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KODAI MATH. J.
3 (1980), 144-161

ON ALMOST CONTACT STRUCTURES BELONGING TO A *CR*-STRUCTURE

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§0. Introduction.

A *CR*-structure on an odd dimensional differentiable manifold is a pair $(\mathcal{D}, J)$ of a 1-codimensional subbundle $\mathcal{D}$ of the tangent bundle and a complex structure J on $\mathcal{D}$ with certain integrability condition. In this paper we shall consider almost contact structures belonging to a *CR*-structure.

CR-structures are recently developed by Burns-Shneider [3], Burns-Diederich-Shneider [2], Chern-Moser [4], Tanaka [10], Webster [11] [12] and so on. In particular, Tanaka [10] has treated almost contact structures with certain conditions belonging to a *CR*-structure and found canonical connections associated with them. Ishihara [5] has also considered almost contact structures in the *CR*-category and studied pseudo-conformal mappings. Our standpoint is similar to [5]. Our main purpose is to give a change of canonical connections associated with almost contact structures belonging to a *CR*-structure. An almost contact structure (ϕ, ξ, θ) defines a hyperdistribution $\mathcal{D}$ and a complex structure J on $\mathcal{D}$. We shall also show that there is an affine connection with respect to which all structure tensors are parallel and whose torsion tensor is proportional to the Nijenhuis tensor formally defined by J when they are restricted to $\mathcal{D}$.

In §1, we shall recall definitions of a *CR*-structure and its integrability. Some facts about almost contact structures belonging to a *CR*-structure will also given. §2 will be devoted to the study of affine connections associated with almost contact structures. In §3, we shall obtain a change of canonical connections.

The authors wish to express their hearty thanks to Professor S. Ishihara for his constant encouragement and valuable suggestions.

§1. CR-structures and almost contact structures.

Let M be a connected C^∞-manifold of dimension $2n+1$ $(n \geqq 1)$. Let $\mathcal{D}$ denote a 1-codimensional subbundle of the tangent bundle TM, what is called a hyper-

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Received April 10, 1979

distribution. A cross-section J of the bundle $\mathcal{D}\otimes\mathcal{D}^*$ satisfying $J^2=-I$ is called a complex structure on $\mathcal{D}$ where $\mathcal{D}^*$ is the dual bundle of $\mathcal{D}$ and I is the identity transformation. Let $(\mathcal{D}, J)$ be a pair of a hyperdistribution $\mathcal{D}$ and a complex structure J on $\mathcal{D}$. Then the complexification $\boldsymbol{C}TM$ of the tangent bundle TM is decomposed as $\boldsymbol{C}TM=\boldsymbol{C}\mathcal{D}\oplus\mathcal{L}$ where $\boldsymbol{C}\mathcal{D}$ is the complexification of $\mathcal{D}$ and $\mathcal{L}$ is a line bundle isomorphic with $TM/\boldsymbol{C}\mathcal{D}$. It is clear that if M is orientable, then $\mathcal{L}$ is a trivial line bundle. The complex structure J on $\mathcal{D}$ can be uniquely extended to a complex linear endomorphism of $\boldsymbol{C}\mathcal{D}$ and the extended endomorphism, denoted also by J, satisfies the equation $J^2=-I$. Thus J has two eigenvalues i and $-i$. Let $\mathcal{D}^{1,0}$ (resp. $\mathcal{D}^{0,1}$) be a subbundle of $\boldsymbol{C}\mathcal{D}$ composed of the eigenspaces corresponding to i (resp. $-i$). Then we have

$$\boldsymbol{C}TM=\mathcal{D}^{1,0}\oplus\mathcal{D}^{0,1}\oplus\mathcal{L}\,.$$

A pair $(\mathcal{D}, J)$ is said to be *integrable* if $\mathcal{D}^{1,0}$ is involutive and integrable pair $(\mathcal{D}, J)$ is called a *CR-structure* on M. Since a cross-section $\tilde{X}$ of $\mathcal{D}^{1,0}$ can be uniquely written as

$$\tilde{X}=X-iJX$$

for some vector field X contained in $\mathcal{D}$, the pair $(\mathcal{D}, J)$ is integrable if and only if the following two conditions hold:

$$[X, Y]-[JX, JY]\in\Gamma(\mathcal{D})\,, \tag{1.1}$$

$$[JX, JY]-[X, Y]-J([X, JY]+[JX, Y])=0 \tag{1.2}$$

for any $X, Y\in\Gamma(\mathcal{D})$ where $\Gamma(\mathcal{D})$ denotes the set of all vector fields contained in $\mathcal{D}$. If M admits a CR-structure $(\mathcal{D}, J)$, then $(M, \mathcal{D}, J)$ is called a *CR-manifold*. Let θ be a local 1-form annihilating the hyperdistribution $\mathcal{D}$, which is determined up to non-vanishing smooth functions. Noting that

$$-2d\theta(X, Y)=\theta([X, Y]) \tag{1.3}$$

for every $X, Y\in\Gamma(\mathcal{D})$, we see that the condition (1.1) for the pair $(\mathcal{D}, J)$ is equivalent to

$$d\theta(JX, JY)=d\theta(X, Y) \tag{1.4}$$

for arbitrary $X, Y\in\Gamma(\mathcal{D})$. Moreover we have

$$d(\alpha\theta)(X, Y)=\alpha d\theta(X, Y) \tag{1.5}$$

for every $X, Y\in\Gamma(\mathcal{D})$ and smooth function α, which allows us to call $(\mathcal{D}, J)$ a *non-degenerate* pair if $d\theta$ is non-degenerate on $\mathcal{D}$.

Now let there be given a pair $(\mathcal{D}, J)$ of a hyperdistribution $\mathcal{D}$ on M and a complex structure J on $\mathcal{D}$. An almost contact structure (ϕ, ξ, θ) is a triplet of (1,1) tensor field ϕ, a vector field ξ and 1-form θ defined on M satisfying

$$\theta(\xi)=1, \quad \phi\xi=0, \quad \theta\circ\phi=0, \tag{1.6}$$
$$\phi^2=-I+\theta\otimes\xi, \quad \operatorname{rank}\phi=2n.$$

If the 1-form θ annihilates $\mathcal{D}$ and the restriction of ϕ to $\mathcal{D}$ coincides with J, then we say that the almost contact structure (ϕ, ξ, θ) *belongs* to the pair $(\mathcal{D}, J)$. Defining a 2-form ω by

$$\omega=-2d\theta, \tag{1.7}$$

we see from (1.3) and (1.4) that if the almost contact structure (ϕ, ξ, θ) belongs to the pair $(\mathcal{D}, J)$, then the equation

$$\omega(X, Y)=\theta([X, Y]) \tag{1.8}$$

hold and the condition (1.1) is equivalent with

$$\omega(JX, JY)=\omega(X, Y) \tag{1.9}$$

for every $X, Y\in\Gamma(\mathcal{D})$. Also it is easily verified that

$$\omega(\xi, X)=\theta([\xi, X]) \tag{1.10}$$

for any $X\in\Gamma(\mathcal{D})$. Define a tensor field $\hat{g}$ of type (0, 2) by

$$\hat{g}(X, Y)=\omega(\phi X, Y), \tag{1.11}$$

X and Y being vectors tangent to M and denote its restriction to $\mathcal{D}$ by g. Then the equation (1.9) implies that if $X, Y\in\Gamma(\mathcal{D})$, then

$$g(X, Y)=g(Y, X) \tag{1.12}$$

and

$$g(JX, JY)=g(X, Y), \tag{1.13}$$

that is, g is symmetric and Hermitian when $(\mathcal{D}, J)$ satisfies (1.1).

Next we give following lemma for later use.

LEMMA 1.1. *Two almost contact structures (ϕ, ξ, θ) and (ϕ', ξ', θ') belong to the same pair $(\mathcal{D}, J)$ of a hyperdistribution $\mathcal{D}$ and a complex structure J on $\mathcal{D}$ if and only if they satisfy*

$$\theta'=\varepsilon e^{\lambda}\theta, \quad \xi'=\varepsilon e^{-\lambda}(\phi A+\xi), \quad \phi'=\phi+\theta\otimes A \tag{1.14}$$

for some smooth function λ and vector field $A\in\Gamma(\mathcal{D})$ where $\varepsilon=\pm1$.

Proof. The if part is clear. Thus suppose that (ϕ, ξ, θ) and (ϕ', ξ', θ') belong to the same pair $(\mathcal{D}, J)$. Since the 1-form θ' does not vanish and is proportional to θ, we find

$$\theta'=\varepsilon e^{\lambda}\theta, \quad (\varepsilon=\pm1).$$

Note that $\varepsilon e^{\lambda}=\theta'(\xi)$. Putting $A=-\varepsilon e^{\lambda}\phi\xi'$, we have

$$\phi A=-\varepsilon e^{\lambda}\phi^2\xi'=-\varepsilon e^{\lambda}(-\xi'+\theta(\xi')\xi)$$
$$=\varepsilon e^{\lambda}\xi'-\theta'(\xi')\xi=\varepsilon e^{\lambda}\xi'-\xi$$

which implies $\xi'=\varepsilon e^{-\lambda}(\phi A+\xi)$. Furthermore we have

$$\phi'\xi=\phi'(\varepsilon e^{\lambda}\xi'-\phi A)=A\,.$$

If $X_{\mathscr{D}}$ denotes the $\mathscr{D}$-component of a tangent vector X with respect to ξ, i.e., $X_{\mathscr{D}}=X-\theta(X)\xi$, then we see that

$$(\phi'-\phi)X=(\phi'-\phi)(X_{\mathscr{D}}+\theta(X)\xi)=\theta(X)A\,,$$

which shows $\phi'=\phi+\theta\otimes A$. Q.E.D.

Remark. Given an almost contact structure (ϕ, ξ, θ) on M, it is easy to verify that ϕ', ξ' and θ' defined by (1.14) satisfy the equations (1.6).

PROPOSITION 1.2. *Let* $(\mathscr{D}, J)$ *be a non-degenerate pair and* (ϕ, ξ, θ) *be an almost contact structure belonging to* $(\mathscr{D}, J)$. *Then there exists an almost contact structure* (ϕ', ξ', θ') *belonging to the same non-degenerate pair* $(\mathscr{D}, J)$ *such that* $[\xi', \Gamma(\mathscr{D})]\subset\Gamma(\mathscr{D})$ *holds.*

Proof. Putting $\xi'=\xi+B$ for some $B\in\Gamma(\mathscr{D})$, we sees that $[\xi', \Gamma(\mathscr{D})]\subset\Gamma(\mathscr{D})$ holds if and only if $\omega(B, X)=\omega(X, \xi)$ for every $X\in\Gamma(\mathscr{D})$, because we have

$$\theta([\xi', X])=\theta([\xi+B, X])$$
$$=\omega(B, X)-\omega(X, \xi)\,.$$

Our assumption ω is non-degenerate when it is restricted to $\mathscr{D}$ shows that such B uniquely exists. Therefore if we put

$$A=-\phi B\,,\qquad \theta'=\theta\,,\qquad \xi'=\xi+\phi A\,,\qquad \phi'=\phi+\theta\otimes A\,,$$

then we obtain an almost contact structure (ϕ', ξ', θ') belonging to the same non-degenerate pair $(\mathscr{D}, J)$ by virtue of Lemma 1.1 and Remark. Of course it satisfies $[\xi', \Gamma(\mathscr{D})]\subset\Gamma(\mathscr{D})$. Q.E.D.

The condition

$$[\xi, \Gamma(\mathscr{D})]\subset\Gamma(\mathscr{D}) \tag{1.15}$$

for an almost contact structure (ϕ, ξ, θ) belonging to a pair $(\mathscr{D}, J)$ is equivalent to

$$\omega(X, \xi)=0 \tag{1.16}$$

or

$$\mathscr{L}_{\xi}\theta=0 \tag{1.17}$$

Let (ϕ, ξ, θ) be an almost contact structure belonging to a pair $(\mathcal{D}, J)$. We introduce a tensor field N of type (1,2) defined by

$$N(X, Y)=[\phi X, \phi Y]-[X, Y]-\phi[X, \phi Y]-\phi[\phi X, Y]. \tag{1.18}$$

We have

PROPOSITION 1.3. *A pair $(\mathcal{D}, J)$ is a CR-structure if and only if*

$$N(X, Y)=0$$

for every $X, Y\in\Gamma(\mathcal{D})$.

Proof. Note that when $[X, JY]+[JX, Y]$ is contained in $\Gamma(\mathcal{D})$, we have

$$N(X, Y)=[JX, JY]-[X, Y]-J([X, JY]+[JX, Y]).$$

Thus if $(\mathcal{D}, J)$ is a CR-structure, then clearly $N(X, Y)=0$ because of (1.2). Conversely, suppose that N vanishes on $\mathcal{D}$. Since ξ-component of $N(X, Y)$ is given by

$$\theta(N(X, Y))=\theta([JX, JY]-[X, Y]),$$

we have the condition (1.1). It follows that $[X, JY]+[JX, Y]$ is contained in $\Gamma(\mathcal{D})$. Therefore we obtain (1.2). Q. E. D.

Let (ϕ', ξ', θ') be another almost contact structure belonging to the pair $(\mathcal{D}, J)$. Then the difference between N and N' is

$$N'(X, Y)-N(X, Y)=-\theta([X, JY]+[JX, Y])A$$

when $X, Y\in\Gamma(\mathcal{D})$. Taking account of the above proof, we see that the vanishing property of N on $\mathcal{D}$ is independent of the choice of almost contact structures belonging to the pair $(\mathcal{D}, J)$.

In the sequel we shall denote the $\mathcal{D}$-component of $N(X, Y)$ $(X, Y\in\mathcal{D})$ by $N_{\mathcal{D}}(X, Y)$, so that $N_{\mathcal{D}}$ is a cross-section of the bundle $\Lambda^2\mathcal{D}^*\otimes\mathcal{D}$. It can be easily shown that if $(\mathcal{D}, J)$ satisfies the condition (1.1), then $N_{\mathcal{D}}$ does not depend on the choice of the almost contact structure (ϕ, ξ, θ) which belongs to the pair $(\mathcal{D}, J)$.

§2. Connections associated with almost contact structures.

In this section, we shall study the relation between the integrability of a pair $(\mathcal{D}, J)$ and the existence of linear connections with certain torsion conditions associated with almost contact structures belonging to $(\mathcal{D}, J)$. Let H be a Lie group

$$\left\{ \begin{pmatrix} u & 0 \\ X & C \end{pmatrix} : C\in \text{real rep. of } GL(n, \boldsymbol{C}),\ X\in\boldsymbol{C}^n=\boldsymbol{R}^{2n},\quad u\in\boldsymbol{R}-\{0\} \right\}.$$

It is clear that pairs of a hyperdistribution $\mathcal{D}$ defined on M and a complex structure on $\mathscr{D}$ are in a one to one correspondence with reductions of the linear frame bundle $L(M)$ to the group H. Let $\mathfrak{H}$ be a H-subbundle in $L(M)$ corresponding to a pair $(\mathscr{D}, J)$. Let ∇ be a covariant differentiation with respect to a linear connection reducible to a connection of $\mathfrak{H}$, in other words, whose connection form takes values in the Lie algebra of H (cf. [7]). Then ∇ satisfies

$$\nabla_X \Gamma(\mathscr{D}) \subset \Gamma(\mathscr{D}) \tag{2.1}$$

for every vector field X tangent to M and hence we have an induced connection in the vector bundle $\mathscr{D}$ whose covariant differentiation will be denoted by D. Under this notation, we have

$$D_X J = 0 \qquad (X \in TM). \tag{2.2}$$

Conversely if a linear connection satisfies (2.1) and (2.2), then it is reducible to a connection of $\mathfrak{H}$. Moreover there is a one to one correspondence between almost contact structures belonging to $(\mathscr{D}, J)$ and reduction of $\mathfrak{H}$ to a Lie group H^* which is defined by

$$H^* = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix} : C \in GL(n, \boldsymbol{C}) \right\}.$$

If (ϕ, ξ, θ) is an almost contact structure belonging to $(\mathscr{D}, J)$ and $\mathfrak{H}^*$ is the corresponding H^*-subbundle of $\mathfrak{H}$, then we see that a linear connection is reducible to a connection of $\mathfrak{H}^*$ when and only when all structure tensors ϕ, ξ, θ satisfy

$$\nabla \phi = 0, \qquad \nabla \xi = 0, \qquad \nabla \theta = 0. \tag{2.3}$$

We say that a linear connection is *associated* with an almost contact structure (ϕ, ξ, θ) when the equations (2.3) hold.

Well let there be given a pair $(\mathscr{D}, J)$. We have the following

PROPOSITION 2.1. *If there exists a linear connection reducible to a connection of* $\mathfrak{H}$ *such that the torsion tensor* T *satisfies*

$$T(JX, JY) = T(X, Y), \qquad X, Y \in \mathscr{D}, \tag{2.4}$$

then $(\mathscr{D}, J)$ *is integrable.*

Proof. From (2.1) and (2.4), we have (1.1). Furthermore we see from (2.2) that the left-hand-side of the equation (1.2) is equal to

$$T(X, Y) - T(JX, JY) + J(T(X, JY) + T(JX, Y))$$

which vanishes. Q. E. D.

Let (ϕ, ξ, θ) be an almost contact structure belonging to the pair $(\mathscr{D}, J)$. The torsion tensor field T of a linear connection associated with (ϕ, ξ, θ) satisfies

$$\theta(T(X, Y))=-\omega(X, Y), \qquad X, Y\in\mathcal{D} \tag{2.5}$$

because of (1.8). Thus the $\mathcal{D}$-component $T_{\mathcal{D}}(X, Y)$ of $T(X, Y)$ is given by

$$T_{\mathcal{D}}(X, Y)=T(X, Y)+\omega(X, Y)\xi, \qquad X, Y\in\mathcal{D}. \tag{2.6}$$

Here we note that $T_{\mathcal{D}}$ is a cross-section of the bundle $\Lambda^2\mathcal{D}^*\otimes\mathcal{D}$.

As a corollary of Proposition 2.1, we obtain

COROLLARY. *Assume that the pair* $(\mathcal{D}, J)$ *satisfies the condition* (1.1) *and there is a linear connection associated with* (ϕ, ξ, θ) *such that* $T_{\mathcal{D}}=0$. *Then* $(\mathcal{D}, J)$ *is integrable.*

Next we shall show the existence of a linear connection associated with arbitrary almost contact structure (ϕ, ξ, θ) belonging to the pair $(\mathcal{D}, J)$ such that $4T_{\mathcal{D}}=N_{\mathcal{D}}$. First we prepare

LEMMA 2.2. *There is a linear connection such that* $\nabla\xi=0$, $\nabla\theta=0$ *and*

$$T(X, Y)=-\omega(X, Y)\xi \qquad (X, Y\in\mathcal{D}).$$

Proof. It is well-known that there is a torsion-free linear connection such that the vector field ξ is parallel (cf. [6]). Let $\overset{*}{\nabla}$ denote the covariant differentiation of such connection. Putting

$$\nabla_X Y=\overset{*}{\nabla}_X Y+(\overset{*}{\nabla}_X\theta)(Y)\xi$$

for each vector fields X, Y tangent to M, we easily see that $\nabla\xi=0$ and $\nabla\theta=0$. Since the auxiliary connection is torsionfree, we have $T(X, Y)=-\omega(X, Y)\xi$ $X, Y\in\mathcal{D}$. In this way, we have obtained a linear connection satisfying the required conditions. Q.E.D.

Secondly we need

LEMMA 2.3. *Let* ∇ *be the covariant differentiation with respect to such a linear connection as in Lemma* 2.2. *Put*

$$\nabla'_X Y=\nabla_X Y+S(X, Y)$$

where S *is a tensor field of type* (1,2). *Then* ∇' *defines a linear connection associated with the almost contact structure* (ϕ, ξ, θ) *such that* $4T'_{\mathcal{D}}=N_{\mathcal{D}}$ *if and only if* S *satisfies*

$$S(X, \xi)=0, \quad \theta(S(X, Y))=0 \qquad X, Y\in TM, \tag{2.7}$$

$$(\nabla_X\phi)Y=\phi S(X, Y)-S(X, \phi Y), \qquad X, Y\in TM \tag{2.8}$$

and

$$S(X, Y)-S(Y, X)=\frac{1}{4}N_{\mathcal{D}}(X, Y), \qquad X, Y\in\mathcal{D}. \tag{2.9}$$

Proof. Straightforward computation shows that ∇' also satisfies $\nabla'\xi=0$ and $\nabla'\theta=0$ if and only if (2.7) holds. Since

$$\begin{aligned}(\nabla'_X\phi)Y&=\nabla'_X\phi Y-\phi\nabla'_XY\\ &=\nabla_X\phi Y+S(X,\ \phi Y)-\phi\nabla_XY-\phi S(X,\ Y)\\ &=(\nabla_X\phi)Y+S(X,\ \phi Y)-\phi S(X,\ Y),\end{aligned}$$

we see that $\nabla'\phi=0$ if and only if (2.8) holds. Taking the equation $T(X,\ Y)=-\omega(X,\ Y)\xi$ into account, we have, for $X,\ Y\in\Gamma(\mathcal{D})$,

$$\begin{aligned}T'(X,\ Y)&=\nabla'_XY-\nabla'_YX-[X,\ Y]\\ &=-\omega(X,\ Y)\xi+S(X,\ Y)-S(Y,\ X)\end{aligned}$$

which implies that $T'_{\mathcal{D}}(X,\ Y)=\mathcal{D}$-component of $S(X,\ Y)-S(Y,\ X)$. From these facts, we conclude our assertion. Q. E. D.

Finally we state

THEOREM 2.4. *Let $(\phi,\ \xi,\ \theta)$ be an arbitrary almost contact structure belonging to a pair $(\mathcal{D}, J)$. Then there exists a linear connection associated with $(\phi,\ \xi,\ \theta)$ such that the torsion tensor field T satisfies $4T_{\mathcal{D}}=N_{\mathcal{D}}$.*

Proof. Let ∇ be the covariant differentiation of a linear connection as in Lemma 2.2. By virtue of Lemma 2.3, we have only to find a tensor field S satisfying the equations (2.7), (2.8) and (2.9). We define S as following

$$\begin{aligned}4S(X,\ Y)=\theta(Y)\phi(\nabla_\xi\phi)X-(\nabla_{\phi Y}\phi)X-\phi(\nabla_Y\phi)X\\ -2\phi(\nabla_X\phi)Y,\qquad X,\ Y\in TM.\end{aligned}$$

The remainder of the proof is devoted to verify that the above tensor field S satisfies (2.7), (2.8) and (2.9). Let X and Y be arbitrary vector fields. We have

$$4S(X,\ \xi)=-2\phi(\nabla_X\phi)\xi=0$$

and

$$\begin{aligned}4\theta(S(X,\ Y))&=-\theta((\nabla_{\phi Y}\phi)X)\\ &=-\theta(\nabla_{\phi Y}\phi X-\phi\nabla_{\phi Y}X)\\ &=(\nabla_{\phi Y}\theta)(\phi X)=0,\end{aligned}$$

which prove the equation (2.7). The equation (2.8) can be obtained as follows:

$$\begin{aligned}&4\{\phi S(X,\ Y)-S(X,\ \phi Y)\}\\ &=\theta(Y)\phi^2(\nabla_\xi\phi)X-\phi(\nabla_{\phi Y}\phi)X-\phi^2(\nabla_Y\phi)X-2\phi^2(\nabla_X\phi)Y\\ &\quad+(\nabla_{\phi^2Y}\phi)X+\phi(\nabla_{\phi Y}\phi)X+2\phi(\nabla_X\phi)\phi Y\end{aligned}$$

$$=-\theta(Y)(\nabla_\xi\phi)X+(\nabla_Y\phi)X+2(\nabla_X\phi)Y+(\nabla_{-Y+\theta(Y)\xi}\phi)X+2\phi(\nabla_X\phi)\phi Y$$

$$=2(\nabla_X\phi)Y+2\phi(\nabla_X\phi)\phi Y=4(\nabla_X\phi)Y,$$

where we have used the fact that $\theta\circ\nabla_Z\phi=0$ for every vector field Z and the equality $\phi(\nabla_X\phi)\phi=\nabla_X\phi$. If X and Y are vector fields contained in $\mathscr{D}$, then we see

$$\begin{aligned}&4\{S(X, Y)-S(Y, X)\}\\&\qquad=(\nabla_{\phi X}\phi)Y-(\nabla_{\phi Y}\phi)X-\phi(\nabla_X\phi)Y+\phi(\nabla_Y\phi)X.\end{aligned}$$

On the other hand, from the defining equation of the tensor field N we have

$$\begin{aligned}N(X, Y)&=\nabla_{\phi X}\phi Y-\nabla_{\phi Y}\phi X+\omega(\phi X, \phi Y)\xi\\&\quad-\nabla_X Y+\nabla_Y X-\omega(X, Y)\xi\\&\quad-\phi(\nabla_X\phi Y-\nabla_{\phi Y}X+\omega(X, \phi Y)\xi)\\&\quad-\phi(\nabla_{\phi X}Y-\nabla_Y\phi X+\omega(\phi X, Y)\xi)\\&=(\nabla_{\phi X}\phi)Y-(\nabla_{\phi Y}\phi)X-\phi(\nabla_X\phi)Y+\phi(\nabla_Y\phi)X\\&\quad+\{\omega(\phi X, \phi Y)-\omega(X, Y)\}\xi\end{aligned}$$

for every $X, Y\in\Gamma(\mathscr{D})$, which implies that the $\mathscr{D}$-component $N_{\mathscr{D}}(X, Y)$ of $N(X, Y)$ is equal to $4\{S(X, Y)-S(Y, X)\}$. Thus we obtain the equation (2.9)
Q. E. D.

COROLLARY. *If a pair $(\mathscr{D}, J)$ is a CR-structure, then any almost contact structure belonging to $(\mathscr{D}, J)$ admits an associated linear connection such that $T_{\mathscr{D}}=0$.*

§ 3. Change of associated linear connections.

Let there be given a pair $(\mathscr{D}, J)$ on the manifold M. We note that if M is orientable, then there always exists an almost contact structure belonging to $(\mathscr{D}, J)$. First we shall be concerned with associated linear connections which induce the same connection D on $\mathscr{D}$. Let (ϕ, ξ, θ) be any almost contact structure belonging to (D, J). It is trivial that for a given connection D on the vector bundle $\mathscr{D}$ such that $DJ=0$ there is a unique linear connection ∇ associated with (ϕ, ξ, θ) whose restriction to $\mathscr{D}$ coincides with D. Thus if (ϕ', ξ', θ') is another almost contact structure belonging to $(\mathscr{D}, J)$, then we have an explicit formula for the change of associated linear connections ∇ and ∇' extending D.

PROPOSITION 3.1. *Let ∇ (resp. ∇') be a linear connection associated with (ϕ, ξ, θ) (resp. (ϕ', ξ', θ')) and suppose that they induce the same connection D on $\mathscr{D}$. Then the difference*

$$S(X, Y)=\nabla'_X Y-\nabla_X Y$$

is given by $S(X, Y)=\theta(Y)QX$ *where* Q *is a tensor field of type* (1, 1) *defined by*

$$QX=d\lambda(X)\phi A-\phi\nabla_X A+d\lambda(X)\xi \tag{3.1}$$

λ *and* A *appearing in Lemma* 1.1.

Proof. From our assumption we have $S(X, \mathcal{D})=0$ for each $X\in TM$. It follows that

$$S(X, Y)=S(X, Y_{\mathcal{D}}+\theta(Y)\xi)=\theta(Y)S(X, \xi).$$

Putting $QX=S(X, \xi)$, we obtain

$$\begin{aligned}QX&=\nabla'_X\xi=\nabla'_X(\varepsilon e^{\lambda}\xi'-\phi A)\\&=d\lambda(X)(\phi A+\xi)-\phi\nabla_X A\end{aligned}$$

which proves our assertion. Q. E. D.

From now we study canonical connections and their changes. We assume that the pair $(\mathcal{D}, J)$ is non-degenerate. Let $\mathfrak{H}$ be the H-bundle corresponding to $(\mathcal{D}, J)$ and π be the projection of $\mathfrak{H}$ to M. The canonical form η of $\mathfrak{H}$ is defined by

$$\eta_z(\tilde{X})=z^{-1}\pi_*\tilde{X}, \quad \tilde{X}\in T_z\mathfrak{H}, \quad z\in\mathfrak{H}$$

where the linear frame z is considered as a linear map from $\boldsymbol{R}\oplus\boldsymbol{C}^n$ to $T_{\pi(z)}M$ (cf. [7]). Let $\tilde{\theta}$ be the $\boldsymbol{R}$-component of η. Define a 2-form $\tilde{\omega}_z$: $\mathcal{D}_{\pi(z)}\Lambda\mathcal{D}_{(z)}\rightarrow\boldsymbol{R}$ by

$$\tilde{\omega}_z(X, Y)=-2d\tilde{\theta}(\tilde{X}, \tilde{Y})$$

where $\tilde{X}$, $\tilde{Y}\in T_z\mathfrak{H}$ such that $\pi_*\tilde{X}=X$ and $\pi_*\tilde{Y}=Y$. It is clear that $\tilde{\theta}(\tilde{X})=0$ if and only if $\pi_*\tilde{X}\in\mathcal{D}$. Hence if $\pi_*\tilde{X}\in\mathcal{D}$ and $\tilde{Y}$ is vertical, then $d\tilde{\theta}(\tilde{X}, \tilde{Y})=0$. This fact implies that $\tilde{\omega}$ is well-defined. For any $h=\begin{pmatrix} u & 0 \\ X & C\end{pmatrix}\in H$, we easily obtain $\tilde{\omega}_{zh}=u^{-1}\tilde{\omega}_z$. It follows that if we define $\tilde{g}_z$: $\mathcal{D}_{\pi(z)}\times\mathcal{D}_{\pi(z)}\rightarrow\boldsymbol{R}$ by

$$\tilde{g}_z(X, Y)=\tilde{\omega}_z(JX, Y),$$

then $\tilde{g}$ is non-degenerate and satisfies

$$\tilde{g}_{zh}=u^{-1}\tilde{g}_z. \tag{3.2}$$

We moreover assume that the condition (1.1) holds. Then the bilinear form $\tilde{g}_z$ is symmetric and Hermitian. Let $a(z)$ (resp. $b(z)$) be the dimension of the maximal subspace in $\mathcal{D}_{\pi(z)}$ on which $\tilde{g}_z$ is positive definite (resp. negative definite). These numbers are necessary even and we see from the equation (3.2) that $2\gamma=\min(a(z), b(z))$ depends only on $\pi(z)$. So γ is a function on M. Since M is connected, we easily see that γ is constant. This fact allows us to say that a non-degenerate CR-structure $(\mathcal{D}, J)$ is of index γ.

In the sequel, the orientability of M will be assumed and the pair $(\mathcal{D}, J)$ will be a non-degenerate CR-structure of index γ. Thus the corresponding H-bundle $\mathfrak{H}$ has two connected components. Let E_γ be a $2n\times 2n$ matrix defined by

$$E_\gamma=\begin{pmatrix} F_\gamma & 0 \\ 0 & F_\gamma \end{pmatrix}, \qquad F_\gamma=\begin{pmatrix} \overbrace{-1 \ \ddots \ -1}^{\gamma} & & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}$$

Then we have a principal subbundle $\mathfrak{g}^\varepsilon$ of $\mathfrak{H}$;

$$\mathfrak{g}^\varepsilon=\{z\in\mathfrak{H}:\ g_z(X,\ Y)=\varepsilon{}^t\boldsymbol{x}E_\gamma\boldsymbol{y},\ z^{-1}X=\boldsymbol{x},\ z^{-1}Y=\boldsymbol{y}\}$$

with structure group

$$G=\left\{\begin{pmatrix} u & 0 \\ X & C \end{pmatrix}\in H:\ {}^tCE_\gamma C=uE_\gamma,\ u>0\right\}$$

where we remark that $C\in CU_\gamma=GL(n,\ \boldsymbol{C})\cap CO(2\gamma,\ 2n-2\gamma)$. Let $(\phi,\ \xi,\ \theta)$ be an almost contact structure belonging to $(\mathcal{D}, J)$. The intersection $\mathfrak{g}^*=\mathfrak{H}^*\cap\mathfrak{g}^\varepsilon$ is a principal bundle with structure group

$$G^*=\left\{\begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}\in H^*:\ {}^tCE_\gamma C=E_\gamma\right\}.$$

It is easily verified that a linear connection ∇ is reducible to a connection of $\mathfrak{g}^*$ if and only if ∇ satisfies

$$(*)\qquad \nabla\xi=0,\qquad \nabla\theta=0,\qquad \nabla\phi=0,\qquad Dg=0.$$

Let $(\phi',\ \xi',\ \theta')$ be another almost contact structure belonging to $(\mathcal{D}, J)$. Taking account of the equation (1.14), we see that the Levi metrics g and g' are related by

$$(3.3)\qquad g'=\varepsilon e^\lambda g$$

Given two linear connections ∇ and ∇', as before we define a tensor field S by $S(X,\ Y)=\nabla'_XY-\nabla_XY$.

LEMMA 3.2. *Let ∇ and ∇' be linear connections satisfying* (*) (*for* ∇' *one needs dash in the equations of* (*)). *Then the tensor field S satisfies*

$$(3.4)\qquad S(X, JA)+S(X,\ \xi)=QX,$$

$$(3.5)\qquad \theta(S(X,\ Y))=d\lambda(X)\theta(Y),$$

$$(3.6)\qquad S(X, JV)=JS(X,\ V),$$

$$g(S(X, V), W)+g(V, S(X, W))=d\lambda(X)g(V, W) \tag{3.7}$$

where $X, Y\in TM$ *and* $V, W\in\mathcal{D}$. *The tensor field* Q *of type* (1, 1) *is defined as* (3.1).

Proof. Let X, Y be arbitrary vector fields and V, W be elements of $\Gamma(\mathcal{D})$. Since

$$\begin{aligned}0&=\nabla'_X\xi'=\nabla_X\xi'+S(X, \xi')\\&=\nabla_X(\varepsilon e^{-\lambda}(\phi A+\xi))+S(X, \varepsilon e^{-\lambda}(\phi A+\xi))\\&=\varepsilon e^{-\lambda}\{S(X, \phi A)+S(X, \xi)-QX\}\ ,\end{aligned}$$

we have the equation (3.4). Similarly we have

$$\begin{aligned}0&=(\nabla'_X\theta')(Y)=X\cdot\theta'(Y)-\theta'(\nabla'_XY)\\&=X\cdot(\varepsilon e^{\lambda}\theta(Y))-\varepsilon e^{\lambda}\theta(\nabla_XY+S(X, Y))\\&=\varepsilon e^{\lambda}\{d\lambda(X)\theta(Y)-\theta(S(X, Y))\}\end{aligned}$$

which proves (3.5). Moreover

$$\begin{aligned}0&=(\nabla'_X\phi')Y=\nabla'_X(\phi'Y)-\phi'\nabla'_XY\\&=\nabla_X(\phi'Y)+S(X, \phi'Y)-\phi'\nabla_XY-\phi'S(X, Y)\\&=S(X, \phi Y)-\phi S(X, Y)+\theta(Y)\{\nabla_XA+S(X, A)-d\lambda(X)A\}\ .\end{aligned}$$

In particular if we put $Y=V$, then the equation (3.6) follows. Finally from (3.3) we have

$$\begin{aligned}0&=(D'_Xg')(V, W)=X\cdot g'(V, W)-g'(\nabla'_XV, W)-g'(V, \nabla'_XW)\\&=\varepsilon e^{\lambda}\{d\lambda(X)g(V, W)-g(S(X, V), W)-g(V, S(X, W))\end{aligned}$$

and hence (3.7) holds. Q.E.D.

Next we want to restrict our attention to linear connections with condition (*) whose torsion tensors satisfy $T_{\mathcal{D}}=0$. However the existence of such connections imposes restrictions on the class of almost contact structures belonging to $(\mathcal{D}, J)$. In the following Lemma we show that ∇_XY $(X, Y\in\Gamma(\mathcal{D}))$ is determined by the equations $Dg=0$ and $T_{\mathcal{D}}=0$.

LEMMA 3.3. *If* ∇ *is a linear connection such that* $Dg=0$ *and* $T_{\mathcal{D}}=0$, *then*

$$\begin{aligned}2g(\nabla_XY, Z)&=X\cdot g(Y, Z)+Y\cdot g(X, Z)-Z\cdot g(X, Y)\\&\qquad-g(X, [Y, Z]_{\mathcal{D}})-g(Y, [X, Z]_{\mathcal{D}})+g(Z, [X, Y]_{\mathcal{D}})\end{aligned} \tag{3.8}$$

where $X, Y, Z\in\Gamma(\mathcal{D})$ *and* $[\ ,\ ]_{\mathcal{D}}=[\ ,\ ]-\omega(\ ,\ )\xi$, *that is* $\mathcal{D}$-*component of* $[\ ,\ ]$ *with respect to* ξ.

Proof. Since $Dg=0$, we have

$$X\cdot g(Y, Z)=g(\nabla_X Y, Z)+g(Y, \nabla_X Z)$$

for every $X, Y, Z\in\Gamma(\mathcal{D})$. Using usual technique (recall Levi-Civita connection), we obtain (3.8). Q. E. D.

Therefore three more conditions $\nabla\xi=0$, $\nabla\theta=0$, $\nabla\phi=0$ impose a restriction on the almost contact structure (ϕ, ξ, θ). The following proposition gives a necessary condition to the existence of linear connections satisfying (*) and $T_{\mathcal{D}}=0$.

PROPOSITION 3.4. *Assume that* $n\geqq 2$. *If there exists a linear connection satisfying the condition* (*) *and* $T_{\mathcal{D}}=0$, *then we have* (1.16).

Proof. First we have

$$\begin{aligned} g(D_X J)Y, Z) &= g(\nabla_X(JY), Z)-g(J\nabla_X Y, Z) \\ &= g(\nabla_X(JY), Z)+g(\nabla_X Y, JZ) \end{aligned}$$

To the tirms on the right we apply the equation (3.8). We note that (3.8) can be written as

$$\begin{aligned} 2g(\nabla_X Y, Z)=&X\cdot\omega(JY, Z)+Y\cdot\omega(JX, Z)-Z\cdot\omega(JX, Y) \\ &-\omega(\phi[Y, Z], X)-\omega(\phi[X, Z], Y)+\omega(\phi[X, Y], Z)\,. \end{aligned}$$

Using the formula

$$\begin{aligned} 3d\omega(X, Y, Z)=&X\cdot\omega(Y, Z)+Y\cdot\omega(Z, X)+Z\cdot\omega(X, Y) \\ &-\omega([X, Y], Z)-\omega([Z, X], Y) \\ &-\omega([Y, Z], X)\,, \end{aligned}$$

direct calculation shows

$$\begin{aligned} 2g((D_X J)Y, Z)=&3d\omega(X, JY, JZ)-3d\omega(X, Y, Z) \\ &+\omega(N(Y, Z), X) \\ &+\omega(X, Z)\omega(\xi, Y)-\omega(X, Y)\omega(\xi, Z) \\ &+\omega(X, JY)\omega(\xi, JZ)-\omega(X, JZ)\omega(\xi, JY) \end{aligned}$$

for every $X, Y, Z\in\Gamma(\mathcal{D})$. We now have $D_X J=0$, $d\omega=0$ and $N=0$. It follows that

$$\begin{aligned} &\omega(X, Z)\omega(\xi, Y)-\omega(X, Y)\omega(\xi, Z)+\omega(X, JY)\omega(\xi, JZ) \\ &-\omega(X, JZ)\omega(\xi, JY)=0 \end{aligned}$$

and hence we have

$$\omega(\xi, Y)Z-\omega(\xi, Z)Y+\omega(\xi, JZ)JY-\omega(\xi, JY)JZ=0$$

for any $Y, Z\in\Gamma(\mathcal{D})$. Our assumption $n\geqq 2$ implies that $\omega(\xi, Y)=0$. Q. E. D.

Remark. As will be shown in later, the equation (1.16) is also a sufficient condition to the existence of linear connections satisfying (*) and $T_{\mathcal{D}}=0$.

According to Proposition 3.4, we from now consider only almost contact structures with condition (1.15), which always exist in virtue of Proposition 1.2. Let define $\Lambda\in\Gamma(\mathcal{D})$ by

$$g(\Lambda, X)=d\lambda(X), \qquad X\in\mathcal{D}. \tag{3.9}$$

Note that the vector field Λ is uniquely determined because of the non-degeneracy of the Levi metric g. We need

LEMMA 3.5. *Let* (ϕ, ξ, θ) *and* (ϕ', ξ', θ') *be almost contact structures satisfying the condition* (1.15). *Then the vector field* A *in Lemma* 1.1 *is given by*

$$A=-\Lambda. \tag{3.10}$$

Proof. Let $X\in\Gamma(\mathcal{D})$. We have

$$\begin{aligned}0=\theta'([\xi', X])&=\varepsilon e^{\lambda}\theta([\varepsilon e^{-\lambda}(\phi A+\xi), X])\\ &=\theta([\phi A+\xi, X])+d\lambda(X)\\ &=g(A, X)+d\lambda(X)\end{aligned}$$

from which (3.10) follows. Q. E. D.

Let (ϕ, ξ, θ) be an almost contact structure with condition (1.15) which belongs to the non-degenerate *CR*-structure $(\mathcal{D}, J)$. Let ∇ be a linear connection satisfying (*) and $T_{\mathcal{D}}=0$. But such connection is not yet uniquely determined. Thus we shall demand a condition for $T(\xi, X)$. To do this we need

LEMMA 3.6. *Let* F *be a tensor field of type* (1, 1) *defined by*

$$FX=T(\xi, X), \qquad X\in TM. \tag{3.11}$$

Then F *satisfies*

$$\nabla_{\xi}X=FX+[\xi, X] \qquad X\in\Gamma(\mathcal{D}), \tag{3.12}$$

$$F\xi=0, \qquad \theta\circ F=0, \tag{3.13}$$

$$F\phi-\phi F=-\mathcal{L}_{\xi}\phi, \tag{3.14}$$

$$g(FX, Y)+g(X, FY)=g(-\phi(\mathcal{L}_{\xi}\phi)X, Y), \qquad X, Y\in\Gamma(\mathcal{D}). \tag{3.15}$$

Proof. The equations (3.12) and (3.13) are immediately derived from the definition of F and (2.1). Since

$$0=(\nabla_\xi\phi)X=\nabla_\xi(\phi X)-\phi\nabla_\xi X$$
$$=F\phi X+[\xi,\ \phi X]-\phi FX-\phi[\xi,\ X],$$

we have (3.14). To the formula

$$0=\xi\cdot g(X,\ Y)-g(\nabla_\xi X,\ Y)-g(X,\ \nabla_\xi Y)$$

we apply the equation (3.12). The resulting equation is

$$g(FX,\ Y)+g(X,\ FY)=\xi\cdot\omega(\phi X,\ Y)-\omega(\phi\mathcal{L}_\xi X,\ Y)-\omega(\phi X,\ \mathcal{L}_\xi Y).$$

The right of the above equation becomes

$$-2(d\mathcal{L}_\xi\theta)(\phi X,\ Y)+\omega((\mathcal{L}_\xi\phi)X,\ Y).$$

Using the equations (1.11) and (1.17), we obtain (3.15). Q. E. D.

Now we demand for F the condition that F is symmetric with respect to g. Then F must be

$$F=-\frac{1}{2}\phi\mathcal{L}_\xi\phi. \tag{3.16}$$

Remark. Tanaka [10] demanded the condition that F anticommutes with ϕ. From this condition and equations (3.13) (3.14), we have also (3.16).

Tanaka [10] has mentioned

THEOREM 3.7. *Let $(\phi,\ \xi,\ \theta)$ be an almost contact structure with condition* (1.15) *which belongs to the non-degenerate CR-structure $(\mathcal{D},\ J)$. Then there is a unique linear connection ∇ satisfying* (*), $T_{\mathcal{D}}=0$ *and* (3.16), *which is given by* (3.8) *and* (3.12).

Proof. Define ∇ by the equations (3.8), (3.12) and $\nabla\xi=0$. Then direct calculation shows that it satisfies (*), $T_{\mathcal{D}}=0$ and (3.16).

The linear connection stated in Theorem 3.7 is called a *canonical connection* of an almost contact structure with condition (1.15) (cf. [10]).

Before mentioning the change of canonical connections, we give

LEMMA 3.8. *Let $(\phi,\ \xi,\ \theta)$ and $(\phi',\ \xi',\ \theta')$ be almost contact structures with condition* (1.15) *which belong to $(\mathcal{D},\ J)$. Let ∇ and ∇' be canonical connections of $(\phi,\ \xi,\ \theta)$ and $(\phi',\ \xi',\ \theta')$ respectively. Then we have*

$$\begin{aligned}2\varepsilon e^\lambda F'X-2FX \\ =\nabla_{JX}A-J\nabla_X A-g(A,\ JX)A+g(A,\ X)JA\end{aligned} \tag{3.17}$$

for every $X\in\Gamma(\mathcal{D})$.

Proof. From the equation (3.16) we have

$$2\varepsilon e^\lambda F'X=-\varepsilon e^\lambda\phi'(\mathcal{L}_\xi\phi')X$$
$$=-\varepsilon e^\lambda\phi[\xi',\ \phi X]-\varepsilon e_\lambda[\xi',\ X]$$

where we note that $[\xi', X] \in \Gamma(\mathcal{D})$ for $X \in \Gamma(D)$. Substituting (1.14) into the right of the above equation, we obtain

$$2\varepsilon e^{\lambda} F'X = -\phi[\xi, \phi X] - [\xi, X] - \phi[\phi A, \phi X] - [\phi A, X]$$
$$+ d\lambda(\phi X)A - d\lambda(X)(\phi A + \xi).$$

Apply the equation

$$[Y, Z] = \nabla_Y Z - \nabla_Z Y + \omega(Y, Z)\xi, \qquad Y, Z \in \Gamma(\mathcal{D})$$

to the third and fourth tirms of the right. Using the equation (3.10) we get the equation (3.17). Q. E. D.

Finally we state

THEOREM 3.9. *Let* (ϕ, ξ, θ) *and* (ϕ', ξ', θ') *be almost contact structures with condition* (1.15) *which belong to the non-degenerate CR-structure* $(\mathcal{D}, J)$. *Let* ∇ *and* ∇' *be canonical connections of* (ϕ, ξ, θ) *and* (ϕ', ξ', θ') *respectively. Then the tensor field S is given by*

$$2S(X, Y) = d\lambda(X)Y + d\lambda(Y)X - g(X, Y)A \tag{3.18}$$
$$+ g(X, JA)JY + g(Y, JA)JX - g(JX, Y)JA,$$

$$2S(\xi, X) = \nabla_{JX}A + J\nabla_X A + g(A, JX)A \tag{3.19}$$
$$+ g(A, X)JA + g(A, A)JX$$

where $X, Y \in \Gamma(\mathcal{D})$.

Proof. Let $X, Y \in \Gamma(\mathcal{D})$. The equation $T_{\mathcal{D}} = 0$ implies that

$$T'(X, Y) = \nabla'_X Y - \nabla'_Y X - [X, Y]$$
$$= -\omega(X, Y)\xi + S(X, Y) - S(Y, X).$$

On the other hand, noting that $\omega'(X, Y) = \varepsilon e^{\lambda}\omega(X, Y)$ we have

$$T'(X, Y) = -\omega'(X, Y)\xi'$$
$$= -\omega(X, Y)JA - \omega(X, Y)\xi.$$

From these equations the equation

$$S(X, Y) - S(Y, X) = -\omega(X, Y)JA \tag{3.20}$$

follows. Rewrite (3.7) as

$$g(S(X, Y), Z) + g(Y, S(X, Z)) = d\lambda(X)g(Y, Z)$$

in which we permute the letters cyclically and subtract one from the sum of the other two. Applying (3.20) to the obtained equation, we have the equation

(3.18). Next we compute $S(\xi, X)$. Noting that $\xi=\varepsilon e^{\lambda}\xi'+\phi\Lambda$, we have

$$\begin{aligned} S(\xi, X)&=\nabla'_{\xi}X-\nabla_{\xi}X=\nabla'_{\varepsilon e^{\lambda}\xi'+\phi\Lambda}X-\nabla_{\xi}X \\ &=\varepsilon e^{\lambda}(F'X+[\xi', X])+\nabla_{\phi\Lambda}X+S(\phi\Lambda, X)-FX-[\xi, X] \\ &=\varepsilon e^{\lambda}F'X-FX+[\phi A, X]+d\lambda(X)\phi A+d\lambda(X)\xi+\nabla_{\phi\Lambda}X+S(\phi\Lambda, X). \end{aligned}$$

Substituting (3.17) into the right, we get

$$2S(\xi, X)=\nabla_{JX}\Lambda+J\nabla_{X}\Lambda-g(\Lambda, JX)\Lambda-g(\Lambda, X)J\Lambda+2S(J\Lambda, X)$$

where we have used $[\phi A, X]=-\nabla_{J\Lambda}X+J\nabla_{X}\Lambda-g(\Lambda, X)\xi$. Moreover (3.18) implies that

$$2S(J\Lambda, X)=2g(\Lambda, X)J\Lambda+2g(\Lambda, JX)\Lambda+g(\Lambda, \Lambda)JX.$$

Therefore we obtain (3.19). Q. E. D.

Remark. By virtue of the equation (3.4) we can calculate the components $S(X, \xi)$ $(X\in\mathscr{D})$ and $S(\xi, \xi)$.

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