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THE REALIZATION OF ABSTRACT STRATIFIED SETS

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§ 1. Introduction.

It is well-known that n-dimensional differentiable manifolds can be realized in the $(2n+1)$-dimensional Euclidean space.

Recently, R. Thom and J. Mather have introduced the notion of abstract stratified set, modelling after variety (manifold with singuralities). In this paper, we shall realize n-dimensional stratified sets in the $(2n+1)$-dimensional Euclidean space.

DEFINITION 1 ([1]). A stratification $\mathcal{S}$ for a subset V in R^N is a locally finite family of pairwise disjoint submanifolds X of R^N, satisfying the following conditions.

S1. Each $X\in\mathcal{S}$ lies in V, which is called a stratum of $\{V, \mathcal{S}\}$.
S2. Each point of V is contained in the interior of some stratum.
S3. The frontier condition: for each stratum X, if a stratum Y intersects with the closure $\bar{X}$ in V of X, then $Y\subset\bar{X}$.

If $Y\subset\bar{X}$, Y is said to be incident to X and we write $Y<X$ or $X>Y$.

A topological space V with a stratification $\mathcal{S}$ is called a stratified set. H. Whitney in [1] considered stratified sets with the following condition.

Whitney condition. For each pair of strata (X, Y) such that $X>Y$, if both series of points $\{x_i\}$ in X and $\{y_i\}$ in Y converge to a point y in Y and the line through x_i and y_i converges to some line l and the tangent space of X at x_i converges to some plane P, then $l\subset P$.

R. Thom ([3]) and J. Mather ([2]), axiomatizing stratified set together with a tubular neighbourhood system, introduced the following notion of abstract stratified sets.

DEFINITION 2 ([2]). Let V be a subset of R^N with a stratification $\mathcal{S}$. A family $\mathcal{T}=\{(T_X, \pi_X, \rho_X)\}_{X\in\mathcal{S}}$ is called a tubular neighbourhood system for $\mathcal{S}$, if it satisfies the following conditions:

T1. T_X is an open neighbourhood of X in R^N.
T2. $\pi_X: T_X\to X$ is a bundle such that there exists a bundle isomorphism $\varphi_X: T_X\to B_X$ where B_X is the open unit ball bundle in an inner product bundle over X.

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T3. $\rho_X: T_X \to [0, 1)$ is the function defined by $\rho_X(v) = \|\varphi_X(v)\|$ for $v \in T_X$.
T4. For $Y < X$, $\rho_Y \circ \pi_X(v) = \rho_Y(v)$ for $v \in T_X \cap T_Y$.

DEFINITION 3. Let V be a subset of R^N with a stratification $\mathcal{S}$. A family $\mathcal{T} = \{(T_X, \pi_X, \rho_X)\}_{X \in \mathcal{S}}$ is called a closed tubular neighbourhood system, if T_X is a closed tubular neighbourhood of X in R^N and there exists a tubular neighbourhood system $\mathcal{T}' = \{T'_X, \pi'_X, \rho'_X)\}_{X \in \mathcal{S}}$ for $\mathcal{S}$ such that $T'_X \supset T_X$, $\pi'_X | T_X = \pi_X$ and $\rho'_X | T_X = \rho_X$ for any X in $\mathcal{S}$.

DEFINITION 4 ([2]). An abstract stratified set is a triple $\{V, \mathcal{S}, \mathcal{T}\}$ satisfying the following conditions.

A1. V is a Hausdorff, locally compact topological space with a countable basis.
A2. $\mathcal{S}$ is a family of locally closed subsets of V such that V is the disjoint union of members of $\mathcal{S}$.
A3. Each stratum of $\{V, \mathcal{S}, \mathcal{T}\}$ is a smooth mainfold.
A4. The family $\mathcal{S}$ is locally finite.
A5. The frontier condition: the same as S3 in Definition 1.
If $Y \subset \bar{X}$, Y is also said to be incident to X and we write $Y < X$ or $X > Y$.
A6. $\mathcal{T}$ is a family of triples $\{(T_X, \pi_X, \rho_X)\}_{X \in \mathcal{S}}$, where T_X is a closed neighbourhood of X in V such that there exists an open neighbourhood $T'_X \supset T_X$ of X in V, π_X is a continuous retraction of T'_X to X and ρ_X is a non-negative continuous function of T'_X.
A7. $X = \rho_X^{-1}(0)$.
A8. If $X > Y$, then the mapping $\pi_{Y,X} \times \rho_{Y,X}: T_{Y,X} \to Y \times (0, \infty)$ is a smooth submersion, where $T_{Y,X} = T'_Y \cap X$, $\pi_{Y,X} = \pi_Y | T_{Y,X}$, $\rho_{Y,X} = \rho_Y | T_{Y,X}$ and the inverse image of $Y \times (0, 1]$ by $\pi_{Y,X} \times \rho_{Y,X}$ is equal to $T_Y \cap X$.
A9. For arbitrary strata X, Y and Z, we have $\pi_{Z,Y} \circ \pi_{Y,X} = \pi_{Z,X}$ and $\rho_{Z,Y} \circ \pi_{Y,X} = \rho_{Z,X}$ whenever both sides of these equations are defined.

DEFINITION 5. We say that a stratified set $\{V, \mathcal{S}, \mathcal{T}\}$ is equivalent to $\{V', \mathcal{S}', \mathcal{T}'\}$ if the following conditions hold.
(1) $V = V'$, $\mathcal{S} = \mathcal{S}'$ and for each stratum X of $\mathcal{S} = \mathcal{S}'$, the two smoothness structures on X are the same.
(2) If $\mathcal{T} = \{(T_X, \pi_X, \rho_X)\}$ and $\mathcal{T}' = \{(T'_X, \pi'_X, \rho'_X)\}$, then for each stratum X of $\mathcal{S}$, there exists a neighbourhood T''_X of X in $T_X \cap T'_X$ such that $\rho_X | T''_X = \rho'_X | T''_X$ and $\pi_X | T''_X = \pi'_X | T''_X$.
Now, we introduce the notion of the relization of abstract stratified sets as follows:

DEFINITION 6. An abstract stratified set $\{V, \mathcal{S}, \mathcal{T}\}$ is called to be realized in the Euclidean space R^N if there is a homeomorphism $F: V \to R^N$ and an equivalent abstract stratified set $\{V, \mathcal{S}, \mathcal{T}'\}$ to $\{V, \mathcal{S}, \mathcal{T}\}$ satisfying the following conditions:
(1) $V_1 = F(V)$ is a stratified set with a stratification $\mathcal{S}_1 = \{F(X); X \in \mathcal{S}\}$ and a

closed tubular neighbourhood system

$$\mathcal{T}_1=\{(T_{F(X)},\ \pi_{F(X)},\ \rho_{F(X)})\}\ .$$

(2) For each triple $(T_X, \pi_X, \rho_X)\in\mathcal{T}'$, F is a diffeomorphism of T_X into $T_{F(X)}$, compatible with the retractions π_X and $\pi_{F(X)}$, that is, $F\circ\pi_X=\pi_{F(X)}\circ F$.

(3) $\rho_X=\rho_{F(X)}\circ F$.

F is said to be a realization of $\{V, \mathcal{S}, \mathcal{T}\}$ in R^N.

DEFINITION 7. Two realizations F_0 and F_1 of $\{V, \mathcal{S}, \mathcal{T}\}$ in R^N are called isotopic if there exists a realization H of $\{V, \mathcal{S}, \mathcal{T}\}\times I$ in R^{N+1} which satisfies the following conditions:

(1) $H(x, t)=(H_t(x), t)$ where H_t is a realization of $\{V, \mathcal{S}, \mathcal{T}\}$ in R^N for each $t\in[0, 1]$.

(2) $H_0=F_0$ and $H_1=F_1$.

Remark. We can define naturally a stratification on $\{V, \mathcal{S}, \mathcal{T}\}\times I$ with a tubular neighbourhood system ([3]).

Our main result is the following.

THEOREM. *Every paracompact abstract stratified set* $\{V, \mathcal{S}, \mathcal{T}\}$ *with* $\dim V=n$ *can be realized in* R^{2n+1} *such that the image of the realization is the stratified set satisfying the Whitney condition. All realizations of* $\{V, \mathcal{S}, \mathcal{T}\}$ *in* R^N *are isotopic if* $N\geqq 2n+2$.

Remark 1. The dimension of V is the topological dimension.

Remark 2. More generally, every continuous mapping F from an n-dimensional abstract stratified set $\{V, \mathcal{S}, \mathcal{T}\}$ into R^{2n+1} can be approximated by a realization.

§ 2. **Preliminary lemmas.**

In this section, we prove several lemmas for the proof of the theorem.

LEMMA 1. *Let* $\pi: E\to M$ *be a fibre bundle with a compact manifold* F *as the fibre,* h *a diffeomorphism of* M *onto* M', S *a sphere bundle of* M' *and* p *the projection of* S *to* M'. *Suppose that* $\dim S\geqq 2\dim E+2$, *then we have an embedding* $\tilde{h}$ *of* E *into* S *such that* $h\circ\pi=p\circ\tilde{h}$. *Moreover, such an embedding* $\tilde{h}$ *can be obtained by extending any given embedding* $\tilde{h}_0: \pi^{-1}(A)\to p^{-1}(h(A))$ *for a closed set* A *in* M *with* $h\circ\pi=p\circ\tilde{h}_0$.

Proof. Let A be a closed set in M and $\tilde{h}_0$ a given embedding defined on $\pi^{-1}(A)$ into $p^{-1}(h(A))$ with $h\circ\pi=p\circ\tilde{h}_0$. Let U be an open set in M, and V a compact set with $V\subset U$ such that both fibre bundles π and p are trivialized in U and $h(U)$. Then, it is sufficient to prove that $\tilde{h}_0$ can be extended to an

embedding $\tilde{h}$ of a closed set $\pi^{-1}(A\cup V)$. Let φ be a trivialization $\pi^{-1}(U)\cong U\times F$ and ψ a trivialization $p^{-1}(h(U))\cong h(U)\times S^n$ with $n=\dim S-\dim M$. Define $\tilde{k}_0: (A\cap U)\times F\to S^n$ by $\tilde{k}_0=p'\circ\psi\circ h_0\circ\varphi^{-1}$ where p' is a natural projection from $h(U)\times S^n$ to S^n. Then $\tilde{k}_0$ can be extended to a smooth map $\tilde{k}_1$ on $U\times F$ because $\pi_j(S^n)=0$ for any $j<\dim E$. Moreover Thom's transversality theorem ([4]) assures that $\tilde{k}_1$ can be approximated by a mapping $\tilde{k}$ with $\tilde{k}=\tilde{k}_1$ on $(A\cap U)\times F$ and satisfying the following conditions:
(1) The cross-section $j_1\tilde{k}$ is transverse to Σ where Σ is the subset in $J^1(U\times F, S^n)$ consisting of the first jets of those mappings g which are of full rank as mappings of F into S^n by fixing points of U.
(2) The mapping $j_0\tilde{k}\times j_0\tilde{k}$ is transeverse to Σ_1 where Σ_1 is the subset in $J^0(U\times F, S^n)\times J^0(U\times F, S^n)$ consisting of all elements of the form $((u, f, s), (u, f', s))$.

As easy calculation shows, the transversality in the conditions (1) or (2) means the disjointness of $j_1\tilde{k}(U\times F)$ from Σ or $j_0\tilde{k}(U\times F)\times j_0\tilde{k}(U\times F)$ from Σ_1. Now, the desired embedding is obtained by defining $\tilde{h}=\tilde{h}_0$ on $\pi^{-1}(A)$ and $\tilde{h}=\psi^{-1}\circ(h\times\tilde{k})\circ\varphi$ on $\pi^{-1}(V)$ with $(h\times\tilde{k})(u, f)=(h(u), \tilde{k}(u, f))$. This completes the proof of Lemma 1.

We introduce the following two conditions for convenience.

DEFINITION 8. Let X, Y and Z be strata with $X>Y>Z$, ξ_Y a vector field on $T_Z\cap Y$ and ξ_X a vector field on $T_Z\cap X$. We say that ξ_X and ξ_Y are $\pi_{Y,X}$-related or ξ_X is $\pi_{Y,X}$-related to ξ_Y if $(\pi_{Y,X})_*\xi_X=\xi_Y$.

Now, let $T_X(t)$ denote the subset $\rho_X^{-1}(t)$ of T_X. Let X, Y and Z be strata with $X>Y>Z$, ξ a vector field on $T_Z\cap X$ and $\{\sigma_t\}$ one parameter family of local transformations defined by ξ. Suppose that for each point x in $T_Z\cap X$ there is a point x_0 in $T_Z(1)\cap X$ and a real number t with $\sigma_t(x_0)=x$.

DEFINITION 9. Under the above situation, a vector field η on $T_Z\cap T_Y\cap X$ is said to be a sliding of a vector field η_0 on $T_Z(1)\cap T_Y\cap X$ along ξ if $\eta(\sigma_t(x_0))=\sigma_{t*}(\eta_0(x_0))$ for every $x_0\in T_Z(1)$ and $t\in[0, T]$.

As J. Mather has shown in [2], we can choose, for a given abstract stratified set, an equivalent abstract stratified set satisfying the following condition A10 and A11.
A10. If X and Y are strata and $T_{Y,X}\neq\Phi$, then $Y<X$.
A11. If X and Y are strata and $T_X\cap T_Y\neq\Phi$, then $X<Y$, $Y<X$, or $X=Y$.

LEMMA 2. Let $\{V, \mathcal{S}, \mathfrak{T}\}$ be an abstract stratified set, satisfying A10 and A11. Then, there exists a family $\{\sigma_t^X\}$ of one parameter families of local transformations on T_X for each stratum X in $\mathcal{S}$ satisfying the following conditions:
(1) σ_t^Y is smooth on $T_Y\cap X$ for each X with $X>Y$.
(2) $\pi_X\circ\sigma_t^X(x)=\pi_X(x)$ for each x in T_X.
(3) $\rho_X\circ\sigma_t^X(x)=\rho_X(x)-t$ for $x\in T_X$ and $t\in[0, 1)$ whenever both sides of this equation are defined.

(4) If $X>Y$, $\pi_X\circ\sigma_t^Y(x)=\sigma_t^Y(\pi_X(x))$ for each x in $T_X\cap T_Y$.
(5) If $X>Y$, $\rho_Y\circ\sigma_t^X(x)=\sigma_Y(x)$ and $\sigma_s^Y\circ\sigma_t^X(x)=\sigma_t^X\circ\sigma_s^Y(x)$ for any x in $T_X\cap T_Y$.

Proof. It is sufficient to show that if the lemma is true for $\dim V\leqq k$, then it is true for $\dim V=k+1$. So we assume that $\dim V=k+1$. Let $\{\sigma_t^Z\}$ denote the one parameter family on $T_Z\cap V^{(k)}$ for each Z in $V^{(k)}$ satisfying the conditions (1)$\sim$(5). Here $V^{(k)}$ denotes the union of all strata Y of V with $\dim Y\leqq k$. Let $\eta_{Z,Y}$ be the vector field on $T_Z\cap Y$ which define $\{\sigma_t^Z\}$ on $T_Z\cap Y$. Let X be an stratum in $\mathcal{S}$ with $\dim X=k+1$ and $\mathcal{S}_X$ the set of all strata Y with $Y<X$. We define a smooth vector field $\eta_{Y,X}$ on $T_Y\cap X$ for each $Y\in\mathcal{S}_X$ satisfying the following conditions:
(a) $\eta_{Y,X}$ is tangential to the fibre of the projection $\pi_{Y,X}$.
(b) $\rho_Y\circ\sigma_t^Y(x)=\rho_Y(x)-t$ for each x in $T_Y\cap X$ and $t\in[0, 1)$.
(c) If $Z<Y<X$, $\eta_{Z,X}$ and $\eta_{Z,Y}$ are $\pi_{Y,X}$-related on $T_Z\cap T_Y\cap X$.
(d) If $Z<Y<X$, $\eta_{Y,X}$ is a sliding of $\eta_{Y,X}|T_Z(1)\cap T_Y\cap X$ along $\eta_{Z,X}$ restricted on $T_Z\cap T_Y\cap X$.

Let $\mathcal{S}_X$ be $\{Y_1, Y_2, \cdots, Y_q\}$ where the suffixes are chosen such that $i<j$ if $\dim Y_i<\dim Y_j$. For an element x in $T_{Y_1}\cap X$, $\mathcal{S}(x)$ denotes the set of strata Y in $\mathcal{S}_X$ such that $x\in T_Y$. Let $(c)_{Z,Y}$ denote the condition (c) for the triple $Z<Y<X$.

Now, we assign for each $x\in X\cap T_{Y_1}$ a neighbourhood $U(x)$ on which a vector field satisfying the conditions (a), (c) and (d) can be constructed. If $\mathcal{S}(x)=\{Y_1\}$, there is a neighbourhood $U(x)$ such that $\mathcal{S}(y)=\mathcal{S}(x)$ for each $y\in U(x)$. We construct, in this case, a vector field satisfying the condition (a) on $U(x)$. If $\mathcal{S}(x)\neq\{Y_1\}$, then there exists $Y_m\in\mathcal{S}(x)$ to which each stratum in $\mathcal{S}(x)$ is incident. Then we assign a neighbourhood $U(x)$ such that $\mathcal{S}(y)=\mathcal{S}(x)$ for each $y\in U(x)$ and construct a vector field on $U(x)$ satisfying conditions (a) and $(c)_{Y_1,Y_m}$. Let $\{U_\lambda\}$ be a locally finite refinement of the covering $\{U(x)\}$. We construct $\eta_{Y_1,X}$ satisfying the conditions (a), (c) and (d) by means of a partition of unity associated to $\{U_\lambda\}$ and a certain normalization such that $\eta_{Y_1,X}$ satisfies the condition (b). Now we assume that $\eta_{Y_j,X}$ is defined for each Y_j with $j<p$. Now we assign for each $x\in X\cap T_{Y_p}$ a neighbourhood on which a vector field satisfying the conditions (a), (c) and (d) can be constructed. Let A be the set of those points $x\in X\cap T_{Y_p}$ such that Y_p is incident to each stratum in $\mathcal{S}(x)$. We assign $U(x)$ for each $x\in A$ such that $\mathcal{S}(y)$ is equal to $\mathcal{S}(x)$ for each point y in $U(x)$ and construct a vector field on $U(x)$ satisfying (a) and $(c)_{Y_p,Y_m}$. Here Y_m denote the stratum in $\mathcal{S}(x)$ to which each stratum in $\mathcal{S}(x)$ is incident. If $\mathcal{S}(x)$ contains a stratum which is incident to Y_p, we put B the set of points $x\in T_{Y_{i_1}}(1)\cap\cdots\cap T_{Y_{i_r}}(1)\cap T_{Y_p}\cap T_{Y_{j_1}}\cap\cdots\cap T_{Y_{j_m}}$ with $\mathcal{S}(x)=\{Y_{i_1}, \cdots, Y_{i_r}, Y_p, Y_{j_1}, \cdots, Y_{j_m}\}$. For each point $x\in B$, we choose a neighbourhood $V(x)$ in $T_{Y_{i_1}}(1)\cap\cdots\cap T_{Y_{i_r}}(1)\cap T_{Y_{j_1}}\cap\cdots\cap T_{Y_{j_m}}$ such that each point y in $V(x)$ satisfies the condition A9 and $\mathcal{S}(y)=\mathcal{S}(x)$ and construct a vector field η_x on $V(x)$ satisfying (a) and $(c)_{Y_p,Y_m}$. We slide each η_x along

$\eta_{Y_{i_r},X}, \cdots, \eta_{Y_{i_2},X}$ and $\eta_{Y_{i_1},X}$ successively. Let $V^*(x)$ be the open set of all the elements of the form $\sigma_{t_1}^{Y_{i_1},X} \circ \cdots \circ \sigma_{t_r}^{Y_{i_r},X}(y)$ with $y \in V(x)$. We extend η_x over $V^*(x)$. It is clear that the family $\{U(x): x \in A\} \cup \{V^*(x): x \in B\}$ is an open covering of $T_{Y_p} \cap X$. Let $\{U_\lambda\}$ be a locally finite refinement of this covering. We construct $\eta_{Y_p,X}$ by means of a partition of unity associated to $\{U_\lambda\}$ and a certain normalization such that $\eta_{Y_p,X}$ satisfies (a)~(d). The required one parameter family $\{\sigma_t^X\}$ on $T_Y \cap X$ is generated by $\eta_{Y,X}$ for each $X \in \mathcal{S}$ and $Y \in \mathcal{S}_X$. By the construction, $\{\sigma_t^X\}$ satisfies the conditions (1)~(5). This completes the proof of the lemma.

§ 3. The proof of the theorem.

Let $\{V, \mathcal{S}, \mathcal{T}\}$ be an n-dimensional stratified set satisfying A10 and A11. Let $V^{(k)}$ denote the union of all strata X in $\mathcal{S}$ with $\dim X \leqq k$. Then, there is the number k_0 such that $V^{(k_0)}$ is the disjoint union of manifolds. Let $F^{(k_0)}$ be an embedding of $V^{(k_0)}$ into R^N, $\mathcal{S}(F^{(k_0)}) = \{F^{(k_0)}(X): X \subset V^{(k_0)}\}$ and $\mathcal{N}$ the closed tubular neighbourhood system $\{(N(X), p_X, \tau_X): X \subset V^{(k_0)}\}$ for $\mathcal{S}(F^{(k_0)})$ in R^N. We assume that there is a stratified realization $F^{(k)}$ of $V^{(k)}$ into R^N and that $\mathcal{S}(F^{(k)}) = \{F^{(k)}(X): X \subset V^{(k)}\}$. Let $\mathcal{N}$ be the closed tubular neighbourhood system $\{(N(X), p_X, \tau_X): X \subset V^{(k)}\}$ for $\mathcal{S}(F^{(k)})$ in R^N. As shown by Lemma 2, there exists a family of one parameter families of local transformations $\{\sigma_t^X\}_{X \in \mathcal{S}}$ for $\{V, \mathcal{S}, \mathcal{T}\}$ and there exists a family of one parameter families of local transformations $\{\beta_t^X\}_{X \in \mathcal{S}}$ for $\{F^{(k)}(V^{(k)}), \mathcal{S}(F^{(k)}), \mathcal{N}\}$. Let X be a stratum in $\mathcal{S}$ with $\dim X = k+1$. Now we show that there exists an embedding F_Y of $T_Y \cap X$ into $N(Y)$ for each $Y \in \mathcal{S}_X$, satisfying the following conditions:

(1) If $Z < Y$, then $F_Z = F_Y$ on $T_Z \cap T_Y \cap X$.

(2) $p_Y \circ F_Y = F^{(k)} \circ \pi_Y$.

(3) $\rho_Y(x) = \tau_Y \circ F_Y(x)$ for each x in $T_Y \cap X$. We denote the elements of $\mathcal{S}_X$ by $Z_1, Z_2, \cdots, Z_q$ where we have $i < j$ whenever $Z_i < Z_j$. Now we construct the embedding F_{Z_i} of $T_{Z_i} \cap X$ to N_{Z_i} successively. A sequence $(Y_1, \cdots, Y_l)$ of elements in $\mathcal{S}_X$ is said to be a chain of the length l if $Y_1 < Y_2 < \cdots < Y_l$. For a chain $C = (Y_1, \cdots, Y_l)$, we apply Lemma 1 to the bundle $\pi_{Y_1}: T_{Y_1}(1) \cap \cdots \cap T_{Y_l}(1) \cap X \to Y_1 \cap T_{Y_2}(1) \cap \cdots \cap T_{Y_l}(1) \cap X$ to obtain the embedding $F_C^{(1)}$ of $T_{Y_1}(1) \cap \cdots \cap T_{Y_l}(1) \cap X$ into $N_{Y_1}(1) \cap \cdots \cap N_{Y_l}(1)$. We can extend $F_C^{(1)}$ to the embedding F_C of $T_{Y_1} \cap \cdots \cap T_{Y_l} \cap X$ by $F_C(\sigma_{t1}^{Y_1} \cdots \sigma_t^Y(x)) = \beta_{t1}^{Y_1} \cdots \beta_t^Y(F_C^{(1)}(x))$. We define F_{Z_1} on $T_{Z_1} \cap T_{Y_1} \cap \cdots \cap T_{Y_k} \cap X$ by F_C for every $C = (Z_1, Y_1, \cdots, Y_k)$ with the maximum length. Now we define F_{Z_1} on $T_{Z_1} \cap T_{Y_1} \cap \cdots \cap T_{Y_{i-1}} \cap T_{Y_{i+1}} \cap \cdots \cap T_{Y_k} \cap X$. We apply Lemma 1 to the bundle $\pi_{Z_1}: T_{Z_1}(1) \cap T_{Y_1}(1) \cap \cdots \cap T_{Y_{i-1}}(1) \cap T_{Y_{i+1}}(1) \cap \cdots \cap T_{Y_k}(1) \cap X \to Z_1 \cap T_{Y_1}(1) \cap \cdots \cap T_{Y_{i-1}}(1) \cap T_{Y_{i+1}}(1) \cap \cdots T_{Y_k}(1) \cap X$ to extend the embedding F_{Z_1} on the union of $T_{Z_1}(1) \cap T_{Y_1}(1) \cap \cdots \cap T_{Y_{i-1}}(1) \cap T_Y \cap T_{Y_{i+1}}(1) \cap \cdots \cap T_{Y_k}(1) \cap X$ with $Y_{i-1} < Y < Y_{i+1}$ to $T_{Z_1}(1) \cap T_{Y_1}(1) \cap \cdots \cap T_{Y_{i-1}}(1) \cap T_{Y_{i+1}}(1) \cap \cdots \cap T_{Y_k}(1) \cap X$. Then we define F_{Z_1} on $T_{Z_1} \cap T_{Y_1} \cap \cdots \cap T_{Y_{i-1}} \cap$

$T_{Y_{i+1}} \cap \cdots \cap T_{Y_k} \cap X$ by means of the same procedure as in the extension of $F_C^{(1)}$ to F_C. In this way, we can extend F_{Z_1} step by step to reach finally the whole domain $T_{Z_1} \cap X$. Next we define F_{Z_2}. We define F_{Z_2} on $T_{Z_1} \cap T_{Z_2} \cap X$ by F_{Z_1}. Let $(Z_2, Y_1, \cdots, Y_m)$ be a chain with the maximum length beginning with Z_2. We define F_{Z_2} on $T_{Z_2} \cap T_{Y_1} \cap \cdots \cap T_{Y_m} \cap X$ by applying Lemma 1 to the bundle $\pi_{Z_2}: T_{Z_2}(1) \cap T_{Y_1}(1) \cap \cdots \cap T_{Y_m}(1) \cap X \to Z_2 \cap T_{Y_1}(1) \cap \cdots \cap T_{Y_m}(1) \cap X$ to extend F_{Z_2} on $T_{Z_1} \cap T_{Z_2}(1) \cap \cdots \cap X$ and by means of σ_t and β_t. The extension of F_{Z_2} to whole $T_{Z_2} \cap X$ can be otained by the same procedure as in the case of F_{Z_1}. Similarly we can define the embedding F_{Z_i} of $T_{Z_i} \cap X$. This completes the proof of the existence of $\{F_Y\}_{Y \in \mathcal{S}_X}$ and it is obvious by the construction that F_Y satisfies the conditions (1)~(3). We define the stratified realization F by $F|V^{(k)} = F^{(k)}$ and $F = F_Y$ on T_Y for each $Y \in \mathcal{S}_X$ with $\dim X = k+1$. Let $\underline{X}$ denote the closure of the difference $X - \cup \{T_Y : Y \in \mathcal{S}_X\}$. The set $\underline{X}$ is a cornered manifold in the sense of J. Cerf ([5]). As shown in [5], we can extend F to whole X and we can choose the closed tubular neighbourhood $N(X)$ of $F(X)$ in R^N with the projection p_X and the tubular function τ_X.

Now, we show that the image of the stratified realization satisfies the Whitney condition.

For any strata X and Y with $X > Y$ and each point y in Y, by the definition of the tubular neighbourhood and the construction of the realization F, there exists a diffeomorphism h from a neighbourhood U of $F(y)$ in R^N to R^N such that $h(F(X) \cap U) \cong h(U) \times R^{N-c_1}$ and $h(F(Y) \cap U) = h(U) \times R^{N-c_2}$ where $c_1 = \operatorname{codim} X$ and $c_2 = \operatorname{codim} Y$. Since the pair (R^{N-c_1}, R^{N-c_2}) with $c_1 < c_2$ obviously satisfies the Whitney condition, the pair (X, Y) also satisfies the same condition.

This completes the proof of the first part of the realization theorem. The remaining part of the theorem can be proved similarly.

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