD}$. For $\tilde{O}_{QL^p}\cap O_{HD}$, take $\alpha=1/(N-2)$. For $\tilde{O}_{QL^p}\cap\tilde{O}_{HD}$, pick $\alpha=0$. In each case we only have to use Lemma 2 and Proposition 5.

§ 4. Application to quasiharmonic classification

We proceed to an application of our O_{QL^p} characterization to quasiharmonic classification. Again, we will be brief and choose only the class O_{QD} as an illustration.

THEOREM 2. *For $p\geqq 1$, $N>2$, the family $\mathscr{B}$ of the Poincaré N-balls decomposes into the following three disjoint nonempty sets.*

$$\mathscr{B}=O_{QL^p}\cap O_{QD}\oplus\tilde{O}_{QL^p}\cap O_{QD}\oplus\tilde{O}_{QL^p}\cap O_{QD}.$$

Proof. By Lemma 2 and Proposition 5, for each $N>2$, the values $\alpha=-1$, $1/(N-2)$ and 0 furnish examples for $O_{QL^p}\cap O_{QD}$, $\tilde{O}_{QL^p}\cap O_{QD}$, and $\tilde{O}_{QL^p}\cap\tilde{O}_{QD}$, respectively. To see that $O_{QL^p}\cap\tilde{O}_{QD}$ is empty, assume that $B_\alpha^N\in O_{QL^p}\cap\tilde{O}_{QD}$. By Proposition 4, $-3/(N+2)<\alpha<1/(N-2)$. By Lemma 2, $-3/(N+2)<\alpha<-1/N$. Again by Lemma 2, $2p(\alpha+1)+N\alpha+1\leqq 0$. We want to show that for this α, there is no solution for the last inequality of $p\geqq 1$ and $N>2$. Again as in the preceding section, we have $\alpha\leqq-(2p+1)/(2p+N)\leqq -3/(2+N)$, in violation of $-3/(N+2)<\alpha<1/(N-2)$.

§ 5. The classes QL^p

We proceed to study relations between QL^s and QL^t with $1<s<t<\infty$.

LEMMA 4. (a) *$B_\alpha^N\in O_{QL^s}\cap O_{QL^t}$ if $\alpha\leqq -1$.*

(b) $B_\alpha^N \in \tilde{O}_{QL^s} \cap O_{QL^t}$ *if* $-1/N \leqq \alpha \leqq 1/(N-2)$.

The proof is by Lemma 2.

LEMMA 5. (a) *If* $N > 2s/(s-1)$, *then* $B_\alpha^N \in \tilde{O}_{QL^s} \cap O_{QL^t}$ *if and only if* $(t+1)/[(N-2)t-N] \leqq \alpha < (s+1)/[(N-2)s-N]$.

(b) *If* $2t/(t-1) < N \leqq 2s/(s-1)$, *then* $B_\alpha^N \in \tilde{O}_{QL^s} \cap O_{QL^t}$ *if and only if* $1/(N-2) < \alpha \leqq (t+1)/[(N-2)t-N]$.

(c) *If* $N \leqq 2t/(t-1)$, *then no* $B_\alpha^N \in \tilde{O}_{QL^s} \cap O_{QL^t}$.

Proof. By Lemma 2, necessary and sufficient conditions for $B_\alpha^N \in \tilde{O}_{QL^s} \cap O_{QL^t}$ are either (i) $\alpha > 1/(N-2)$, $\alpha[(N-2)s-N] < s+1$, and $\alpha[(N-2)t-N] \geqq t+1$; or (ii) $-1 < \alpha < -1/N$, $2s(\alpha+1)+N\alpha+1 > 0$, and $2t(\alpha+1)+N\alpha+1 \leqq 0$. The case (ii), however, never occurs. In fact, the last two inequalities in (ii) imply $(1+2s)/(N+2s) > \alpha \geqq (1+2t)/(N+2t)$, which is impossible for $s < t$. As to the case (i), we argue as follows:

(a) If $(N-2)s-N > 0$ or $N > 2s/(s-1)$, we may choose an α such that $(t+1)/[(N-2)t-N] \leqq \alpha < (s+1)/[(N-2)s-N]$. In view of $(t+1)/[(N-2)t-N] > 1/(N-2)$ for $(N-2)t-N > 0$, we have, by (i), $B_\alpha^N \in \tilde{O}_{QL^s} \cap O_{QL^t}$ if and only if α is as claimed.

(b) If $(N-2)s-N \leqq 0$ and $(N-2)t-N > 0$ or $2t/(t-1) < N \leqq 2s/(s-1)$, then $(t+1)/[(N-2)t-N] > 1/(N-2)$ and $\alpha[(N-2)s-N] < s+1$. By (i), we have $B_\alpha^N \in \tilde{O}_{QL^s} \cap O_{QL^t}$ if and only if $1/(N-2) < \alpha \leqq (t+1)/[(N-2)t-N]$.

(c) If $(N-2)t-N \leqq 0$ or $N \leqq 2t/(t-1)$, then no α satisfies $\alpha > 1/(N-2)$ and $\alpha[(N-2)t-N] \geqq t+1$. Again by (i), there exists no $B_\alpha^N \in \tilde{O}_{QL^s} \cap O_{QL^t}$.

LEMMA 6. *For each pair of real numbers* $s, t \geqq 1$, *with* $s < t$, *there exists a* $B_\alpha^N \in O_{QL^s} \cap \tilde{O}_{QL^t}$.

Proof. The equation

$$2s(\alpha+1)+N\alpha+1=0$$

gives $\alpha = -(1+2s)/(N+2s)$. Clearly, $-1 < \alpha$. The relation

$$\frac{1+2s}{N+2s} - \frac{1}{N} = \frac{N+2sN-N-2s}{N(N+2s)} = \frac{2s(N-1)}{N(N+2s)} > 0$$

yields $-1 < \alpha < -1/N$. In view of $s < t$, we obviously have $2t(\alpha+1)+N\alpha+1 > 0$. Thus, by Lemma 2, $B_\alpha^N \in O_{QL^s} \cap \tilde{O}_{QL^t}$.

We have reached the following result:

THEOREM 3. *The family* $\mathscr{B}$ *of Poincaré N-balls decomposes into the following disjoint nonempty sets: If s and t do not satisfy conditions* (a) *or* (b) *of Lemma* 5, *then*

$$\mathscr{B} = O_{QL^s} \cap O_{QL^t} \oplus O_{QL^s} \cap \tilde{O}_{QL^t} \oplus \tilde{O}_{QL^s} \cap \tilde{O}_{QL^t}.$$

For other s, t,

$$\mathscr{B}=O_{QL^s}\cap O_{QL^t}\oplus O_{QL^s}\cap\tilde{O}_{QL^t}\oplus\tilde{O}_{QL^s}\cap O_{QL^t}\oplus\tilde{O}_{QL^s}\cap\tilde{O}_{QL^t}.$$

The above results have implications to the general classification theory. These questions, as well as applications of our QL^p criterion to the biharmonic classification of $\mathscr{B}$, will be discussed in other studies.

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