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ANTI-INVARIANT SUBMANIFOLDS SATISFYING A CERTAIN CONDITION ON NORMAL CONNECTION

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§1. Introduction.

In a previous paper [3] the present author studied anti-invariant submanifolds of a $(2m+1)$-dimensional Sasakian manifold $\bar{M}$ with structure $(\phi, \xi, \eta, \bar{g})$ when the structure vector field ξ is tangent to the submanifolds everywhere.

An n-dimensional Riemannian manifold M isometrically immersed in $\bar{M}$ is said to be anti-invariant in $\bar{M}$ if $\phi T_x(M) \subset T_x(M)^{\perp}$ for each point x of M, where $T_x(M)$ and $T_x(M)^{\perp}$ denote respectively the tangent and the normal spaces to M at x. Thus, for any vector X tangent to M, ϕX is normal to M because of the definition given above. ϕ is necessarily of rank $2m$ and hence $n \leqq m+1$.

The purpose of the present paper is to study n-dimensional anti-invariant submanifolds normal to the structure vector field ξ of a $(2m+1)$-dimensional Sasakian manifold $\bar{M}$. If a submanifold M of $\bar{M}$ is normal to the structure vector field ξ, then M is anti-invariant in $\bar{M}$ as a consequence of Lemma 3.1. So, in this paper, we mean, by an anti-invariant submanifold M of a Sasakian manifold $\bar{M}$, a submanifold M normal to the structure vector field ξ of a Sasakian manifold $\bar{M}$.

§2. Sasakian manifolds.

First, we would like to recall definitions and some fundamental properties of Sasakian manifolds. Let $\bar{M}$ be a $(2m+1)$-dimensional differentiable manifold of class C^{∞} and ϕ, ξ, η be a tensor field of type (1, 1), a vector field, a 1-form on $\bar{M}$ respectively such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1 \tag{2.1}$$

for any vector field X on $\bar{M}$, where I denotes the identity tensor of type (1, 1). Then $\bar{M}$ is said to admit an *almost contact structure* (ϕ, ξ, η) and called an *almost contact manifold*. The almost contact structure is said to be *normal* if

$$N + d\eta \otimes \xi = 0, \tag{2.2}$$

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where N denotes the Nijenhuis tensor formed with ϕ. If there is given in $\bar{M}$ a Riemannian metric $\bar{g}$ satisfying

$$\bar{g}(\phi X, \phi Y)=\bar{g}(X, Y)-\eta(X)\eta(Y), \qquad \eta(X)=\bar{g}(X, \xi) \tag{2.3}$$

for any vector fields X and Y on $\bar{M}$, then the set $(\phi, \xi, \eta, \bar{g})$ is called a *almost contact metric structure* and $\bar{M}$ an *almost contact metric manifold*. If

$$d\eta(X, Y)=\bar{g}(\phi X, Y) \tag{2.4}$$

for any vector fields X and Y on $\bar{M}$, then the almost contact metric structure is called a *contact metric structure*. If the structure is moreover normal, then the contact metric structure is called a *Sasakian structure* and $\bar{M}$ a *Sasakian manifold*. As is well known, in a Sasakian manifold $\bar{M}$ with structure $(\phi, \xi, \eta, \bar{g})$ the equations

$$\bar{\nabla}_X\xi=\phi X, \qquad (\bar{\nabla}_X\phi)Y=-\bar{g}(X, Y)+\eta(Y)X \tag{2.5}$$

are established for any vector fields X and Y on $\bar{M}$, where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to $\bar{g}$.

A plane section σ in the tangent space $T_x(\bar{M})$ of a Sasakian manifold $\bar{M}$ at x is called a *ϕ-section* if it is spanned by vectors X and ϕX, where X is assumed to be orthogonal to ξ. The sectional curvature $K(\sigma)$ with respect to a ϕ-section σ is called a *ϕ-sectional curvature*. When the ϕ-sectional curvature $K(\sigma)$ is independent of the ϕ-section σ at each point of $\bar{M}$, as is well known, the function $K(\sigma)$ defined in $\bar{M}$ is necessarily a constant c. A Sasakian manifold $\bar{M}$ is called a *Sasakian space form* and denoted by $\bar{M}(c)$ if it has constant ϕ-sectional curvature c (see [4]). The curvature tensor K of a Sasakian space form $\bar{M}(c)$ is given by

$$\begin{aligned} K(X, Y)Z=\frac{1}{4}(c+1)(\bar{g}(Y, Z)X-\bar{g}(X, Z)Y)-\frac{1}{4}(c-1)(&\eta(Y)\eta(Z)X \\ &-\eta(X)\eta(Z)Y+\bar{g}(Y, Z)\eta(X)\xi-\bar{g}(X, Z)\eta(Y)\xi \\ &-\bar{g}(\phi Y, Z)\phi X+\bar{g}(\phi X, Z)\phi Y+2\bar{g}(\phi X, Y)\phi Z). \end{aligned}$$

EXAMPLE 1. Let S^{2n+1} be a $(2n+1)$-dimensional unit sphere, i.e.,

$$S^{2n+1}=\{z\in C^{n+1} : |z|=1\},$$

where C^{n+1} is a complex $(n+1)$-space. For any point $z\in S^{2n+1}$, we put $\xi=Jz$, J being the complex structure of C^{n+1}. Considering the orthogonal projection

$$\pi : T_z(C^{n+1}) \longrightarrow T_z(S^{2n+1}),$$

at each point z in S^{2n+1} and putting $\phi=\pi\circ J$, we have a Sasakian structure (ϕ, ξ, η, g) on S^{2n+1}, where η is a 1-form dual to ξ and g the standard metric tensor field on S^{2n+1}. Obviously, S^{2n+1} is of constant ϕ-sectional curvature 1.

EXAMPLE 2. Let E^{2n+1} be a Euclidean space with cartesian coordinates $(x^1, \cdots, x^n, y^1, \cdots, y^n, z)$. Then a Sasakian structure on E^{2n+1} is defined by ϕ, ξ, η and g such that

$$\xi=(0, \cdots, 0, 2), \qquad 2\eta=(-y^1, \cdots, -y^n, 0, \cdots, 0, 1),$$

$$(g_{AB})=\begin{pmatrix} \frac{1}{4}(\delta_{ij}+y^i y^j) & 0 & -\frac{1}{4}y^i \\ 0 & \frac{1}{4}\delta_{ij} & 0 \\ -\frac{1}{4}y^j & 0 & \frac{1}{4} \end{pmatrix},$$

$$(\phi^A_B)-\begin{pmatrix} 0 & \delta^i_j & 0 \\ -\delta^i_j & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}.$$

Then E^{2n+1} with such a structure (ϕ, ξ, η, g) is of constant ϕ-sectional curvature -3 and denoted by $E^{2n+1}(-3)$.

§ 3. Fundamental properties.

Let $\bar{M}^{2m+1}$ be a Sasakian manifold of dimension $2m+1$ with structure $(\phi, \xi, \eta, \bar{g})$. An n-dimensional Riemannian manifold M isometrically immerssed in $\bar{M}^{2m+1}$ is said to be *anti-invariant* in $\bar{M}^{2m+1}$ if $\phi T_x(M) \subset T_x(M)^\perp$ for each point x of M. Throughout the paper, we now restrict ourselves only to anti-invariant submanifolds of a Sasakian manifold such that the structure vector field ξ of the ambient manifold is normal to the submanifolds.

Let g be the induced metric tensor field of M. We denote by $\bar{\nabla}$ (resp. ∇) the operator of covariant differentiation with respect to $\bar{g}$ (resp. g). Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y=\nabla_X Y+B(X, Y), \qquad \bar{\nabla}_X N=-A_N(X)+D_X N$$

for any vector fields X, Y tangent to M and any vector field N normal to M, where D is the operator of covariant differentiation with respect to the liner connection induced in the normal bundle. Both A and B are called the second fundamental form of M. They satisfy $\bar{g}(B(X, Y), N)=g(A_N(X), Y)$.

First of all, we prove

LEMMA 3.1. ([2, 5]) *Let M be an n-dimensional submanifold of a Sasakian manifold M^{2m+1}. If the structure vector field ξ of the ambient manifold is normal to M everywhere, then M is an anti-invariant submanifold of $\bar{M}^{2m+1}$ and $n \leqq m$.*

Proof. Since the structure vector field ξ is normal to M everywhere, we have

$$\bar{g}(\phi X, Y)=\bar{g}(\bar{\nabla}_X\xi, Y)=g(-A_\xi(X), Y)+\bar{g}(D_X\xi, Y)=-g(A_\xi(X), Y)$$

for any vector fields X and Y tangent to M. Since A_ξ is symmetric and ϕ is skew-symmetric, we have $A_\xi=0$ and ϕX is normal to M. Thus M is anti-invariant and $n\leqq m$.

Throughout the paper, by an anti-invariant submanifold M of a Sasakian manifold $\bar{M}^{2m+1}$, we mean a submanifold M such that the structure vector field ξ of the ambient manifold is normal to M.

We choose a local field of orthonormal frames $e_1, \cdots, e_n; e_{n+1}, \cdots, e_m; e_{0^*}=\xi, e_{1^*}=\phi e_1, \cdots, e_{n^*}=\phi e_n; e_{(n+1)^*}=\phi e_{n+1}, \cdots, e_{m^*}=\phi e_m$ in $\bar{M}^{2m+1}$ in such a way that $e_1, \cdots, e_n$ are along M tangent to M. Taking such a field of frames of $\bar{M}^{2m+1}$, we denote the dual coframes by $\omega^1, \cdots, \omega^n; \omega^{n+1}, \cdots, \omega^m; \omega^{0^*}=\eta, \omega^{1^*}, \cdots, \omega^{n^*}; \omega^{(n+1)^*}, \cdots, \omega^{m^*}$. Unless otherwise stated, let the range of indices be as follows:

$$A, B, C, D=1, \cdots, m, 0^*, 1^*, \cdots, m^*,$$
$$i, j, k, l, s, t=1, \cdots, n,$$
$$a, b, c, d=n+1, \cdots, m, 0^*, 1^*, \cdots, m^*,$$
$$p, q, r=n+1, \cdots, m, 1^*, \cdots, m^*,$$
$$\lambda, \mu, \nu=n+1, \cdots, m, 0^*, (n+1)^*, \cdots, m^*,$$
$$x, y, z=n+1, \cdots, m, (n+1)^*, \cdots, m^*,$$
$$\alpha, \beta, \gamma=n+1, \cdots m,$$

and use the so-called summation convention for these systems of indices. Then the structure equations of the Riemannian manifold $\bar{M}^{2m+1}$ are given by

$$d\omega^A=-\omega_B^A\wedge\omega^B, \qquad \omega_B^A+\omega_A^B=0, \tag{3.1}$$

$$d\omega_B^A=-\omega_C^A\wedge\omega_B^C+\Phi_B^A, \qquad \Phi_B^A=\frac{1}{2}K_{BCD}^A\omega^C\wedge\omega^D, \tag{3.2}$$

where K_{BCD}^A are components of the curvature tensor of $\bar{M}^{2m+1}$ with respect to $\{e_A\}$ and ω_B^A satisfy

$$\begin{aligned} &\omega_j^i=\omega_{j^*}^{i^*}, && \omega_j^{i^*}=\omega_i^{j^*}, && \omega^i=\omega_{0^*}^{i^*}, && \omega^{i^*}=-\omega_{0^*}^i, \\ &\omega_\beta^\alpha=\omega_{\beta^*}^{\alpha^*}, && \omega_\beta^{\alpha^*}=\omega_\alpha^{\beta^*}, && \omega^\alpha=\omega_{0^*}^{\alpha^*}, && \omega^{\alpha^*}=-\omega_{0^*}^\alpha, \\ &\omega_\alpha^i=\omega_{\alpha^*}^{i^*}, && \omega_\alpha^{i^*}=\omega_i^{\alpha^*}. \end{aligned} \tag{3.3}$$

Thus we have along M

$$\omega^a=0, \tag{3.4}$$

which implies $0=d\omega^a=-\omega_i^a\wedge\omega^i$ along M. Thus, by Cartan's lemma, we obtain along M

$$\omega_i^a = h_{ij}^a \omega^j, \qquad h_{ij}^a = h_{ji}^a, \tag{3.5}$$

which imply the following structure equations of the submanifold M;

$$d\omega^i = -\omega_j^i \wedge \omega^j, \qquad \omega_j^i + \omega_i^j = 0, \tag{3.6}$$

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k + \Omega_j^i, \qquad \Omega_j^i = \frac{1}{2} R_{jkl}^i \omega^k \wedge \omega^l, \tag{3.7}$$

$$R_{jkl}^i = K_{jkl}^i + \sum_a (h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a), \tag{3.8}$$

$$d\omega_b^a = -\omega_c^a \wedge \omega_b^c + \Omega_b^a, \qquad \Omega_b^a = \frac{1}{2} R_{bkl}^a \omega^k \wedge \omega^l, \tag{3.9}$$

$$R_{bkl}^a = K_{bkl}^a + \sum_i (h_{ik}^a h_{il}^b - h_{il}^a h_{ik}^b), \tag{3.10}$$

where R_{jkl}^i are components of the curvature tensor of M with respect to $\{e_i\}$ and R_{bkl}^a components of the curvature tensor of the normal bundle with respect to $\{e_i\}$ and $\{e_a\}$. The equations (3.8) and (3.10) are called respectively the equations of Gauss and those of Ricci for the submanifold M. The forms (ω_j^i) define the Riemannian connection of M and the forms (ω_b^a) define the connection induced in the normal bundle of M.

From (3.3), (3.4) and (3.5) we have along M

$$h_{jk}^i = h_{ki}^j = h_{ij}^k, \qquad h_{ij}^{0^*} = 0, \tag{3.11}$$

where we donote $h_{jk}^{i^*}$ simply by h_{jk}^i.

The second fundamental form $h_{ij}^a \omega^i \omega^j e_a$ is sometimes denoted by its components h_{ij}^a. If the second fundamental form vanishes identically, i. e., $h_{ij}^a = 0$ for all indices, then the submanifold is as usual said to be *totally geodesic*. If h_{ij}^a have the form $h_{ij}^a = \frac{1}{n}(\sum_k h_{kk}^a)\delta_{ij}$ for a fixed index a, then the submanifold is said to be *umbilical* with respect to the normal vector e_a. If the submanifold M is umbical with respect to all e_a, then M is said to be *totally umbilical*. The vector field $\frac{1}{n}(\sum_k h_{kk}^a e_a)$ normal to M is called the *mean curvature vector* of M. The submanifold M is said to be *minimal* if its mean curvature vector vanishes identically, i. e., $\sum_k h_{kk}^a = 0$ for all a. We define the covariant derivative h_{ijk}^a of h_{ij}^a by

$$h_{ijk}^a \omega^k = dh_{ij}^a - h_{lj}^a \omega_i^l - h_{il}^a \omega_j^l + h_{ij}^b \omega_b^a. \tag{3.12}$$

If $h_{ijk}^a = 0$ for all indices, the second fundamental form of M is said to be *parallel*. If the mean curvature vector of M is parallel with respect to the connection in the normal bundle, then the mean curvature vector of M is said to be *parallel*. From (3.3), (3.4), (3.11) and (3.12), we obtain

$$h_{ijk}^{0^*} = -h_{ij}^k. \tag{3.13}$$

Thus, we have

LEMMA 3.2. ([6]) *Let M be an n-dimensional anti-invariant submanifold of a Sasakian manifold $\bar{M}^{2n+1}$. If the second fundamental form of M is parallel, then M is totally geodesic.*

Using (3.13), we obtain

$$\sum_k h^{0^*}_{kki} = -\sum_k h^i_{kk}. \tag{3.14}$$

Thus, we have

LEMMA 3.3. ([7]) *Let M be an n-dimensional anti-invariant submanifold of a Sasakian manifold $\bar{M}^{2n+1}$. If the mean curvature vector of M is parallel, then M is minimal.*

Because of Lemmas 3.2 and 3.3, the conditions that the second fundamental form is parallel and that the mean curvature vector is parallel are not interesting for anti-invariant submanifolds, when $m=n$. Therefore we shall now introduce some new notions as follows. On an anti-invariant submanifold M of a Sasakian manifold $\bar{M}^{2m+1}$, if $h^p_{ijk}=0$ for all indices, then we say that the second fundamental form of M is *η-parallel*. If $\sum_k h^p_{kki}=0$ for all indices i and p, then the mean curvature vector said to be *η-parallel*.

We now define the Laplacian Δh^a_{ij} of h^a_{ij} by

$$\Delta h^a_{ij} = \sum_k h^a_{ijkk}, \tag{3.15}$$

where we have defined h^a_{ijkl} by

$$h^a_{ijkl}\omega^l = dh^a_{ijk} - h^a_{ljk}\omega^l_i - h^a_{ilk}\omega^l_j - h^a_{ijl}\omega^l_k + h^b_{ijk}\omega^a_b. \tag{3.16}$$

We shall establish a formula containing the Laplacian of h^a_{ij}. In the sequel the second fundamental form of M is assumed to satisfy the equation of Codazzi type, i. e.,

$$h^a_{ijk} - h^a_{ikj} = 0. \tag{3.17}$$

Then, from (3.16), we have

$$h^a_{ijkl} - h^a_{ijkl} = h^a_{tj}R^t_{ikl} + h^a_{it}R^t_{jkl} - h^b_{ij}R^a_{bkl}. \tag{3.18}$$

On the other hand, (3.15) and (3.17) imply

$$\Delta h^a_{ij} = \sum_k h^a_{ijkk} = \sum_k h^a_{kijk}. \tag{3.19}$$

From (3.17), (3.18) and (3.19), we obtain

$$\Delta h^a_{ij} = \sum_k (h^a_{kkij} + h^a_{kt}R^t_{ijk} + h^a_{ti}R^t_{kjk} - h^b_{ki}R^a_{bjk}). \tag{3.20}$$

Therefore for any submanifold M satisfying the equation (3.17) of Codazzi type we have the formula

$$\sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a = \sum_{a,i,j,k} (h_{ij}^a h_{kkij}^a + h_{ij}^a h_{kt}^a R_{ijk}^t + h_{ij}^a h_{ti}^a R_{kjk}^t - h_{ij}^a h_{ki}^b R_{bjk}^a). \tag{3.21}$$

If the ambient manifold $\bar{M}^{2m+1}$ is of constant ϕ-sectional curvature c, then the Riemannian curvature tensor of $\bar{M}^{2m+1}$ has the form

$$\begin{aligned} K_{BCD}^A = \frac{1}{4}(c+3)(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) + \frac{1}{4}(c-1)(&\eta_B\eta_C\delta_{AD} \\ &- \eta_B\eta_D\delta_{AC} + \eta_A\eta_D\delta_{BC} - \eta_A\eta_C\delta_{BD} + \phi_{AC}\phi_{BD} \\ &- \phi_{AD}\phi_{BC} + 2\phi_{AB}\phi_{CD}), \end{aligned} \tag{3.22}$$

and the second fundamental form of M satisfies the equation (3.17) of Codazzi type.

§ 4. Normal connection.

In this section we study the normal connection of an n-dimensional anti-invariant submanifold M of a $(2m+1)$-dimensional Sasakian space form $\bar{M}^{2m+1}(c)$ when the structure vector field ξ is normal to M. The curvature tensor of the normal connection of M is assumed to have the form

$$R_{bkl}^a = -(\delta_{ak^*}\delta_{bl^*} - \delta_{al^*}\delta_{bk^*}). \tag{4.1}$$

LEMMA 4.1. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian manifold $\bar{M}^{2m+1}$. If the curvature tensor of the normal connnection of M is of the form* (4.1), *then*

$$R_{jkl}^i = \sum_x (h_{ik}^x h_{jl}^x - h_{il}^x h_{jk}^x). \tag{4.2}$$

Proof. (3.2) and (3.3) imply

$$K_{jkl}^i = K_{j^*kl}^{i^*} + (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{4.3}$$

Moreover, from (3.8), (3.10) and (3.11), we obtain

$$\begin{aligned} R_{jkl}^i &= K_{jkl}^i + \sum_a (h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a) \\ &= K_{jkl}^i + \sum_t (h_{tk}^i h_{tl}^j - h_{tl}^i h_{tk}^j) + \sum_x (h_{ik}^x h_{il}^x - h_{il}^x h_{jk}^x) \\ &= R_{i^*kl}^{j^*} + (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_x (h_{ik}^x h_{jl}^x - h_{il}^x h_{jk}^x), \end{aligned} \tag{4.4}$$

which proves Lemma 4.1.

By (3.22) we obtain

$$(4.5)\qquad K^{\lambda}_{j^*kl}=0\,,\qquad K^{\lambda}_{jkl}=0\,,\qquad K^{\lambda}_{\mu kl}=0\,,$$

$$K^{i^*}_{j^*kl}=\frac{1}{4}(c-1)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\,.$$

If the curvature tensor of the normal connection of M is of the form (4.1), then (3.10) and (4.5) imply

$$(4.6)\qquad \sum_t (h^x_{tk}h^t_{tl}-h^x_{tl}h^t_{tk})=0\,,\qquad \sum_t (h^x_{tk}h^y_{tl}-h^x_{tl}h^y_{tk})=0\,,$$

$$(4.7)\qquad \sum_t (h^i_{tk}h^j_{tl}-h^i_{tl}h^j_{tk})=-\frac{1}{4}(c+3)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\,.$$

PROPOSITION 4.2. *Let M be an n-dimensional ($n>1$) anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$. If the curvature tensor of the normal connection of M has the form* (4.1) *and M is umbilical with respect to some e_{t^*}, then $c=-3$.*

Proof. If M is umbilical with respect to e_{t^*}, then the second fundamental form h^t_{ij} is of the form $h^t_{ij}=\frac{1}{n}(\sum_k h^t_{kk})\delta_{ij}$. Thus we have

$$\sum_i (h^t_{ik}h^s_{il}-h^t_{il}h^s_{ik})=0\,.$$

From this and (4.7) we find $c=-3$.

For each fixed index a, we consider a symmetric (n, n)-matrix $A_a=(h^a_{ij})$ composed of components of the second fundamental form.

LEMMA 4.3. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ ($c\neq-3$). If the curvature tensor of the normal connection of M is of the form* (4.1), *then M is umbilical with respect to all e_x.*

Proof. From (4.6) we obtain $A_xA_y=A_yA_x$ and $A_xA_1=A_1A_x$ for all x and y. Therefore we can choose a local field of orthonormal frames with respect to which A_1 and all A_x are diagonal, i. e.,

$$(4.8)\qquad A_1=\begin{pmatrix} h^1_{11} & & 0\\ & \ddots & \\ 0 & & h^1_{nn}\end{pmatrix},\qquad A_x=\begin{pmatrix} h^x_{11} & & 0\\ & \ddots & \\ 0 & & h^x_{nn}\end{pmatrix}.$$

Putting $i=l$ and $k=l$ in the first equation of (4.6) and using (3.11) and (4.8), we find

$$(4.9)\qquad (h^x_{11}-h^x_{ii})h^1_{ii}=0\,.$$

On the other hand, putting $i=k=1$ and $j=l\neq1$ in (4.7) and using (3.11) and (4.8), we have

(4.10) $$(h^1_{11}-h^1_{jj})h^1_{jj}=-\frac{1}{4}(c+3).$$

Since $c\neq -3$, (4.10) implies that $h^1_{jj}\neq 0$ ($j=2, \cdots, n$). From this fact and (4.9) we find that $h^x_{ii}=h^x_{jj}$ for all x. Thus M is umbilical with respect to all e_x. This proves Lemma 4.3.

LEMMA 4.4. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space from $\bar{M}^{2m+1}(c)$ ($c\neq -3$). If the curvature tensor of the normal connection of M is of the form* (4.1), *then*

(4.11) $$R^i_{jkl}=\frac{1}{n^2}\sum_x (\mathrm{Tr}\, A_x)^2(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}).$$

Proof. Lemma 4.3 implies $h^x_{ij}=\frac{1}{n}(\mathrm{Tr}\, A_x)\delta_{ij}$ for all x. Therefore (4.2) implies (4.11).

In the Lemma 4.4, if $n\geqq 3$, then $\sum_x (\mathrm{Tr}\, A_x)^2$ is constant. Therefore we have

PROPOSITION 4.5 *Let M be an n-dimensional ($n\geqq 3$) anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ ($c\neq -3$). If the curvature tensor of the normal connection of M has the form* (4.1), *then M is of constant curvature.*

If M is minimal, then $\mathrm{Tr}\, A_x=0$ for all x. Thus Lemma 4.4 implies immediately

PROPOSITION 4.6. *Let M be an n-dimensional anti-invariant minimal submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ ($c\neq -3$). If the curvature tensor of the normal connection of M has the form* (4.1), *then M is flat.*

§ 5. η-parallel mean curvature vector.

Using the results obtain in the previous section, we have

THEOREM 1. *Let M be an n-dimensional ($n\geqq 3$) anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ ($c\neq -3$) with η-parallel mean curvature vector. If the curvature tensor of the normal connection of M is of the form* (4.1), *then there is in $\bar{M}^{2m+1}(c)$ a totally geodesic and invariant submanifold $\bar{M}^{2n+1}(c)$ of dimension $2n+1$ in such a way that M is immersed in $\bar{M}^{2n+1}(c)$ as a flat anti-invariant minimal submanifold.*

Proof. First of all, $\sum_a (\mathrm{Tr}\, A_a)^2$ is constant because the mean curvature vector is η-parallel. Since $n\geqq 3$, $\sum_x (\mathrm{Tr}\, A_x)^2$ is constant. On the other hand, from (3.8), (3.22) and (4.11), we have

$$(5.1)\qquad \frac{n-1}{n}\sum_x (\mathrm{Tr}\, A_x)^2=\frac{1}{4}n(n-1)(c+3)+\sum_a (\mathrm{Tr}\, A_a)^2-\sum_{a,i,j}(h_{ij}^a)^2 .$$

Therefore the square of the length of the second fundamental form of M is constant, i. e., $\sum_{a,i,j}(h_{ij}^a)^2$ is constant. From this we have

$$(5.2)\qquad \sum_{a,i,j,k}(h_{ijk}^a)^2+\sum_{a,i,j}h_{ij}^a\Delta h_{ij}^a=\frac{1}{2}\Delta\sum_{a,i,j}(h_{ij}^a)^2=0 .$$

By assumption, (3.11) and (3.21), we have

$$(5.3)\qquad \sum_{a,i,j}h_{ij}^a\Delta h_{ij}^a=\sum_{a,i,j,k}(h_{ij}^a h_{kt}^a R_{ijk}^t+h_{ij}^a h_{ti}^a R_{kjk}^t)$$
$$+\sum_{i,j,k}(h_{ii}^k h_{jj}^k-(h_{ij}^k)^2) .$$

Moreover substituting (4.11) into (5.3) and using (5.2), we obtain

$$(5.4)\qquad \sum_{a,i,j,k}(h_{ijk}^a)^2=-\frac{1}{n^2}\sum_x (\mathrm{Tr}\, A_x)^2\sum_{a,i,j}(n(h_{ij}^a)^2-h_{ii}^a h_{jj}^a)$$
$$-\sum_{i,j,k}(h_{ii}^k h_{jj}^k-(h_{ij}^k)^2) .$$

From Lemma 4.3 and (3.13), we have

$$(5.5)\qquad \sum_{p,i,j,k}(h_{ijk}^p)^2=-\frac{1}{n^2}\sum_x (\mathrm{Tr}\, A_x)^2\sum_{i,j,k}(n(h_{ij}^k)^2-h_{ii}^k h_{jj}^k)-\sum_{i,j,k}h_{ii}^k h_{jj}^k$$
$$=-\frac{1}{n^2}\sum_x (\mathrm{Tr}\, A_x)^2\sum_k\Big(\sum_{i<j}(h_{ii}^k-h_{jj}^k)^2+n\sum_{i\neq j}(h_{ij}^k)^2\Big)$$
$$-\sum_k (\mathrm{Tr}\, A_{k^*})^2 .$$

Since $c\neq -3$ by assumption, Proposition 4.2 implies $\sum_{i<j}(h_{ii}^k-h_{jj}^k)^2>0$. Thus (5.5) implies $\mathrm{Tr}\, A_x=0$, $\mathrm{Tr}\, A_{k^*}=0$ and $h_{ijk}^p=0$, that is, the second fundamental form is η-parallel. Lemma 4.4, $\mathrm{Tr}\, A_x=0$, $\mathrm{Tr}\, A_{k^*}=0$ and (3.11) mean that M is flat and minimal. On the other hand, by Lemma 4.3, $\mathrm{Tr}\, A_x=0$ implies $A_x=0$ for all x. From (3.5) and $A_x=0$, we obtain $\omega_i^x=0$ and hence $\omega_{i^*}^x=0$ along M, by (3.3). Moreover, (3.3) and (3.4) imply $\omega_{0^*}^x=0$ along M. From the arguments above, taking account of a fundamental theorem in the theory of submanifolds, we see that M is an anti-invariant submanifold immersed in some totally geodesic and $(2n+1)$-dimensional submanifold $\bar{M}^{2n+1}(c)$ of $\bar{M}^{2m+1}(c)$ (see §6 in [3]). And the submanifold $\bar{M}^{2n+1}(c)$ is invariant (see §6 in [3]). Thus Theorem 1 is proved.

In Theorem 1, the case where $n=2$, i. e., where M is 2-dimensional, is excluded. However, the same conclusions will be established even if $n=2$, provided that M is compact. To establish this fact, we now prove

THEOREM 2. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c\neq -3)$ with η-parallel mean curvature vector and assumed to be compact. If the curvature tensor of the normal connection of M is of the form* (4.1), *then M is a flat anti-invariant minimal submanifold of a certain $(2n+1)$-dimensional totally geodesic submanifold $\bar{M}^{2n+1}(c)$ of $\bar{M}^{2m+1}(c)$.*

Proof. Since M is compact, we have

$$\int_M \sum_{a,i,j,k} (h^a_{ijk})^2 *1 = -\int_M \sum_{a,i,j} h^a_{ij} \Delta h^a_{ij} *1\,,$$

where $*1$ denotes the volume element of M (see (5.2)). Using this formula, we can prove Theorem 2 by a same way as taken to prove Theorem 1.

We shall now consider the case where $c=-3$.

PROPOSITION 5.1. *Let M be an n-dimensional $(n\geqq 3)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(-3)$ with η-parallel mean curvature vector and the curvature tensor of the normal connection of M be of the form* (4.1). *If M is umbilical with respect to all e_x, then M is a totally umbilical anti-invariant submanifold.*

Proof. From (3.10), (3.22) and (4.1), we obtain

$$\sum_t (h^a_{tk} h^b_{tl} - h^a_{tl} h^b_{tk}) = 0\,.$$

Therefore we can choose a local field of orthonormal frames with respect to which all A_a are simultaneously diagonal, i. e.,

$$A_a = \begin{pmatrix} h^a_{11} & & 0 \\ & \ddots & \\ 0 & & h^a_{nn} \end{pmatrix}$$

Moreover, (3.11) implies that $h^i_{jk}=0$ unless $i=j=k$. On the other hand, from the assumption and (4.2) we have the equation (4.11). Therefore the equation (5.5) holds and hence we have

$$\sum_{p,i,j,k} (h^p_{ijk})^2 = -\sum_k (h^k_{kk})^2 \Big(\frac{n-1}{n} \sum_x (\mathrm{Tr}\, A_x)^2 + 1\Big)\,. \tag{5.6}$$

Therefore we have $h^k_{kk}=0$, that is, $A_{k^*}=0$ for all k. Thus M is totally umbilical.

Remark. In Proposition 5.1, the case where $n=2$, that is, where M is 2-dimensional, is excluded. However, the same conclusions are established even if $n=2$, provided that M is compact.

EXAMPLE 5.2. Let J be the almost complex structure of the complex $(n+1)$-space C^{n+1} given by

$$J=\begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix}$$

Let S^{2n+1} be a $(2n+1)$-dimensional unit sphere in C^{n+1} with standard Sasakian structure $(\phi, \xi, \eta, \bar{g})$. Let S^1 be a circle of radius 1. Let us consider

$$T^n=S^1\times\cdots\times S^1 .$$

Then we construct an isometric minimal immersion of T^n into S^{2n+1} which is anti-invariant in the following way. Let $X: T^n\to S^{2n+1}$ be a minimal immersion represented by

$$X=\frac{1}{\sqrt{n+1}}(\cos u^1, \sin u^1, \cdots, \cos u^n, \sin u^n, \cos u^{n+1}, \sin u^{n+1}),$$

where we have put $u^{n+1}=-(u^1+\cdots+u^n)$. We may regard X as a position vector of S^{2n+1} in C^{n+1}. The structure vector field ξ of S^{2n+1}, restricted to T^n, is then given by

$$\xi=JX=\frac{1}{\sqrt{n+1}}(-\sin u^1, \cos u^1, \cdots, -\sin u^n, \cos u^n, -\sin u^{n+1}, \cos u^{n+1}).$$

Putting $X_i=\partial X/\partial u^i$, we have

$$X_i=\frac{1}{\sqrt{n+1}}(0, \cdots, 0, -\sin u^i, \cos u^i, 0, \cdots, 0, \sin u^{n+1}, -\cos u^{n+1}),$$

where $i=1, \cdots, n$. Thus X_i, $i=1, \cdots, n$, are linearly independent and $\eta(X_i)=0$ for $i=1, \cdots, n$. Therefore the immersion X is anti-invariant.

The integral curves of the structure vector field ξ are great circles S^1 in S^{2n+1} which are the fibres of the standard fibration $\pi: S^{2n+1}\to CP^n$, where CP^n is a complex projective space of complex dimension n and of constant holomorphic sectional curvature 4. Now we consider the following diagram:

$$\begin{array}{ccc} T^n & \xrightarrow{\ X\ } & S^{2n+1} \\ & \searrow & \downarrow \pi \\ & & CP^n . \end{array}$$

We easily see that $\pi|_{X(T^n)}$ is one to one. Consequently T^n is imbedded in CP^n by $\pi\circ X$.

By Theorems 1, 2 and Example 5.2, we have

THEOREM 3. *Let M be an n-dimensional compact orientable anti-invariant submanifold of a Sasakian space form S^{2m+1} with η-parallel mean curvature vector. If the curvature tensor of the normal connection of M is of the form* (4.1), *then M is a torus $S^1\times\cdots\times S^1$ in some S^{2n+1} in S^{2m+1}.*

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