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NOTES ON LINKS OF COMPLEX ISOLATED SINGULAR POINTS

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§1. Introduction.

Let $V\subset C^{n+1}$ be an algebraic variety of codimension d with isolated singular point at the origin, and K be the intersection with a small sphere S^{2n+1} centered at the origin. Then the cobordism class of K in S^{2n+1} is the obstruction to topological smoothing of the singularity and is represented by the element of $\pi_{2n+1}(MU(d))$.

D. Sullivan [3] gave an interesting conjecture that the link K cannot represent an element involving higher order whitehead product in $\pi_{2n+1}(MU(d))\otimes Q$.

In this note we consider the case when V is a cone over smooth projective variety. Then using M. Larsen's result [2], we shall give an affirmative answer to Sullivan's conjecture. Moreover we can describe the represented class in $\pi_{2n+1}(MU(d))\otimes Q$ in terms of the chern class of the normal bundle of V in CP^n. This shows that the classical Thom's condition is the only rational obstruction to topological smoothing of the singularity.

§2. The rational homotopy type of $MU(d)$.

In this section we shall construct the minimal model of universal Thom space $MU(d)$ according to Sullivan's idea.

Let c_i be the i-th chern class of the universal complex vector bundle, then $H^*(BU(d), Q)\cong Q[c_1\cdots c_d]$. If $I=(i_1\cdots i_{d-1})$ is a multi-index, we denote by c_I the monomial $c_1^{i_1}\cdots c_{d-1}^{i_{d-1}}$. Let S be the polynomial ring generated by c_d and c_dc_I, and T be the ideal of S generated by forms $(c_dc_I)(c_dc_J)-(c_d)(c_dc_{I+J})$. Then from the cofibration

$$BU(d-1)\to BU(d)\to MU(d)$$

we have the ring isomorphism:

$$H^*(MU(d))\cong S/T$$

Since the rational homotopy type of $MU(d)$ is the formal consequence of its cohomology ring ([3]), we can construct the minimal model of $MU(d)$ from

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S/T with the trivial differential as follows:

$$m^*(MU(d)) \cong \Lambda(c_d \cdots c_d c_I \cdots \theta_{I,J} \cdots \lambda_k \cdots)$$

where $\Lambda(xy \cdots)$ means a free graded commutative algebra generated by $x, y, \cdots$ over Q. And the differential is as follows:

$d(c_d)=o,\ d(c_d c_I)=o$

$d(\theta_{I,J})=(c_d c_I)(c_d c_J)-(c_d)(c_d c_{I+J})$

$d(\lambda_k)$ is not a linear combination of products of closed elements.

In view of the theory of minimal model, $c_d c_I$, $\theta_{I,J}$ and λ_k correspond to the terms dual to Hurewicz image, single Whitehead product and higher order Whitehead product respectively in $\pi_*(MU(d)) \otimes Q$. According to Sullivan, we denotes elements corresponding to the dual of $c_d c_I$, $\theta_{I,J}$ by $c_d c_I$, $[c_d c_I, c_d c_J]$ respectively.

§ 3. Sullivan's conjecture.

In this section we study the effect of connectivity of manifolds on their cobordism classes. Then we have

PROPOSITION 3.1. *Let $N \subset S^{2n+1}$ be a closed submanifold of codimension d with complex normal bundle. If N is n-$2d$ connected, then N does not represent an element involving a higher order Whitehead product in $\pi_{2n+1}(MU(d)) \otimes Q$. Especially in case n is even the represented class in $\pi_{2n+1}(MU(d)) \otimes Q$ is zero.*

Proof. Let ν be the normal bundle of N in S^{2n+1}. Since $BU(d)$ has the same rational homotopy type as $\prod_{i=1}^{d} K(Q, 2i)$, by the assumption of connectivity, the localization of the classifying map of ν factors as follows:

$$N_{(0)} \to \prod_{i>[n-2l/2]}^{d} K(Q, 2i) \to \prod_{i=1}^{d} K(Q, 2i)$$

Therefore the localization of the Thom map factors as follows:

$$(※) \qquad S^{2n+1}_{(0)} \to T(\nu)_{(0)} \to M \to MU(d)_{(0)}$$

where $T(\nu)$ is a Thom space of ν, and M is the

$$\text{cofiber}\left(\prod_{i>[n-2l/2]}^{d-1} K(Q, 2i) \to \prod_{i>[n-2l/2]}^{d} K(Q, 2i)\right).$$

In the same way as the case of $MU(d)$, we can construct the minimal model of M as follows:

$$m^*(M) \cong \Lambda(c_d \cdots c_d c_I \cdots \theta_{I,J} \cdots \lambda_k \cdots)$$

where the element $c_d c_I$ appears if and only if degree of $c_I > n-2d$. Hence there is no element in degree $2n+1$ if n is even, and $\theta_{I,I}$ with $I=(0 \cdots 1, 0 \cdots)$ (1 appears in the $n-2d+1/2$-th component) is the only generator in degree $2n+1$ if n is odd. Thus the proof is completed.

Now we consider the link of the cone over smooth projective varieties. First we quote the following theorem obtained by M. Larsen.

THEOREM 3.2. *Let $K \subset S^{2n+1}$ be the link of the cone over smooth projective variety of codimension d. Then K is n-$2d$ connected.*

Thus by Theorem 3.2 and Proposition 3.1 we have Sullivan's conjecture in our case as follows

THEOREM 3.3. *The link $K \subset S^{2n+1}$ of the cone over smooth projective variety cannot represent an element involving higher order Whitehead product in $\pi_{2n+1}(MU(d)) \otimes Q$.*

§ 4. Thom's condition.

In this section we shall describe elements in $\pi_{2n+1}(MU(d)) \otimes Q$ represented by links.

Let $j: W \to CP^n$ be the inclusion of a smooth projective variety W in CP^n and $c_i(\mu) \in H^{2i}(W, Z)$ be the i th chern class of the normal bundle μ. We denote by $\deg c_I(\mu)$ be the integer obtained as the Gysin-image $j_!(c_I(\mu)) \in H^{2|I|+2d}(CP^n, Z)$, which is identified with Z, where $|I| = \sum_{k=1}^{d-1} k i_k$ with $I=(i_1 \cdots i_{d-1})$. Then we have

THEOREM 4.1. *The link of the affiine cone over W represents in $\pi_{2n+1}(MU(d)) \otimes Q$ some multiple of*

$[c_d c_{n-2d+1/2}, c_d c_{n-2d+1/2}]$, *if n is odd and*

$$\{\deg (C_{n-2d+1/2}(\mu)\}^2 \neq \deg (C_{n-2d+1}{}^2(\mu)) \cdot \deg W$$

0 , *otherwise.*

Proof. If n is even it is immediate from Proposition 3.1. So let n be odd. The Thom map of the link is the composition of the Hopf map $\pi: S^{2n+1} \to CP^n$ and the Thom map $\varphi: CP^n \to MU(d)$ of W. We translate these maps to D.G.A. maps between minimal models, and denote them by adding "$\wedge$". We recall the minimal model of S^{2n+1} and CP^n.

$$m^*(S^{2n+1}) \cong \Lambda(\alpha) \quad \deg \alpha = 2n+1 \quad \text{and} \quad d\alpha = 0$$

$$m^*(CP^n) \cong \Lambda(\beta, \gamma) \quad \deg \beta = 2 \deg \gamma = 2n+1 \quad \text{and} \quad d\beta = 0 \quad d\gamma = \beta^{n+1}$$

and $\hat{\pi}(\gamma) = \alpha$. Then by the commutative diagram

$$\begin{array}{ccc} H^{n+1}(CP^n, Z) & \xleftarrow{\quad\varphi^*\quad} & H^{n+1}(MU(d), Z) \\ j!\uparrow & & \uparrow \text{Thom iso.} \\ H^{n-2d+1}(W, Z) & \xleftarrow[\mu]{\qquad} & H^{n-2d+1}(BU(d), Z) \end{array}$$

where horizontal maps are induced by the classifying map of μ. We have

$$\varphi^*(c_d c_{n-2d+1/2}) = j!\,(c_{n-2d+1/2}(\mu))$$

and moreover

$$\hat{\varphi}(c_d c_{n-2d+1/2}) = \deg\,(c_{n-2d+1/2}(\mu)) \cdot \beta^{n+1/2}$$

Hence we obtain

$$\hat{\pi} \circ \hat{\varphi}(\theta_{n-2d+1/2,\, n-2d+1/2}) = \{(\deg C_{n-2d+1/2}(\mu))^2 - (\deg\,(C_{n-2d+1/2}{}^2(\mu)) \cdot (\deg W)\} \cdot \gamma\,.$$

For other generator x, since φ factors as shown (*) in the proof of Proposition 3.1, we have $\hat{\pi} \circ \hat{\varphi}(x) = 0$. Thus the proof is completed.

Thom obtained the number $(\deg C_{n-2d+1/2}(\mu))^2 - (\deg C_{n-2d+1/2}{}^2(\mu)) \cdot (\deg W)$ as a necessary condition for topological smoothing of the cone singularity long ago (refer to Theorem 3.1 in [1]). Theorem 4.1 shows that this number is the only rational obstruction to topological smoothings.

We list some examples.

EXAMPLE 1. Let V be the cone over Segre embedding:

$$CP^1 \times CP^2 \subsetneq CP^5$$

Then its link represents some multiple of $[c_2c_1, c_2c_1]$

EXAMPLE 2. Let V be the cone over Grass (1,4) the Grassman variety of projective line in CP^4 which is embedded in CP^9. Then its link represent some multiple of $[c_3c_2, c_3c_2]$.

These are easily obtained from the computation in [1] and Theorem 4.1. I do not know whether there is a variety which represents $[c_dc_i, c_dc_i]$ for i with $3 \leqq i \leqq d-1$.

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