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RIEMANNIAN MANIFOLDS ADMITTING A PROJECTIVE VECTOR FIELD

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§ 1. Introduction.

Let M be a connected Riemannian manifold of dimension n covered by a system of coordinate neighborhoods $\{U; x^h\}$, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$ and let g_{ji}, $\{_j{}^h{}_i\}$, ∇_j, $K_{kji}{}^h$, K_{ji} and K be respectively the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\{_j{}^h{}_i\}$, the curvature tensor, the Ricci tensor and the scalar curvature of M.

A vector field v^h on M is called a projective vector field if it satisfies

$$L_v\{_j{}^h{}_i\}=\nabla_j\nabla_i v^h+v^k K_{kji}{}^h=\delta_j^h\rho_i+\delta_i^h\rho_j \tag{1.1}$$

for a certain covariant vector field ρ_i, called the associated covariant vector field of v^h, where L_v denotes the operator of Lie derivation with respect to the vector field v^h. In particular, if ρ_i in (1.1) is zero vector field then the projective vector field v^h is called an affine vector field. When we refer in the sequel to a projective vector field v^h, we always mean by ρ_i the associated covariant vector field appearing in (1.1).

Recently, the present author [1, 2] proved a series of integral inequalities in a compact and orientable Riemannian manifold with constant scalar curvature admitting a projective vector field and then obtained necessary and sufficient conditions for such a Riemannian manifold to be isometric to a sphere.

The purpose of the present paper is to continue the work of the present author [1, 2] and to prove the following theorem.

THEOREM A. *If a connected, compact, orientable and simply connected Riemannian manifold M with constant scalar curvature K of dimension $n>1$ admits a non-affine projective vector field v^h, then M is globally isometric to a sphere of radius $\sqrt{n(n-1)/K}$ in the Euclidean $(n+1)$-space.*

In the sequel, we need the following theorem due to Obata [3]. (See also [4].)

THEOREM B. *Let M be a complete, connected and simply connected Rieman-*

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nian manifold. In order for M to admit a non-trivial solution φ of a system of partial differential equations

$$\nabla_j\nabla_i\varphi_h+k(2\varphi_j g_{ih}+\varphi_i g_{jh}+\varphi_h g_{ji})=0\,, \tag{1.2}$$

where $\varphi_h=\nabla_h\varphi$, k being a positive constant, it is necessary and sufficient that M is globally isometric to a sphere of radius $1/\sqrt{k}$ in the Euclidean $(n+1)$-space.

We assume in this paper that Riemannian manifolds under consideration are connected.

§ 2. Preliminaries.

We consider a projective vector field v^h on a Riemannian manifold M of dimension n. From (1.1), we have easily

$$\nabla_j\nabla_t v^t=(n+1)\rho_j \tag{2.1}$$

and

$$\nabla^j\nabla_j v^i+K_j{}^i v^j=2\rho^i\,, \tag{2.2}$$

where $\nabla^j=g^{ji}\nabla_i$ and $\rho^i=g^{ih}\rho_h$. (2.1) shows that the associated covariant vector field ρ_j is gradient. Putting

$$\rho=\frac{1}{n+1}\nabla_t v^t\,, \tag{2.3}$$

we have

$$\rho_j=\nabla_j\rho\,. \tag{2.4}$$

We have, from (1.1),

$$\nabla_j(\nabla_i v_h+\nabla_h v_i)=2\rho_j g_{ih}+\rho_i g_{jh}+\rho_h g_{ji}\,, \tag{2.5}$$

from which

$$\nabla_j L_v g^{ih}=-2\rho_j g^{ih}-\rho^i\delta_j^h-\rho^h\delta_j^i \tag{2.6}$$

and

$$\nabla^j L_v g^{ih}=-2\rho^j g^{ih}-\rho^i g^{jh}-\rho^h g^{ji}\,, \tag{2.7}$$

where $v_h=v^i g_{ih}$.

Substituting (1.1) into the formula [5]

$$L_v K_{kji}{}^h=\nabla_k L_v\{{}_j{}^h{}_i\}-\nabla_j L_v\{{}_k{}^h{}_i\}\,,$$

we find

$$L_v K_{kji}{}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i \,, \tag{2.8}$$

from which

$$L_v K_{ji} = -(n-1) \nabla_j \rho_i \,. \tag{2.9}$$

We define a tensor field G_{ji} [1, 2] on M by

$$G_{ji} = K_{ji} - \frac{K}{n} g_{ji} \,, \tag{2.10}$$

which satisfies

$$G_{ji} = G_{ij} \,, \qquad G_{ji} g^{ji} = 0 \,. \tag{2.11}$$

If $G_{ji}=0$ for $n>2$ then M is an Einstein manifold and K is a constant.
The projective curvature tensor field $P_{kji}{}^h$ on M is defined by

$$P_{kji}{}^h = K_{kji}{}^h - \frac{1}{n-1} (\delta_k^h K_{ji} - \delta_j^h K_{ki}) \,. \tag{2.12}$$

The tensor field $P_{kji}{}^h$ satisfies

$$P_{kji}{}^h = -P_{jki}{}^h \,, \tag{2.13}$$

$$P_{kji}{}^h + P_{ikj}{}^h + P_{jik}{}^h = 0 \,, \tag{2.14}$$

$$P_{tji}{}^t = 0 \,, \qquad P_{kjt}{}^t = 0 \tag{2.15}$$

and

$$P_{kjih} g^{ji} = \frac{n}{n-1} G_{kh} \,, \tag{2.16}$$

where $P_{kjih} = P_{kji}{}^t g_{th}$.

It is known that if $n=2$ then $P_{kji}{}^h=0$ and if $P_{kji}{}^h=0$ for $n>2$ then M is projectively flat.

If the scalar curvature K is a constant, then, using

$$\nabla^j K_{ji} = \frac{1}{2} \nabla_i K = 0$$

and

$$\nabla_t K_{kji}{}^t = \nabla_k K_{ji} - \nabla_j K_{ki} \,,$$

which can be obtained from the Bianchi identity

$$\nabla_l K_{kji}{}^h + \nabla_j K_{lki}{}^h + \nabla_k K_{jli}{}^h = 0 \,,$$

we have

$$\nabla^k P_{kji}{}^h = \frac{n-2}{n-1} \nabla^h G_{ji} - \nabla_i G_j{}^h \,, \tag{2.17}$$

where $G_j{}^h = G_{ji} g^{ih}$.

For a projective vector field v^h on a Riemannian manifold M, we have, for the tensor field G_{ji},

$$L_v G_{ji} = -\nabla_j w_i - \nabla_i w_j \tag{2.18}$$

if the scalar curvature K is a constant, where we have put

$$w^h = \frac{n-1}{2} \rho^h + \frac{K}{n} v^h \tag{2.19}$$

and $w_i = g_{ih} w^h$, and, for the projective curvature tensor field $P_{kji}{}^h$,

$$L_v P_{kji}{}^h = 0 . \tag{2.20}$$

§ 3. Proof of Theorem A.

In this section, we prove Theorem A. For this purpose, we need a series of lemmas.

LEMMA 1. *In a compact and orientable Riemannian manifold M, we have*

$$\int_M (\nabla_i f)(\nabla^i h) dV = -\int_M f \Delta h dV = -\int_M h \Delta f dV \tag{3.1}$$

for any functions f and h on M, where $\Delta = g^{ji} \nabla_j \nabla_i$ and dV denotes the volume element of M.

Proof. This follows from

$$0 = \int_M \nabla_i (f \nabla^i h) dV = \int_M (\nabla_i f)(\nabla^i h) dV + \int_M f \Delta h dV$$

and

$$0 = \int_M \nabla_i (h \nabla^i f) dV = \int_M (\nabla_i h)(\nabla^i f) dV + \int_M h \nabla f dV .$$

LEMMA 2. *If, in a compact and orientable Riemannian manifold M, a non-constant function φ satisfies a system of partial differential equations*

$$\nabla_j \nabla_i \varphi_h + k(2\varphi_j g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji}) = 0 , \tag{3.2}$$

where $\varphi_h = \nabla_h \varphi$, k being a constant, then the constant k is necessarily positive.

Proof. Transvecting (3.2) with g^{ih}, we have

$$\nabla_j \Delta \varphi + 2(n+1) k \varphi_j = 0 ,$$

from which and Lemma 1,

$$k\int_M \varphi_j\varphi^j dV = -\frac{1}{2(n+1)}\int_M (\nabla_j\Delta\varphi)\varphi^j dV$$

$$= \frac{1}{2(n+1)}\int_M (\Delta\varphi)^2 dV,$$

where $\varphi^j = g^{ji}\varphi_i$. Since φ is a non-constant function, two integral inequalities

$$\int_M \varphi_j\varphi^j dV > 0, \qquad \int_M (\Delta\varphi)^2 dV > 0$$

hold and consequently k must be positive.

LEMMA 3. *If a complete and simply connected Riemannian manifold M with positive constant scalar curvature K of dimension $n>1$ admits a non-affine projective vector field v^h and if the vector field w^h defined by* (2.19) *is a Killing vector field, then M is globally isometric to a sphere of radius $\sqrt{n(n-1)/K}$ in the Euclidean $(n+1)$-space.*

Proof. We have, from (1.1),

$$\nabla_j(\nabla_i v_h + \nabla_h v_i) = 2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji}.$$

Since w^h is a Killing vector field, we have

$$\nabla_i w_h + \nabla_h w_i = 0$$

or, equivalently,

$$(n-1)\nabla_i\rho_h + \frac{K}{n}(\nabla_i v_h + \nabla_h v_i) = 0,$$

from which and the above relation, we find

$$\nabla_j\nabla_i\rho_h + \frac{K}{n(n-1)}(2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji}) = 0,$$

and consequently the lemma follows from Theorem B.

Remark. Using Lemma 2, we see that if, in Lemma 3, M is compact and orientable then we can remove the positiveness of the scalar curvature K.

LEMMA 4. *For a projective vector field v^h on a compact and orientable Riemannian manifold M with constant scalar curvature K of dimension $n>1$, we have*

(3.3)
$$\int_M G_{ji}\rho^j w^i dV = \frac{2}{n-1}\int_M (\Delta_t w^t)^2 dV$$

$$-\frac{1}{2(n-1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV.$$

Proof. By using an identity

$$\nabla^j\nabla_j\rho_i-K_{ji}\rho^j-\nabla_i\Delta\rho=0,$$

we have

$$\int_M G_{ji}\rho^j w^i dV=-\int_M(\nabla_i\Delta\rho)w^i dV+\int_M(\nabla^j\nabla_j\rho_i)w^i dV-\frac{K}{n}\int_M\rho_i w^i dV.$$

Here we notice that we have, using (2.1) and (2.19),

$$-\int_M(\nabla_i\Delta\rho)w^i dV=\int_M(\Delta\rho)\nabla_i w^i dV$$

$$=\int_M\left[\nabla_t\left\{\frac{2}{n-1}w^t-\frac{2K}{n(n-1)}v^t\right\}\right]\nabla_i w^i dV$$

$$=\frac{2}{n-1}\int_M(\nabla_t w^t)^2 dV-\frac{2K}{n(n-1)}\int_M(\nabla_t v^t)\nabla_i w^i dV$$

$$=\frac{2}{n-1}\int_M(\nabla_t w^t)^2 dV+\frac{2K}{n(n-1)}\int_M(\nabla_i\nabla_t v^t)w^i dV$$

$$=\frac{2}{n-1}\int_M(\nabla_t w^t)^2 dV+\frac{2(n+1)K}{n(n-1)}\int_M\rho_i w^i dV.$$

Consequently, we have

$$\int_M G_{ji}\rho^j w^i dV$$

$$=\frac{2}{n-1}\int_M(\nabla_t w^t)^2 dV+\int_M(\nabla^j\nabla_j\rho_i)w^i dV+\frac{(n+3)K}{n(n-1)}\int_M\rho_i w^i dV,$$

from which, using

$$\nabla^j(\nabla_j v_i+\nabla_i v_j)=\nabla^j L_v g_{ji}=(n+3)\rho_i,$$

$$\int_M G_{ji}\rho^j w^i dV=\frac{2}{n-1}\int_M(\nabla_t w^t)^2 dV$$

$$+\frac{1}{n-1}\int_M\left[\nabla^j\left\{\nabla_j\left(\frac{n-1}{2}\rho_i+\frac{K}{n}v_i\right)+\nabla_i\left(\frac{n-1}{2}\rho_j+\frac{K}{n}v_j\right)\right\}\right]w^i dV$$

$$=\frac{2}{n-1}\int_M(\nabla_t w^t)^2 dV+\frac{1}{n-1}\int_M\{\nabla^j(\nabla_j w_i+\nabla_i w_j)\}w^i dV$$

$$=\frac{2}{n-1}\int_M(\nabla_t w^t)^2 dV-\frac{1}{n-1}\int_M(\nabla_j w_i+\nabla_i w_j)\nabla^j w^i dV$$

$$=\frac{2}{n-1}\int_M(\nabla_t w^t)^2 dV-\frac{1}{2(n-1)}\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dV.$$

LEMMA 5. *For a projective vector field v^h on a compact and orientable Riemannian manifold M with constant scalar curvature K, we have*

$$\int_M (\nabla^j L_v G_{ji}) w^i dV = \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV. \tag{3.4}$$

Proof. Integrating

$$\begin{aligned}&\nabla^j\{(\nabla_j w_i + \nabla_i w_j) w^i\}\\ &\quad = \{\nabla^j(\nabla_j w_i + \nabla_i w_j)\} w^i + (\nabla_j w_i + \nabla_i w_j)\nabla^j w^i\\ &\quad = \{\nabla^j(\nabla_j w_i + \nabla_i w_j)\} w^i + \frac{1}{2}(\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\end{aligned}$$

over M and using (2.18), we have (3.4).

LEMMA 6. *For a projective vector field v^h on a compact and orientable Riemannian manifold M with constant scalar curvature K of dimension $n>1$, we have*

$$\begin{aligned}\int_M (\nabla^j L_v G_j{}^i) w_i dV = &-\frac{6}{n-1}\int_M (\nabla_t w^t)^2 dV\\ &+\frac{n+2}{2(n-1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV.\end{aligned} \tag{3.5}$$

Proof. Using

$$\nabla^j G_{ji} = \frac{n-2}{2n}\nabla_i K = 0,$$

we have

$$\begin{aligned}(\nabla^j L_v G_j{}^i) w_i &= \{\nabla^j L_v (G_{jt} g^{ti})\} w_i\\ &= (\nabla^j L_v G_{ji}) w^i + G_{jt}(\nabla^j L_v g^{ti}) w_i.\end{aligned}$$

Substituting (2.7) into this, we find

$$(\nabla^j L_v G_j{}^i) w_i = (\nabla^j L_v G_{ji}) w^i - 3 G_{ji}\rho^j w^i.$$

Integrating this over M and using Lemmas 4 and 5, we have (3.5).

LEMMA 7. *For a projective vector field v^h on a campact and orientable Riemannian manifold M with constant scalar curvature K of dimension $n>1$, we have*

$$\begin{aligned}\int_M g^{kj}(L_v \nabla_k G_{ji}) w^i dV = &-\frac{6}{n-1}\int_M (\nabla_t w^t)^2 dV\\ &+\frac{n+2}{2(n-1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV.\end{aligned} \tag{3.6}$$

Proof. We have

$$g^{kj}(L_v\nabla_k G_{ji})w^i=(\nabla^j L_v G_{ji})w^i-g^{kj}(L_v\{_k{}^t{}_j\})G_{ti}w^i-g^{kj}(L_v\{_k{}^t{}_i\})G_{jt}w^i$$
$$=(\nabla^j L_v G_{ji})w^i-3G_{ji}\rho^j w^i .$$

Integrating this over M and using Lemmas 4 and 5, we have (3.6).

Now we prove Theorem A. By using

$$(L_v P_{kji}{}^h)g^{ji}=\frac{n}{n-1}L_v G_k{}^h-P_{kji}{}^h L_v g^{ji},$$

we have

$$(\nabla^k L_v P_{kji}{}^h)g^{ji}w_h=\frac{n}{n-1}(\nabla^k L_v G_k{}^h)w_h$$
$$-(\nabla^k P_{kji}{}^h)(L_v g^{ji})w_h-P_{kji}{}^h(\nabla^k L_v g^{ji})w_h .$$

Substituting (2.7) and (2.17) into this, we find

$$(\nabla^k L_v P_{kji}{}^h)g^{ji}w_h=\frac{n}{n-1}(\nabla^k L_v G_k{}^h)w_h$$
$$-\frac{n-2}{n-1}(\nabla^h G_{ji})(L_v g^{ji})w_h+(\nabla_i G_j{}^h)(L_v g^{ji})w_h+\frac{n}{n-1}G_{ji}\rho^j w^i,$$

from which, integrating over M,

$$\int_M(\nabla^k L_v P_{kji}{}^h)g^{ji}w_h dV$$
$$=\frac{n}{n-1}\int_M(\nabla^j L_v G_j{}^i)w_i dV-\frac{n-2}{n-1}\int_M(\nabla_t G_{ji})(L_v g^{ji})w^t dV$$
$$+\int_M(\nabla_k G_{ji})(L_v g^{kj})w^i dV+\frac{n}{n-1}\int_M G_{ji}\rho^j w^i dV .$$

Here we notice that we have, using (2.6) and (2.18),

$$-\frac{n-2}{n-1}\int_M(\nabla_t G_{ji})(L_v g^{ji})w^t dV$$
$$=\frac{n-2}{n-1}\int_M G_{ji}(\nabla_t L_v g^{ji})w^t dV+\frac{n-2}{n-1}\int_M G_{ji}(L_v g^{ji})\nabla_t w^t dV$$
$$=-\frac{2(n-2)}{n-1}\int_M G_{ji}\rho^j w^i dV-\frac{n-2}{n-1}\int_M(L_v G_{ji})g^{ji}\nabla_t w^t dV$$
$$=-\frac{2(n-2)}{n-1}\int_M G_{ji}\rho^j w^i dV+\frac{2(n-2)}{n-1}\int_M(\nabla_t w^t)^2 dV$$

and, using

$$(\nabla_k G_{ji})g^{kj} = \frac{n-2}{2n}\nabla_i K = 0,$$

$$\int_M (\nabla_k G_{ji})(L_v g^{kj}) w^i dV = -\int_M g^{kj}(L_v \nabla_k G_{ji}) w^i dV.$$

Consequently, we have

$$\int_M (\nabla^k L_v P_{kji}{}^h) g^{ji} w_h dV$$

$$= \frac{2(n-2)}{n-1}\int_M (\nabla_t w^t)^2 dV - \frac{n-4}{n-1}\int_M G_{ji}\rho^j w^i dV$$

$$+ \frac{n}{n-1}\int_M (\nabla^j L_v G_j{}^i) w_i dV - \int_M g^{kj}(L_v \nabla_k G_{ji}) w^i dV,$$

from which, using Lemmas 4, 6 and 7, we have

$$\int_M (\nabla^k L_v P_{kji}{}^h) g^{ji} w_h dV = \frac{2(n-3)}{n-1}\int_M (\nabla_t w^t)^2 dV$$

$$+ \frac{1}{n-1}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV.$$

From this and (2.20), we have

$$\frac{2(n-3)}{n-1}\int_M (\nabla_t w^t)^2 dV + \frac{1}{n-1}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV = 0.$$

If $n > 2$ then we have, from the above relation,

$$\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV = 0,$$

from which

$$\nabla_j w_i + \nabla_i w_j = 0,$$

that is, the vector field w^h defined by (2.19) is a Killing vector field and Theorem A follows from Lemma 3. If $n=2$ then we have $P_{kji}{}^h=0$ and hence, from (2.16), $G_{ji}=0$. Consequently, using Lemma 5, we have

$$\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV = 0,$$

from which

$$\nabla_j w_i + \nabla_i w_j = 0$$

and Theorem A follows from Lemma 3.

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