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COMPLEX ALMOST CONTACT MANIFOLDS

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§ 1. Introduction.

A complex contact manifold is a complex manifold of odd dimensions $2m+1$ ($\geqq 3$) covered by an open covering $\mathcal{A}=\{O, O', \cdots\}$ consisting of coordinate neighborhoods in such a way that

1) In each $O\in\mathcal{A}$ there is a holomorphic 1-form w satisfying $w\wedge(dw)^m\neq 0$ at every point of O;

2) If $O\cap O'\neq\phi$ $(O, O'\in\mathcal{A})$, there is a non-vanshing holomorphic function λ in $O\cap O'$ such that $w'=\lambda w$ in $O\cap O'$, where w' is the holomorphic 1-form given in O' (See Kobayashi [3]).

In a previous paper [2] we have studied complex contact structure $\{(O, w)|O\in\mathcal{A}\}$ which are induced by fiberings of manifolds with (real) normal contact 3-structure and obtained the induced (local) tensor field G of type (1, 1) in each $O\in\mathcal{A}$ such that $G^2=-I+w\otimes W$, $w\circ G=0$, where W is the associated vector field of w. The local structures $\{(O, G, w, W)|O\in\mathcal{A}\}$ are very useful to study curvature properties in the same way as in the real case (See Gray [1]. and Sasaki [4]). In the present paper we first define a system of local structures $\{(O, u, G)|O\in\mathcal{A}\}$ which will be called a complex almost contact structure and next show that such a structure induces a complex contact structure defined by Kobayashi [3], when it satisfies a suitable condition, i.e., to be normal.

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§ 2. Complex almost contact structure.

Let M be a complex manifold with complex structure F and Hermitian metric g and be covered by an open covering $\mathcal{A}=\{O, O', \cdots\}$ consisting of coordinate neighborhoods. Then M is called a *Complex almost contact manifold* if the following conditions **1**) and **2**) are satisfied:

1) In each $O\in\mathcal{A}$ there are given a 1-form u and a tensor field G of type (1, 1) such that*)

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*) Functions vector fields, tensor fields and geometric objects we consider are assumed to be differentiable and of class C^∞, otherwise stated. Throughout this paper, X, Y and Z denote arbitrary vector fields in M.

$$G^2=-I+u\otimes U+v\otimes V,$$
$$GF=-FG, \quad g(GX, Y)=-g(GY, X), \tag{2.1}$$
$$GU=0, \quad g(U, U)=1,$$

I being the identity tensor of type (1, 1) in M, where U and V are respectively the associated vector fields of u and a 1-form v defined in O by

$$v=u\circ F, \tag{2.2}$$

i. e., $g(U, X)=u(X)$ and $g(V, X)=v(X)$;

2) If $O\cap O'\neq\phi$ $(O, O'\in\mathcal{A})$, there are functions a and b in $O\cap O'$ such that

$$u'=au-bv, \quad G'=aG-bH,$$
$$a^2+b^2=1 \tag{2.3}$$
$$v'=bu+av, \quad H'=bG+aH,$$

in $0\cap 0'$, where H is defined in O by

$$H=GF \tag{2.4}$$

and (u', G') are the local structure given in O', v' and H' being defined in O' by (2.2) and (2.4) respectively.

The set $\{(O, u, G)|O\in\mathcal{A}\}$ is called a *complex almost contact structure.* In such a case, M is necesarily of odd complex dimensions $2m+1$ $(\geqq 3)$. For a complex almost structure, we have

$$H^2=-I+u\otimes U+v\otimes V.$$
$$HG=-GH=F+u\otimes V-v\otimes U,$$
$$FH=-HF=G, \quad g(HX, Y)=-g(HY, X),$$
$$GV=HU=HV=0, \quad u\circ G=v\circ G=u\circ H=v\circ H=0, \tag{2.5}$$
$$FU=-V, \quad FV=U, \quad u=-v\circ F,$$
$$g(V, V)=1, \quad g(U, V)=0$$

as consequences of (2.1), (2.2), (2.3), (2.4) and

$$F^2=-I, \quad g(FX, Y)=-g(FY, X).$$

We now have

THEOREM 1. *For a complex almost contact manifold of complex dimensions $2m+1$ $(\geqq 3)$, the structure group of the tangent bundle of M is reducible to $(S_p(m)\cdot S_p(1))\times U(1)$, where $S_p(m)\cdot S_p(1)=S_p(m)\times S_p(1)/\{\pm 1\}$.*

We now put

$$P=u\otimes U+v\otimes V$$

locally in each $O\in\mathcal{A}$. Then, as a consequence of (2.3), P determines a global tensor field, which is also denoted by P, in M such that $P^2=P$. Thus P is a projection tensor of rank 2. The distribution D determined by P is invariant under the action of the complex structure F and called the *vertical distribution* for brevity. We denote by B the vector bundle over M consisting of all vectors belonging to the vertical distribution D.

Let ∇ be the Riemannian connection of (M, g). If we put

$$P\nabla_X U=2\sigma(X)V, \qquad P\nabla_X V=-2\sigma(X)U \tag{2.6}$$

in each $O\in\mathcal{A}$, then we get a local 1-form σ in O. If $O\cap O'\neq\phi$ $(O, O'\in\mathcal{A})$, we have

$$2\sigma'=2\sigma+b^{-1}da=2\sigma-a^{-1}db \tag{2.7}$$

in $O\cap O'$, a and b being the functions appearing in (2.3), where σ' is the local 1-form defined by (2.4) in O'. Then $\{(2\sigma, O)\,|\,O\in\mathcal{A}\}$ defines a linear connection θ in the vertical vector bundle B. The local vector fields defined respectively by

$$D_X U=\nabla_X U-2\sigma(X)V, \qquad D_X V=\nabla_X V+2\sigma(X)U \tag{2.8}$$

are orthogonal to the vertical distribution D. If $O\cap O'\neq\phi$ $(O, O'\in\mathcal{A})$, we get by using (2.3) and (2.6)

$$\begin{cases} D_X U'=aD_X U-bD_X V, \\ D_X V'=bD_X U+aD_X V, \end{cases} \tag{2.9}$$

where $D_X U'$ and $D_X V'$ are defined by (2.8) in O'.

§ 3. Normal complex almost contact structure.

Let M be a complex almost contact manifold with structure $\{(O, u, G)\,|\,O\in\mathcal{A}\}$. Denote by u_i, v_i, u^h, v^h, $G_i{}^h$, $H_i{}^h$ and g_{ji} components of u, v, U, V, G, H, g in O, respectively. Denoting by ∇_i the operator of covariant differentiation in O with respect to the Riemannian connection ∇ of (M, g), we put

$$D_j u_i=\nabla_j u_i-2\sigma_j v_i, \qquad D_j v_i=\nabla_j v_i+2\sigma_j u_i, \tag{3.1}$$

where $\sigma=\sigma_j dx^j$ in O. *) Then we obtain

$$u^k D_j u_k=0, \qquad u^k D_j v_k=0,$$
$$v^k D_j u_k=0, \qquad v^k D_j v_k=0,$$

where $u^h=u_t g^{th}$, $v^h=v_t g^{th}$, g^{ih} being defined by $(g^{ih})=(g_{ih})^{-1}$, since $D_X U$ and $D_X V$ are orthogonal to the vertical distribution D.

The complex almost contact structure is said to be *contact* when

*) The indices $h, i, j, k, \cdots, t, s, \cdots$ run over the range $\{1, \cdots, 4m+2\}$ and the summation convension is used with respect to this system of indices.

$$du-\sigma\wedge v=\hat{G}\,, \qquad dv+\sigma\wedge u=\hat{H}\,,$$

where local 2-forms $\hat{G}$ and $\hat{H}$ are defined in O by

$$\hat{G}(X, Y)=g(GX, Y)\,, \qquad \hat{H}(X, Y)=g(HX, Y)\,,$$

respectively. When the structure is contact,

$$D_j u_i - D_i u_j = 2G_{ji}\,, \qquad D_j v_i - D_i v_j = 2H_{ji} \tag{3.4}$$

hold, where $G_{ji}=G_j{}^t g_{ti}$, and $H_{ji}=H_j{}^t g_{ti}$ are components of $\hat{G}$ and $\hat{H}$ in O respectively.

We now put in O

$$D_j G_i{}^h = \nabla_j G_i{}^h - 2\sigma_j H_i{}^h\,, \qquad D_j H_i{}^h = \nabla_j H_i{}^h + 2\sigma_j G_i{}^h\,, \tag{3.5}$$

which are respectively components of local tensor fields of type (1, 2) in O. If $O\cap O'\neq\phi$ $(O, O'\in\mathcal{A})$, using (2.3) and (2.7), we have in $O\cap O'$

$$D_j G'^h_i = aD_j G_i{}^h - bD_j H_i{}^h\,, \qquad D_j H'^h_i = bD_j G_i{}^h + aD_j H_i{}^h\,, \tag{3.6}$$

where $D_j G'^h_i$ and $D_j H'^h_i$ are defined by (3.5) in O'.

Next, we define in O local tensor fields S, T and W of type (1.2) respectively by their components as followings:

$$\begin{aligned} S_{kj}{}^h &= G_k{}^t D_t G_j{}^h - G_j{}^t D_t G_k{}^h - G_t{}^h (D_k G_j{}^t - D_j G_k{}^t) \\ &\quad + 2(v_j H_k{}^h - v_k H_j{}^h) + 2(G_{kj} u^h - H_{kj} u^h)\,, \\ T_{kj}{}^h &= H_k{}^t D_t H_j{}^h - H_j{}^t D_t H_k{}^h - H_t{}^h (D_k H_j{}^t - D_j H_k{}^t) \\ &\quad + 2(u_j G_k{}^h - u_k G_j{}^h) + 2(H_{kj} v^h - G_{kj} u^h)\,, \\ W_{kj}{}^h &= \frac{1}{2}[G_k{}^t D_t H_j{}^h + H_k{}^t D_t G_j{}^h - G_j{}^t D_t H_k{}^h - H_j{}^t D_t G_k{}^h \\ &\quad - G_t{}^h (D_k H_j{}^t - D_j H_k{}^t) - H_t{}^h (D_k G_j{}^t - D_j G_k{}^t)] \\ &\quad - (u_j H_k{}^h + v_j G_k{}^h - u_k H_j{}^h - v_k G_j{}^h) + 2(G_{kj} v^h + H_{kj} u^h)\,. \end{aligned} \tag{3.7}$$

Then we have in $O\cap O'$

$$\begin{aligned} S' &= a^2 S + 2abW + b^2 T\,, \qquad T' = b^2 S - 2abW + a^2 T\,, \\ W' &= ab(S-T) + (a^2 - b^2)W\,, \end{aligned} \tag{3.8}$$

where S', T' and W' are defined by (3.7) in O'. The set $\{S, T, W\}$ of local tensor fields will be called the *torsion tensor* of the given complex almost contact structure. The equations (3.8) show that if $S=T=W=0$ in O, then $S'=T'=W'=0$ in $O\cap O'$. When a complex almost contact structure is contact and its torsion tensors S, T and W vanish, it is said to be *normal*.

PROPOSITION 1. *If a complex almost contact structure* $\{(O, u, v, G, H) \mid O \in \mathcal{A}\}$ *is normal, then*

$$G_{ji} = D_j u_i, \qquad H_{ji} = D_j v_i. \tag{3.9}$$

Proof. Since the structure is contact, differentiating exteriorly (3.3), we have

$$d\hat{G} - \sigma \wedge \hat{H} = -\Omega \wedge v, \qquad d\hat{H} + \sigma \wedge \hat{G} = \Omega \wedge u, \tag{3.10}$$

where $\Omega = d\sigma$ and 2Ω is the curvature tensor Θ of the linear connection θ induced in the vertical vector bundle B. The equations (3.10) are equivalent to

$$\begin{aligned} &D_k G_{ji} + D_j G_{ik} + D_i G_{kj} = -(\Omega_{kj} v_i + \Omega_{ji} v_k + \Omega_{ik} v_j), \\ &D_k H_{ji} + D_j H_{ik} + D_i H_{kj} = \Omega_{kj} u_i + \Omega_{ji} u_k + \Omega_{ik} u_j. \end{aligned} \tag{3.11}$$

where

$$D_k G_{ji} = \nabla_k G_{ji} - 2\sigma_k H_{ji}, \qquad D_k H_{ji} = \nabla_k H_{ji} + 2\sigma_k G_{ji},$$

$$\Omega_{ji} = \frac{1}{2}(\partial_j \sigma_i - \partial_i \sigma_j), \qquad \partial_j = \frac{\partial}{\partial x^j},$$

$(x^1, \cdots, x^{4m+2})$ being local coordinates in O.

Putting

$$\mathfrak{L}_U G_{ji} = u^k D_k G_{ji} + (D_j u^k) G_{ki} + (D_i u^k) G_{jk},$$

we have from (2.1), (2.2), (2.4), (2.5) and (3.11)

$$\mathfrak{L}_U G_{ji} = u^k (D_k G_{ji} + D_j G_{ik} + D_i G_{kj}) = -u^k (\Omega_{kj} v_i + \Omega_{ik} v_j). \tag{3.12}$$

On the other hand, since $S_{kj}{}^h = 0$, transvecting the first equation of (3.7) with $G_h{}^t u^j$, we get

$$\begin{aligned} 0 &= S_{kj}{}^h G_h{}^t u^j \\ &= -G_k{}^r (D_r u^j) G_j{}^h G_h{}^t + (G_r{}^h G_h{}^t)(D_k u^j) G_j{}^r + (G_r{}^h G_h{}^t) u^j D_j G_k{}^r \\ &= -(u^j D_j G_k{}^t - G_k{}^i D_i u^t + G_j{}^t D_k u^j) \\ &\qquad + G_k{}^r u^t u^j (D_j u_r - D_r u_j) + G_k{}^r v^t u^j (D_j u_r - D_r u_j). \end{aligned} \tag{3.13}$$

Next, we put

$$\mathfrak{L}_U G_j{}^t = u^k D_k G_j{}^t - G_j{}^k D_k u^t + G_k{}^t D_j u^k.$$

Then, using (2.1), (2.2), (2.4), (2.5), (3.4) and (3.13), we have

$$\mathfrak{L}_U G_j{}^t = 0,$$

from which

$$\begin{aligned}0&=(\mathfrak{L}_U G_j{}^t)g_{ti}\\&=(u^k D_k G_j{}^t - G_j{}^k D_k u^t + G_k{}^t D_j u^k)g_{ti}\\&=u^k D_k G_{ji} - G_j{}^k(2G_{ki}+D_i u_k)+G_{ki}D_j u^k\\&=(u^k D_k G_{ji}+G_{ki}D_j u^k+G_{jk}D_i u^k)-2G_j{}^k(G_{ki}+D_i u_k)\\&=\mathfrak{L}_U G_{ji}-2G_j{}^k(G_{ki}+D_i u_k).\end{aligned}$$

Substituting (3.12) into this, we obtain

$$2G_j{}^k(G_{ki}+D_i u_k)=-u^k(\Omega_{kj}v_i+\Omega_{ik}v_j), \tag{3.14}$$

from which, transvecting v^j,

$$u^k\Omega_{ki}=(u^k\Omega_{kj}v^j)v_i. \tag{3.15}$$

Thus, we have from (3.14) and (3.15)

$$G_j{}^k(G_{ki}+D_i u_k)=0$$

and hence, using (3.2),

$$G_{ki}+D_i u_k=0.$$

Consequently, we get

$$G_{ji}=D_j u_i=-D_i u_j. \tag{3.16}$$

Similarly, we obtain

$$v^k\Omega_{ki}=(v^k\Omega_{kj}u^j)u_i \tag{3.17}$$

and

$$H_{ji}=D_j v_i=-D_i v_j. \tag{3.18}$$

Q. E. D.

PROPOSITION 2. *A complex almost contact structure is normal if and only if it is contact and*

$$\begin{aligned}D_j G_i{}^h&=\delta_j^h u_i-g_{ji}u^h+F_j{}^h v_i-F_{ji}v^h,\\D_j H_i{}^h&=\delta_j^h v_i-g_{ji}v^h-F_j{}^h u_i+F_{ji}u^h,\end{aligned} \tag{3.19}$$

where $F_j{}^h$ are components of the complex structure F and $F_{ji}=F_j{}^t g_{ti}$.

Proof. First, the given structure is assumed to be normal. Since it is contact, we have from (3.11)

$$D_k G_{jt}+D_j G_{tk}+D_t G_{kj}=-(\Omega_{kj}v_t+\Omega_{jt}v_k+\Omega_{tk}v_j),$$

from which, transvecting v^t,

$$\begin{aligned}&v^t D_t G_{kj}-G_j{}^t D_k v_t+G_k{}^t D_j v_t\\&\qquad=-(\Omega_{kj}+(v^t\Omega_{jt})v_k+(v^t\Omega_{tk})v_j).\end{aligned} \tag{3.20}$$

On the other hand, since $S_{kj}{}^{h}=0$, transvecting the first equation of (3.7) with $G_{h}{}^{t}v^{j}$ and using (2.1), (2.2), (2.4), (2.5) and (3.2), we have

$$\begin{aligned}0&=S_{kj}{}^{h}G_{h}{}^{t}v^{j}\\&=\{G_{k}{}^{r}G_{h}{}^{t}D_{r}G_{j}{}^{h}-(G_{r}{}^{h}G_{h}{}^{t})(D_{k}G_{j}{}^{r}-D_{j}G_{k}{}^{r})\}v^{j}-2(F_{k}{}^{t}+u_{k}v^{t}-v_{k}u^{t})\\&=-G_{k}{}^{r}(G_{j}{}^{h}G_{h}{}^{t})D_{r}v^{j}-(\delta_{r}^{t}-u_{r}u^{t}-v_{r}v^{t})(G_{j}{}^{r}D_{k}v^{j}+v^{j}D_{j}G_{k}{}^{r})\\&\qquad\qquad-2(F_{k}{}^{t}+u_{k}v^{t}-v_{k}u^{t})\\&=-(v^{j}D_{j}G_{k}{}^{t}-G_{k}{}^{r}D_{r}v^{t}+G_{j}{}^{t}D_{k}v^{j})-2(F_{k}{}^{t}+u_{k}v^{t}-v_{k}u^{t})\end{aligned}$$

and hence

$$v^{j}D_{j}G_{k}{}^{t}-G_{k}{}^{r}D_{r}v^{t}+G_{j}{}^{t}D_{k}v^{j}=-2(F_{k}{}^{t}+u_{k}v^{t}-v_{k}u^{t})\,.$$

This implies together with (3.16), (3.17) and (3.20)

$$\Omega_{kj}=2F_{kj}+(2-\alpha)(u_{k}v_{j}-v_{k}u_{j})\,, \tag{3.21}$$

where $\alpha=\Omega_{kj}v^{k}u^{j}$.

We now put $S_{kji}=S_{kj}{}^{t}g_{ti}$. Then, using (3.9), (3.11) and (3.21), we have

$$\begin{aligned}0&=S_{kji}\\&=G_{k}{}^{r}D_{r}G_{ji}-G_{j}{}^{r}D_{r}G_{ki}-G_{ri}(D_{k}G_{j}{}^{r}-D_{j}G_{k}{}^{r})\\&\qquad\qquad+2(v_{j}H_{ki}-v_{k}H_{ji}+G_{kj}u_{i}-H_{kj}v_{i})\\&=G_{k}{}^{r}(D_{r}G_{ji}-D_{j}G_{ri})-G_{j}{}^{r}(D_{r}G_{ki}-D_{k}G_{ri})\\&\qquad\qquad+v_{j}H_{ki}-v_{k}H_{ji}+u_{k}G_{ji}-u_{j}G_{ki}-4H_{kj}v_{i}\\&=-G_{k}{}^{r}(D_{i}G_{rj}+2F_{rj}v_{i}+2F_{ir}v_{j})+G_{j}{}^{r}(D_{i}G_{rk}+2F_{rk}v_{i}+2F_{ir}v_{k})\\&\qquad\qquad+v_{j}H_{ki}-v_{k}H_{ji}+u_{k}G_{ji}-u_{j}G_{ki}-4H_{kj}v_{i}\\&=-D_{i}(G_{k}{}^{r}G_{rj})+2G_{j}{}^{r}D_{i}G_{rk}-v_{j}H_{ki}+v_{k}H_{ji}-u_{k}G_{ij}-u_{j}G_{ki}\\&=2(G_{j}{}^{r}D_{i}G_{rk}-H_{ij}v_{k}-G_{ij}u_{k})\,,\end{aligned}$$

from which

$$G_{j}{}^{r}D_{i}G_{rk}=u_{k}G_{ij}+v_{k}H_{ij}\,.$$

Thus, transvecting this with $G_{t}{}^{j}$, we get

$$D_{j}G_{tk}=u_{t}g_{jk}-u_{k}g_{it}+v_{t}F_{ik}-v_{k}F_{it}$$

and hence the first equation of (3.19). Similarly, we obtain the second equation of (3.19).

Conversely, the given complex almost contact structure is assumed to satisfy (3.19). Transvecting the first equation of (3.19) with u^{i}, we get

$$(D_jG_i{}^h)u^i=\delta_j^h-u_ju^h-v_jv^h\,.$$

Since $G_i{}^hu^i=0$, we have

$$G_i{}^h(D_ju^i)=-\delta_j^h+u_ju^h+v_jv^h\,,$$

from which

$$D_ju^h=G_j{}^h\,,\quad \text{i.e.,}\quad D_ju_i=G_{ji}\,.$$

This means

$$D_ju_i-D_iu_j=2G_{ji}\,. \tag{3.22}$$

Similarly, we obtain

$$D_jv_i-D_iv_j=2H_{ji}\,. \tag{3.23}$$

From (3.22) and (3.23) we see that the given structure is contact. Next, using (3.19), we can easily verify $S=T=W=0$. Consequently, in this case the given structure is normal. Q. E. D.

PROPOSITION 3. *If a complex almost contact structure is normal, the pair (F, g) is a Kahlerian structure, i.e., $\nabla F=0$.*

Proof. Using $H_i{}^h=G_t{}^hF_i{}^t$, we have

$$D_jH_i{}^h=(D_jG_t{}^h)F_i{}^t+G_t{}^h(\nabla_jF_i{}^t)\,,$$

from which, substituting (3.19),

$$G_t{}^h(\nabla_jF_i{}^t)=0\,.$$

Thus, we get

$$-\nabla_jF_i{}^h+(u_t\nabla_jF_i{}^t)u^h+(v_t\nabla_jF_i{}^t)v^h=0\,,$$

from which, using (2.1) and Proposition 1,

$$\nabla_jF_i{}^h=0\,.$$

Q. E. D.

PROPOSITION 4. *For a complex almost contact structure, which is normal,*

$$\Omega_{jk}=2F_{jk}\,,$$

i.e., the curvature form Θ of the linear connection θ induced in the vertical vector bundle B is given by

$$\Theta(X, Y)=2g(FX, Y)\,.$$

Proof. Since $\Omega=d\sigma$, we have $d\Omega=0$. Thus, using (3.21), we have

$$(2-\alpha)(du\wedge v-u\wedge dv)+d\alpha\wedge u\wedge v=0$$

because $\nabla F=0$. Substituting into this

$$du\wedge v=\hat{H}\wedge v\,,\qquad u\wedge dv=u\wedge\hat{G}$$

which are direct consequences of (3.9), we obtain

$$(2-\alpha)(\hat{H}\wedge v-u\wedge\hat{G})+d\alpha\wedge u\wedge v=0\,. \tag{3.22}$$

Taking account of $(\hat{H}\wedge v)(X, U, V)=(u\wedge\hat{G})(X, U, V)=0$, we have from (3.22)

$$d\alpha=(U\alpha)u+(V\alpha)v\,,$$

from which and (3.22)

$$(2-\alpha)(\hat{H}\wedge u-u\wedge\hat{G})=0$$

and hence $\alpha=2$. Thus, substituting $\alpha=2$ into (3.21), we get

$$\Omega_{jk}=2F_{jk}\,.$$

Q. E. D.

§4. **Curvature properties.** In this section, let (M, G, F) be a complex manifold of (real) dimension n with complex almost contact structure $\{(O, u, v, G, H)\}$ which is normal. Using (3.19) and Ricci formulas gives

$$\begin{aligned}-K_{kji}{}^{s}u_{s}&=u_{j}g_{ki}-u_{k}g_{ji}+v_{j}F_{ki}-v_{k}F_{ji}-2v_{i}F_{kj}+2\Omega_{kj}v_{i}\,,\\ -K_{kji}{}^{s}v_{s}&=v_{j}g_{ki}-v_{k}g_{ji}-u_{j}F_{ki}+u_{k}F_{ji}+2u_{i}F_{kj}-2\Omega_{kj}u_{i}\,,\end{aligned} \tag{4.1}$$

$K_{kji}{}^{h}$ being components of the curvature tensor K of (M, g, F), where $F_{ji}=F_{j}{}^{h}g_{ji}$ and

$$\Omega_{ji}=\frac{1}{2}(\partial_{j}\sigma_{i}-\partial_{i}\sigma_{j})\,. \tag{4.2}$$

Next, (3.19) and Ricci formulas imply

$$\begin{aligned}K_{kjt}{}^{h}G_{i}{}^{t}-K_{kji}{}^{t}G_{t}{}^{h}&=G_{ki}\delta_{j}^{h}-G_{ji}\delta_{k}^{h}+G_{k}{}^{h}g_{ji}+G_{j}{}^{h}g_{ki}\\ &\quad+H_{ki}F_{j}{}^{h}-H_{ji}F_{k}{}^{h}-H_{k}{}^{h}F_{ji}+H_{j}{}^{h}F_{ki}+2\Omega_{kj}H_{i}{}^{h}\,.\end{aligned} \tag{4.3}$$

Changing in (4.3) the index h to s and then transvecting $G_{sh}(=G_{s}{}^{t}g_{th})$, we have by means of (4.1)

$$\begin{aligned}K_{kjih}-K_{kjts}G_{i}{}^{t}G_{h}{}^{s}&=(G_{ki}G_{jh}-G_{ji}G_{kh})+(H_{ki}H_{jh}-H_{ji}H_{kh})\\ &\quad-(F_{ki}F_{jh}-F_{ji}F_{kh})-(g_{ki}g_{jh}-g_{ji}g_{kh})-2\Omega_{kj}F_{ih}\,,\end{aligned} \tag{4.4}$$

where $K_{kjih}=K_{kji}{}^{s}g_{sh}$ and $H_{ji}=H_{j}{}^{s}g_{si}$.

Since (M, g, F) is Kahlerian, we obtain

$$K_{kjih}=K_{kjts}F_{i}{}^{t}F_{h}{}^{s}\,. \tag{4.5}$$

Then, transvecting $G^{ih}(=g^{is}G_{s}{}^{h})$ with (4.5) gives

(4.6) $$K_{kjih}G^{ih}=0.$$

Similarly, we get

(4.7) $$K_{kjih}H^{ih}=0.$$

On the other hand, we have

(4.8) $$K_{ktsh}G^{ts}=\frac{1}{2}(K_{ktsh}-K_{ksth})G^{ts}=-\frac{1}{2}K_{khts}G^{ts},$$

where we used the identity $K_{kjih}+K_{jikh}+K_{ikjh}=0$. Thus (4.6) and (4.8) give

(4.9) $$K_{ktsh}G^{ts}=0.$$

If we transvect g^{ji} with (4.4), then we have by using (4.9)

(4.10) $$K_{kh}=(n-2)g_{kh}+2\Omega_{ks}F_h{}^s.$$

Next, transvecting $F^{ji}(=g^{js}F_s{}^i)$ with (4.4), then we have by using (4.7)

(4.11) $$K_{kjih}F^{ji}=(n-2)F_{kh}+2\Omega_{kh}.$$

Transvecting F^{ih} with (4.4) gives

(4.12) $$K_{kjih}F^{ih}=-(n+2)\Omega_{kj}.$$

On the other hand, (4.5) implies

$$K_{kj}=K_{ktsh}F^{ts}F_j{}^h=-\frac{1}{2}K_{khts}F^{ts}F_j{}^h,$$

which is obtained in the same way as (4.8) done. This equation and (4.12) imply

(4.13) $$K_{kj}=\frac{n+2}{2}\Omega_{ks}F_j{}^s.$$

If we substitute (4.13) in (4.10), we have

(4.14) $$\Omega_{kj}=2F_{kj},$$

which gives Proposition 4. Substituting (4.14) into (4.13), we have

(4.15) $$K_{kj}=(n+2)g_{kj}$$

and hence

(4.16) $$K=K_{kj}g^{kj}=n(n+2),$$

where K denotes the scalar curvature of (M, g, F). Thus we have form (4.15) and (4.16)

THEOREM 2. *If a complex manifold (M, F, g) with Hermitian metric g admits a complex almost contact structure, which is normal, then (M, F, g) is an Einstein Kahlerian space with scalar curvature $n(n+2)$, where* $\dim M=n$ $(\geqq 3)$. *If moreover M is complete, then M is compact.*

We now take complex coordinates $(Z^1, \cdots, Z^{2m+1})$ in O, such that F has components of the form

$$F_i^h=\begin{pmatrix} \sqrt{-1}\,\delta_\lambda^\kappa & 0 \\ 0 & -\sqrt{-1}\,\delta_{\bar{\lambda}}^{\bar{\kappa}} \end{pmatrix}.$$

Putting

$$\pi=u+\sqrt{-1}\,v\,,$$

we see that π is a complex 1-form of type (1, 0), i.e. $\pi=\pi_\kappa dZ^\kappa$. Since (4.14) holds, we can find a holomorphic 1-form $w=f\pi$ in O, where f is a function defined in O, such that $w\wedge(dw)^m\neq 0$ (for proof see [2]). Thus we have

THEOREM 3. *If a complex manifold M with Hermitian metric g admits a complex almost contact structure, which is normal, there is in M a complex contact structure.*

Under the same assumption as in Theorem 3, using Proposition 1, we see that $[U, V]$ belongs to D, i.e. that the vertical distribution D is integrable. A maximal integral submanifold of D will be called a fibre.

PROPOSITION 5. *Under the same assumption as in Theorem 3, the vertical distribution D is integrable and every fibre is a totally geodesic submanifold with complex dimension 1 and with constant curvature 4. If moreover M is complete, every fibre is a 2-dimensional sphere with curvature 4.*

Proof. Proposition 1 implies

$$\nabla_U U=2\sigma(U)V\,,\quad \nabla_V U=2\sigma(V)V\,,\quad \nabla_U V=-2\sigma(U)U\,,\quad \nabla_V V=-2\sigma(V)U\,,$$

which show that every fiber is totally geodesic. Since $FU=-V$, $FV=U$, every fiber is a complex submanifold of complex dimension 1. Next, using (4.1), we have $K_{kjih}V^kU^jV^iU^h=4$, which means that every fiber has constant curvature 4.

Q. E. D.

Remark. For a complex almost contact manifold (M, F, g) with structure $\{(O, u, G)\,|\,O\in\mathcal{A}\}$, the Hermitian manifold (M, F, g) is assumed to be Kahlerian. Then the local tensor field G has components satisfying*)

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*) The indices $\lambda, \mu, \nu, \tau \cdots$ run over the range $\{1, \cdots, 2m+1\}$ and the summation convension is used with respect to this system of indices.

$$G_{\mu}{}^{\lambda}=0, \qquad G_{\bar{\mu}}{}^{\bar{\lambda}}=0$$

with respect to complex coordinates $(Z^1, \cdots, Z^{2m+1})$ in each $O\in\mathcal{A}$. The local tensor fields S, T and W have respectively components satisfying

$$S_{\nu\mu}{}^{\lambda}=0,$$

$$S_{\nu\bar{\mu}}{}^{\lambda}=G_{\nu}{}^{\bar{\tau}}D_{\bar{\tau}}G_{\bar{\mu}}{}^{\lambda}+G_{\bar{\tau}}{}^{\lambda}D_{\bar{\mu}}G_{\nu}{}^{\bar{\tau}}-2u_{\nu}G_{\bar{\mu}}{}^{\lambda},$$

$$S_{\bar{\nu}\bar{\mu}}{}^{\lambda}=G_{\bar{\nu}}{}^{\tau}D_{\tau}G_{\bar{\mu}}{}^{\lambda}-G_{\bar{\mu}}{}^{\tau}D_{\tau}G_{\bar{\nu}}{}^{\lambda}+2(u_{\bar{\nu}}G_{\bar{\mu}}{}^{\lambda}-u_{\bar{\mu}}G_{\bar{\nu}}{}^{\tau})+4G_{\bar{\nu}\bar{\mu}}u^{\lambda};$$

$$T_{\nu\mu}{}^{\lambda}=0, \qquad T_{\nu\bar{\mu}}{}^{\lambda}=S_{\nu\bar{\mu}}{}^{\lambda}, \qquad T_{\bar{\nu}\bar{\mu}}{}^{\lambda}=-S_{\bar{\nu}\bar{\mu}}{}^{\lambda};$$

$$W_{\nu\mu}{}^{\lambda}=0, \qquad W_{\nu\bar{\mu}}{}^{\lambda}=0, \qquad W_{\bar{\nu}\bar{\mu}}{}^{\lambda}=-\sqrt{-1}\,S_{\nu\bar{\mu}}{}^{\lambda}.$$

The equation

$$G^2=-I+u\otimes U+v\otimes V$$

given in (2.1) is equivalent to

$$G_{\bar{\tau}}{}^{\lambda}G_{\mu}{}^{\bar{\tau}}=-\delta_{\mu}^{\lambda}+2u_{\mu}u^{\lambda}.$$

The equations (3.9) are equivalent to

$$G_{\mu\lambda}=D_{\mu}u_{\lambda}, \qquad 0=D_{\mu}u_{\bar{\lambda}}.$$

The $D_jG_i{}^h$ satisfies the identities

$$D_{\nu}G_{\mu}{}^{\lambda}=0, \qquad D_{\bar{\nu}}G_{\mu}{}^{\lambda}=0$$

and the equations (3.19) are equivalent to

$$D_{\nu}G_{\bar{\mu}}{}^{\lambda}=2(\delta_{\mu}^{\lambda}u_{\bar{\mu}}-g_{\nu\bar{\mu}}u^{\lambda}), \qquad D_{\bar{\nu}}G_{\bar{\mu}}{}^{\lambda}=0.$$

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