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THE HORIZONTAL HOLONOMY GROUP OF A FIBRE BUNDLE SPACE

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§1. Introduction.

On a differentiable manifold M, the parallel translation of a vector along a curve C has been studied in many papers and books. A topological group $H(M)$ was assigned to this manifold M. And we call $H(M)$ the linear holonomy group of M. Nijenhuis, in his paper ([1]), found out that the Lie algebra of the restricted holonomy group $H^0(M)$ of $H(M)$ is formed by the curvature tensor of M. On the other hand, if we consider the frame bundle $B(M)$ as a principle fibre bundle over M, then the Nijenhuis's theorem can be restated as follows: The holonomy Lie algebra of $H^0(M)$ is generated by the curvature form Ω of $B(M)$ ([2]).

We know that a principle fibre bundle is only a special case of a fibre bundle space. Hence, the ideal of the linear holonomy group can be extended to the fibre bundle space. Actually, if we consider any fibre bundle space $(\tilde{M}, M, \pi)$ such that in $\tilde{M}$, there is a 1-form ω and ω can determine the horizontal vectors of $\tilde{M}$, then $\tilde{M}$ can have a *horizontal holonomy group* $Hl(\tilde{M})$ (see section 2) associated with this fibre bundle space. And $Hl(\tilde{M})$ is indeed an extended ideal of $H(M)$.

In general, the group $Hl(\tilde{M})$ may not form a Lie group ([3]). Yet, in many cases, $Hl(\tilde{M})$ does form a Lie group. Assume that $Hl(\tilde{M})$ is a Lie group. Let $Hl^0(\tilde{M})$ denote the restricted Lie group of $Hl(\tilde{M})$. In [3], the author studied the structure of the Lie algebra $dHl^0(\tilde{M})$ of $Hl^0(\tilde{M})$. In this paper, we can use the results of [3] and go one step further to find an explicit expression of the structure of $dHl^0(\tilde{M})$. Then, we can easily show that the Nijenhuis's theorem is actually a very special case of the group $Hl^0(\tilde{M})$.

For future use, we state Nijenhuis's theorem as follows:

"Let $h^0(M, p)$ be the restriced holonomy group. Then its Lie algebra $dh^0(M, p)$ is spanned by the matrices that arise from the $R_{\mu\lambda}(x)$; $x \in M$, by parallel transport to p along any curves."

§2. Priliminary.

Let $(\tilde{M}, M, \pi)$ be a fibre bundle space. Assume that there is a 1-form ω on $\tilde{M}$ such that ω can determine the horizontal vectors at every point P of M.

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If a curve $C: I \to \tilde{M}$, I being an interval, has horizontal tangents at all points, then C is called a *horizontal curve*. Consider a curve $C: I \to M$ and let $C(0)=P_0$. Let $\tilde{P}_0$ be a point in $\tilde{M}$ such that $\pi(\tilde{P}_0)=P_0$. Suppose that there is a horizontal curve $\tilde{C}$ passing through $\tilde{P}_0$ and $\pi(\tilde{C})=C$. Then $\tilde{C}$ is unique and called the *horizontal lift of* C passing through $\tilde{P}_0$.

Now, let r be a curve in M joining two points P_0 and P_1 of M. Suppose that there is a horizontal lift C of r. Then there are two neighborhoods $\bar{U}_\lambda$ of the fibre F_{P_λ} containing $\tilde{P}_\lambda$, where $\lambda=0, 1$, in such a way that for any point $Q_0 \in F_{P_0}$ there is a unique horizontal curve passing through Q_0 and joining a point Q_1 in $\bar{U}_1$. Hence, we can define a mapping $\phi_r: \bar{U}_0 \to \bar{U}_1$ by letting $\phi_r(Q_0)=Q_1$. Such a mapping is called a *horizontal mapping covering r*.

Now, take a closed curve C from x_1 to x_1 in the base manifold M. The fibre over x_1 can be mapped onto itself by using horizontal mapping covering C(*). By considering all possible closed curves with finite arc length of x_1, a group of transformations on F_{x_1} is obtained and we call this group the *horizontal holonomy group*. And we denote it by $Hl(\tilde{M}, x_1)$.

Since we have that

$$Hl(M, x_1) \cong Hl(M, x_2),$$

for any two points x_1 and x_2 on M, we write $Hl(\tilde{M})$ to denote the horizontal translation group which is attached to the space $\tilde{M}$.

First, we consider the horizontal holonomy group of a Riemannian fibred space. Let $(\tilde{M}, M, g, \pi)$ be a Riemannian fibred space. And the length ds of a line segment in $\tilde{M}$ is given by

$$ds^2=g_{jk}(y, x)dy^j dy^k+2g_{j\alpha}(y, x)dy^j dx^\alpha+g_{\alpha\beta}(y, x)dx^\alpha dx^\beta,$$

where the Greek letters α, β, γ etc. represent the coordinate system of the base manifold M and the English letters i, j, k etc. represent the coordinate system of the fibre space. Define Γ^i_α as follows:

$$\Gamma^i_\alpha g_{ij}=g_{j\alpha}.$$

And also define

$$K^i_{\alpha\beta}=(\partial_\beta \Gamma^i_\alpha-\partial_\alpha \Gamma^i_\beta)+(\Gamma^j_\alpha \partial_j \Gamma^i_\beta-\Gamma^j_\beta \partial_j \Gamma^i_\alpha).$$

Then $K^i_{\alpha\beta}$ is a skew-symmetric tensor and $K_{\alpha\beta}=K^i_{\alpha\beta}\partial_i$ is a infinitesimal vector field of a infinitesimal translation of $Hl^0(\tilde{M})$ (see [3]).

Since Γ^i_α are functions of (x^α) and (y^i), the vertical vector fields $K_{\beta\alpha}$ are functions of (x^α) and (y^i). We denote them $K_{\beta\alpha}(y, x)$. Let P be a reference point on M. Then we have the followings:

DEFINITION 1. *We define the following set of vector fields on* F_P.

$$S=\{\bar{K}_{\beta\alpha}(y, x, r)\partial_i\,;\ \alpha, \beta=1, 2, \cdots, n, \quad \text{for all} \quad x \text{ and } r\},$$

(*) In this paper, we assume that for any given curve C on M, the horizontal liftings of C always exist. The sufficient condition which makes the above statement true, has been discussed in [3].

where $\bar{K}^i_{\beta\alpha}(y, x, r)\partial_i$ are these vector fields obtained by translating $K_{\beta\alpha}(y, x)$ at the fibre F_x to F_P horizontally along any possible curve r which connects the points x and P.

DEFINITION 2. Let S be the set of vector fields defined in definition 1. If there exists a finite subset $\bar{S}=\{K_1, K_2 \cdots K_e\}$ of S such that

(1) Every element of S is a linear combination of $\bar{S}$ over the real number.

(2) $\bar{S}$ forms a base of an involutive distribution, i.e. at every point b of F_P, $\{K_1(b), \cdots K_e(b)\}$ are linearly independent at T_P and $[K_i, K_j](b)$ belongs to the subspace generated by $\{K_1(b), \cdots, K_e(b)\}$.

In other words, $\bar{S}$ generates a submanifold of F_P at every point b of F_P. *We say that S is integrable.*

DEFINITION 3. Let ϕ_r be an element of $Hl^0(\tilde{M}, P)$. A vector field $X \in F_P$ is said to be *invariant under* ϕ_r *if*

$$\phi_{r*}X = X.$$

A vector field $X \in F_P$ is called *invariant under* $Hl^0(\tilde{M}, P)$ if it is invariant under all $\phi_r \in Hl^0(\tilde{M}, P)$.

DEFINITION 4. Let X be a vector field on F_P. We say that X is tangent to $Hl^0(\tilde{M}, P)$ if (1) X generates a global 1-parameter group ϕ and (2) $\phi_t \in Hl^0(\tilde{M}, P)$ for all $t \in R$ (real number).

From [3], we have the following two theorems.

(A) If $Hl^0(\tilde{M}, P)$ is a Lie group and S is either integrable or invariant under $Hl^0(\tilde{M}, P)$, then its Lie algebra $dHl^0(\tilde{M}, P)$ is spanned by S.

(B) Let U be a neighborhood of P and let all points and curves in the following arguments lie in U. U may be chosen so that the local horizontal holonomy group(*) $Hl^*(P)=Hl^0(U. P)$. Let ∂_α, $\alpha=1, 2, \cdots, n$ be the coordinate vector fields of U. Construct the horizontal lifting ∂^L_α of ∂_α. Let

$$K=\{K_{\alpha\beta}, [K_{\alpha\beta}, \partial^L_\gamma], [[K_{\alpha\beta}, \partial^L_\gamma], \partial^L_\delta], \cdots\}(P)$$

and $R(P)$ be the vector space spanned by K. Then we have that

Suppose that $\tilde{M}$ and M are analytic Riemannian manifolds and Γ^i_α are analytic functions. If $Hl^*(P)$ is a Lie group and K is either integrable or invariant under $Hl^*(P)$, then its Lie algebra $dHl^*(P)=R(P)$. Conversely, if K is tangent to $Hl^*(P)$, either integrable or invariant under $Hl^*(P)$ and forming a finite Lie algebra, then $Hl^*(P)$ is a Lie group.

§ 3. Riemannian Fibred Space.

Let $(\tilde{M}, M, g, \pi)$ be the Riemannian fibred space considered in section two. Define a vector field V_α in a coordinate neighborhood U of $\tilde{M}$ as follows: Let

(*) The local horizontal holonomy group is defined in the same way as local linear holonomy group.

$$V_\alpha=\partial_\alpha-\Gamma^i_\alpha\partial_i\,. \tag{3.1}$$

Consider the inner product of V_α with the vertical vector field ∂_j, i.e. we have that

$$\langle V_\alpha, \partial_j\rangle=\langle\partial_\alpha, \partial_j\rangle-\Gamma^i_\alpha\langle\partial_i, \partial_j\rangle$$
$$=g_{\alpha j}-\Gamma^i_\alpha g_{ij}=g_{\alpha j}-g_{\alpha j}=0\,.$$

Hence, V_α is a horizontal vector field. Since we also have that $\pi_*(V_\alpha)=\partial_\alpha$, V_α is the horizontal lifting of ∂_α. Now, let us calculate the Lie bracket of V_α and V_β. Then we obtain that

$$[V_\beta, V_\alpha]=V_\beta V_\alpha-V_\alpha V_\beta=\partial_\beta\Gamma^i_\alpha\partial_i-\partial_\alpha\Gamma^i_\beta\partial_i-\Gamma^j_\beta\partial_j\Gamma^i_\alpha\partial_i+\Gamma^j_\alpha\partial_j\Gamma^i_\beta\partial_i=K_{\alpha\beta}\,.$$

Therefore, we obtain the following lemma:

LEMMA 1. *Let ∂_α and ∂_β be the coordinate vector fields of the base manifold M. Let $V_\alpha=\partial_\alpha-\Gamma^i_\alpha\partial_i$ and $V_\beta=\partial_\beta-\Gamma^i_\beta\partial_i$ be the horizontal lifting of ∂_α and ∂_β respectively. Then*

$$K_{\alpha\beta}=[V_\beta, V_\alpha]\,.$$

The Geometric Interpretation of $[V_\beta, V_\alpha]$.

First, let us look at a general case. Let M be a differentiable manifold and X and Y be two vector fields on M. Refer to figure one. Let C_1 be the integral curve of X from 0 to 1 with $C_1(t)=1$, C_2 be the integral curve of Y from 1 to 2 with $C_2(t)=2$, C_3 be the integral curve of X from 2 to 3 with $C_3(t)=3$ and C_4 be the integral curve of Y from 3 to 4 with $C_4(t)=4$. By letting $t\to0$, proved by Richard Faber, the tangent vector of the trace of point 4 reprsents the Lie bracket $[X, Y]$. Now, for the Lie bracket $[V_\beta, V_\alpha]$, we have a slightly different figure. Since $[\partial_\beta, \partial_\alpha]=0$, Refering to figure two, we can

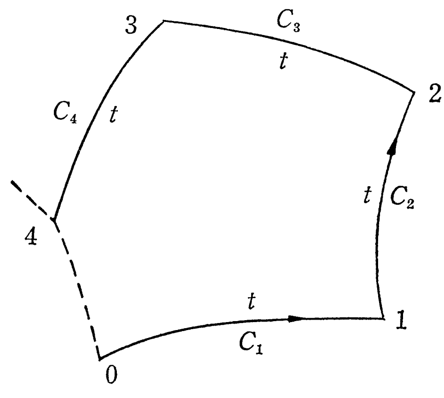

Fig. 1.

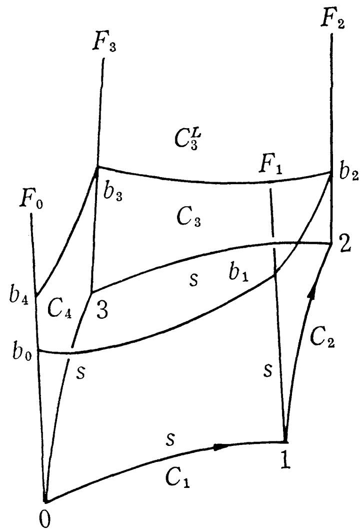

Fig. 2.

always find a closed curve $C_1+C_2+C_3+C_4$ on M such that C_1 and C_3 are the integral curves of ∂_β and C_2 and C_4 are the integral curves of ∂_α. And all $C_\imath$, $\imath=1, \cdots, 4$, are of arc length s. Let b_0 be an arbitrary point on F_0. Consider the horizontal lifting $C_\imath^L$, $\imath=1, \cdots, 4$, of $C_\imath$. Then C_1^L and C_3^L are the integral curves of V_β and C_2^L and C_4^L are the integral curves of V_α. Hence, $[V_\beta, V_\alpha]$ represents the tangent vector of the trace of b_4. On the other hand, by letting $C=C_1+\cdots C_4$, b_4 represents the horizontal translation of b_0 to b_4 along the curve C on M, i.e.

$$b_4=\phi_C(b_0).$$

Hence, by letting $s\to 0$, the tangent vector of the trace of b_4 is an infinitesimal horizontal translation of $Hl^0(\tilde{M}, 0)$. This explains the geometric meaning of lemma 1.

4. Fibre Bundle Space.

By viewing the geometric meaning of $[V_\beta, V_\alpha]$, we know that the Riemannian metric did not play a important role. As long as the horizontal liftings of the coordinate vector fields are defined, the infinitesimal horizontal translations are defined. Besides, if for every ∂_α the horizontal lifting V_α is defined, then from equation 3.1, those quantities Γ_α^i are also defined. Hence, theorem (A) and (B) from Riemannian manifold, stated in section two, can be extended to any fibre bundle space.

Now, we assume that $\tilde{M}$ is a fibre bundle space over M such that the horizontal lifting of the coordinate vector fields are defined and differentiable. Then we obtain that

THEOREM 1. *If $Hl^0(\tilde{M}, P)$ is a Lie group and the following set S' is either integrable or invariant under $Hl^0(\tilde{M}, P)$, then its Lie algebra is spanned by S', where*

$$S'=\{\phi_{r*}[V_\beta, V_\alpha]\,;\ \alpha, \beta=1, 2, \cdots, n,\ r \text{ is any possible curve on } M \text{ which connects the points from } Q \text{ to } P\}.$$

THEOREM 2. *Suppose that $\tilde{M}$ and M are analytis manifolds and the horizontal liftings of ∂_α, $\alpha=1, 2, \cdots, n$, are described by analytic functions. If $Hl^*(P)$ is a Lie group and the following set of vector fields $R(P)$ is either integrable or invariant under $Hl^*(P)$, then its Lie algebra is spanned by $R(P)$, where*

$$R(P)=\{[V_\beta, V_\alpha], [[V_\beta, V_\alpha], V_\gamma], \cdots\}.$$

Conversely, if $R(P)$ is tangent to $Hl^*(P)$, either integrable or invariant under $Hl^*(P)$ and forming a finite Lie algebra, then $Hl^*(P)$ is a Lie group.

Applications:

(1) Let $\tilde{M}$ be the bundle of frames over M and let the connection from ω be given. Then it is known that if X and Y are horizontal vector fields, then the vertical part of $[X, Y]$ is equal to $-2\Omega(X, Y)$ ([4], p. 35). Since the Lie bracket of V_α and V_β is a vertical vector field, we have that

$$[V_\beta, V_\alpha]=-2\Omega(V_\beta, V_\alpha)=R(\partial_\alpha, \partial_\beta).$$

Also notice that the horizontal translation ϕ_r along a curve r, in this case, is the parallel translation along the curve r. Hence, theorem 1 says,

"The Lie algebra of the restricted holonomy group $H^0(\tilde{M}, P)$ is spanned by the set

$$\{\tau_r \circ R(\partial_\alpha, \partial_\beta)\,;\ \text{for any possible curve } r \text{ which connects } Q \text{ and } P\}.$$

This is exactly the same statement as Nijenhuis's theorem.

(2) Let $\tilde{M}$ be a fibre bundle space. We say that $\tilde{M}$ admits holonomy fibres, if at every point P of $\tilde{M}$ there exists at least locally a submanifold of dimension n orthogonal to the fibre passing through the point P.

THEOREM 3. *Let $\tilde{M}$ be a fibre bundle space. $\tilde{M}$ admits holonomy fibres, if and only if $[V_\beta, V_\alpha]=0$, for all $\alpha, \beta=1, 2, \cdots, n$.*

Proof. The sufficient condition is obvious. If $[V_\beta, V_\alpha]=0$, then the horizontal distribution is involutive. Hence, it is integrable. For the necessary condition, since there exists a submanifold orthogonal to the fibre, $[V_\beta, V_\alpha]$ has to equal a horizontal vector. Yet, we know that $[V_\beta, V_\alpha]$ is a vertical vector. Hence it is equal to zero.♯

For a Riemannian fibred space, Muto proved the same theorem in [5].

(3) Let $\tilde{M}$ be a Riemannian fibred space. Since the torsion tensor T is equal to zero, we have that

$$[V_\beta, V_\alpha]=\tilde{\nabla}_{V_\beta} V_\alpha-\tilde{\nabla}_{V_\alpha} V_\beta .$$

Hence, the Lie algebra of $Hl^0(\tilde{M}, P)$ (if it is a Lie group) is spanned by the vector fields of the above form.

(4) Let $\tilde{M}$ be a Riemannian fibred space with projectable metric. Then, $[V_\beta, V_\alpha]=2h^{\iota}_{\beta\alpha}C_\iota=h(V_\beta, V_\alpha)$ ([6]), where h is the second fundamental form of $\tilde{M}$ and C_ι are vertical vector fields. Hence, $dHl^0(\tilde{M}, P)$ is spanned by the vector fields $h(V_\beta, V_\alpha)$.

5. The Associated Lie Algebra.

In this section, we are looking for the connection between the de Rham's decomposition of a manifold and a special Lie algebra associated with the linear holonomy group.

Let $VR(P)$ be the vector space spanned by the set $R(P)$ in theorem 2. Suppose that $VR(P)$ is a finite dimensional Lie algebra. By adding the set of vector fields V_α; $\alpha=1, 2, \cdots, n$ to $VR(P)$, we obtain an enlarged vector space $V\bar{R}(P)$, i.e. $V\bar{R}(P)$ is spanned by the set

$$\{V_\alpha, [V_\beta, V_\alpha], [[V_\beta, V_\alpha], V_\gamma], \cdots\} .$$

It is easy to show that $V\bar{R}(P)$ forms a finite Lie algebra and we call $V\bar{R}(P)$ the *associated Lie algebra* of $Hl^*(U, P)$.

Now, let M be an analytic manifold with an analytic connection. Assume that M is a connected, simply connected and complete manifold. Then from [1] and theorem 2, $VR(P)$ is actually the linear holonomy group of M. Hence, $V\bar{R}(P)$ is the associated Lie algebra of $H(M)$.

In de Rham's theorem, let $T_1(P)$ and $T_2(P)$ be two orthogonal subspaces of the tangent space $T(P)$ and $T(P)=T_1(P)+T_2(P)$. Suppose that $T_1(P)$ (resp. $T_2(P)$) is invariant under the translation of the linear holonomy group of M. By parallel translating $T_1(P)$ (resp. $T_2(P)$) to all other points of M, then this vector distribution is integrable. Denote the integral manifold which passes through the point P by M_1 (resp. M_2). Then de Rham's theorem says that ([7]).

"M is isometric to the direct product $M_1\times M_2$." Let $V\bar{R}_1(P)$ (resp. $V\bar{R}_2(P)$) be the associated Lie algebra of $H(M_1)$ (resp. $H(M_2)$). Then we have the following lemma:

LEMMA 2. *$V\bar{R}_1(P)$ and $V\bar{R}_2(P)$ are ideals of $V\bar{R}(P)$ and*

$$V\bar{R}(P)=V\bar{R}_1(P)+V\bar{R}_2(P) .$$

Proof. Let $\partial_{\bar{\alpha}}$, $\bar{\alpha}=1, 2, \cdots, m_1$ and $\partial_{\beta'}$, $\beta'=1, 2, \cdots, m_2$ be the coordinate system of M_1 and M_2 respectively. Then

$$V\bar{R}_1(P)=\text{the vector space spanned by } \{V_{\bar{\alpha}}, [V_{\bar{\beta}}, V_{\bar{\alpha}}], \cdots\},$$

$$V\bar{R}_2(P)=\text{the vector space spanned by } \{V_{\alpha'}, [V_{\beta'}, V_{\alpha'}], \cdots\}.$$

From de Rham's theorem, we have that

$$V\bar{R}(P)=V\bar{R}_1(P)+V\bar{R}_2(P).$$

Hence, we only have to show that $V\bar{R}_1(P)$ (resp. $V\bar{R}_2(P)$) is an ideal of $V\bar{R}(P)$. It is the same to show that

$$[V\bar{R}_1(P), V\bar{R}_2(P)]=0.$$

(1) $$[V_{\bar{\alpha}}, V_{\beta'}]=0.$$

Referring to figure 3, let C_1 and C_3 be the integral curves of $\partial_{\bar{\alpha}}$ and $\pi(C_3)=C_1$, i.e. C_3 and C_1 have the same M_1 coordinates. Let C_2 and C_4 be the integral curves of $\partial_{\beta'}$ and $\pi(C_2)=C_4$. A vector $v_2\in T_2(P)$ is translated parallelly along the curve $C=C_1+C_2+C_3+C_4$. Then v_2 is invariant along C_1 and C_3 and equal along C_2 and C_4. Hence, $\tau_C v_2=v_2$. Similarly, for any $v_1\in T_1(P)$, $\tau_C v_1=v_1$. Therefore, $\tau_C v=v$, for any $v\in T(P)$. By shrinking C to zero, since $[V_{\bar{\alpha}}, V_{\beta'}]$ represents the infinitesimal parallel translation of a vector along the curve C, we obtain that $[V_{\bar{\alpha}}, V_{\beta'}]=0$.

(2) $$[[V_{\bar{\alpha}}, V_{\bar{\beta}}], V_{\gamma'}]=0.$$

From Jacobi's identity, we have that

$$[[V_{\bar{\alpha}}, V_{\bar{\beta}}], V_{\gamma'}]=-[[V_{\gamma'}, V_{\bar{\alpha}}], V_{\bar{\beta}}]-[[V_{\bar{\beta}}, V_{\gamma'}], V_{\bar{\alpha}}]$$
$$=0+0=0.$$

(3) By similar consideration, we obtain that

$$[V\bar{R}_1(P), V_{\gamma'}]=0.$$

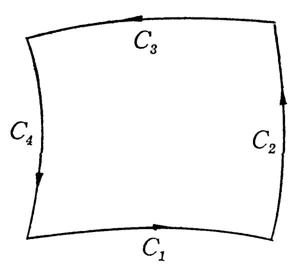

Fig. 3.

Moreover, we have that

$$[V\bar{R}_1(P), [V_{\beta'}, V_{\gamma'}]] = -[V_{\gamma'}, [V\bar{R}_1(P), V_{\beta'}]] - [V_{\beta'}, [V_{\gamma'}, V\bar{R}_1(P)]] = 0. \tag{4}$$

(5) By induction, we can prove this lemma. ♯

Hence, we conclude that

"The de Rham's decomposition of an analytic manifold, is associated with the decomposition of the Lie group $VR(P)$ into the direct sum of ideals."

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