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REPRESENTATION OF ADDITIVE FUNCTIONALS ON VECTOR-VALUED NORMED KÖTHE SPACES

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§ 1. Introduction.

Integral representation theory has been developed by many authors for nonlinear additive functionals and operators on measurable function spaces such as Lebesgue spaces and Orlicz spaces; see Alò and Korvin [1], Drewnowski and Orlicz [3-5], Friedman and Katz [6], Martin and Mizel [11], Mizel [12], Mizel and Sundaresan [13-15], Palagallo [16], Sundaresan [19], and Woyczyński [21]. Representation theorems have been obtained also for additive operators on continuous function spaces; see Batt [2] and references therein. The purpose of this paper is to establish representation theorems for additive functionals on Banach space-valued normed Köthe spaces.

In this paper, let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and X a real separable Banach space. Let $L_\rho(X)$ be an X-valued normed Köthe space equipped with an absolutely continuous function norm ρ. A functional $\Phi: L_\rho(X) \to \bar{R}$ is called to be additive if $\Phi(f+g)=\Phi(f)+\Phi(g)$ for each $f, g \in L_\rho(X)$ such that $\mu(\mathrm{Supp}\, f \cap \mathrm{Supp}\, g)=0$. For several types of additive functionals $\Phi: L_\rho(X) \to \bar{R}$, we shall establish integral representations of the form $\Phi(f)=\int_\Omega \phi(\omega, f(\omega))d\mu$ with certain kernel functions $\phi: \Omega \times X \to \bar{R}$. Representation theorems have been so far obtained for additive functionals which are continuous or rather equicontinuous in some senses. However our method via measurable set-valued functions is applicable to additive lower semicontinuous functionals on $L_\rho(X)$.

In § 2, we give definitions and some elementary facts on function norms and normed Köthe spaces. In § 3, a characterization theorem for closed decomposable subsets in $L_\rho(X) \times L_1$ is established by means of measurable set-valued functions. This characterization will be useful in constructing a set-valued function whose values are closed subsets of $X \times R$ corresponding to epigraphs of an integral kernel function. In § 4, we provide several lemmas on additive functionals and integral functionals on $L_\rho(X)$. Finally in § 5, we discuss integral representations for the following cases:

(1) Additive lower semicontinuous functionals on $L_\rho(X)$.

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(2) Additive continuous functionals on $L_\rho(X)$.
(3) Bounded linear functionals on $L_\rho(X)$.
(4) Additive lower semicontinuous convex functionals on $L_\rho(X)$.

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§ 2. Preliminaries.

Throughout this paper, let $(\Omega, \mathcal{A}, \mu)$ be a fixed σ-finite measure space and $\bar{\mathcal{A}}$ the completion of $\mathcal{A}$ with respect to μ. Let M^+ be the collection of all nonnegative real-valued measurable functions on Ω. A mapping ρ on M^+ to $\bar{R}=[-\infty, \infty]$ is called a *function norm* if ρ satisfies the following conditions:

(i) $\rho(\xi)\geqq 0$ and $\rho(\xi)=0$ if and only if $\xi(\omega)=0$ a.e.,

(ii) $\rho(\xi+\zeta)\leqq\rho(\xi)+\rho(\zeta)$,

(iii) $\rho(\alpha\xi)=\alpha\rho(\xi)$ for $\alpha\geqq 0$,

(iv) $\xi(\omega)\leqq\zeta(\omega)$ a.e. implies $\rho(\xi)\leqq\rho(\zeta)$.

Let ρ be a fixed function norm, and let X be a real separable Banach space with dual space X^*. Note that the notions of strong and weak measurability of functions $f: \Omega\to X$ are identical, since X is separable. Let $L_\rho(X)= L_\rho(\Omega, \mathcal{A}, \mu; X)$ denote the space of all measurable functions $f: \Omega\to X$ such that $\rho(\|f\|)<\infty$ where $\|f\|=\|f(\cdot)\|$. Then $L_\rho(X)$ becomes a normed linear space with the norm $\rho(\|f\|)$ where μ-almost everywhere equal functions are identified. For $X=R$, the space $L_\rho=L_\rho(R)$ is called a *normed Köthe space,* and also called a *Banach function space* if it is complete. Usual $L_p(1\leqq p\leqq\infty)$ spaces and Orlicz spaces are Banach function spaces. The function norm ρ is said to have the *Fatou property* if $\rho(\xi_n)\uparrow\rho(\xi)$ whenever $\xi_n\in M^+$ and $\xi_n\uparrow\xi$, and said to have the *weak Fatou property* if $\rho(\xi)<\infty$ whenever $\xi_n\in M^+$, $\xi_n\uparrow\xi$, and $\sup\rho(\xi_n)<\infty$. The weak Fatou property implies the completeness of L_ρ and $L_\rho(X)$. In this paper, we shall *not* require ρ to have the weak Fatou property.

The characteristic function of a set $A\in\mathcal{A}$ is denoted by 1_A. A set $A\in\mathcal{A}$ with $\mu(A)>0$ is called *unfriendly* relative to ρ if $\rho(1_B)=\infty$ for every $B\in\mathcal{A}$ with $B\subset A$ and $\mu(B)>0$. The function norm ρ is called *saturated* if $\mathcal{A}$ contains no unfriendly sets. There exists a maximal (up to μ-null sets) unfriendly set Ω_∞ and so $\xi(\omega)=0$ a.e. on Ω_∞ for every $\xi\in L_\rho$. In order to give representations of additive functionals on $L_\rho(X)$, we may assume by removing Ω_∞ from Ω without loss of generality that ρ is saturated. As a consequence of this assumption, there exists a *ρ-admissible sequence,* i.e., a sequence $\{\Omega_n\}$ in $\mathcal{A}$ with $\Omega_n\uparrow\Omega$ such that $\mu(\Omega_n)<\infty$ and $\rho(1_{\Omega_n})<\infty$ for all n. The *associate norm* ρ' is defined by

$$\rho'(\zeta)=\sup\left\{\int_\Omega\xi\zeta\,d\mu:\ \xi\in M^+,\ \rho(\xi)\leqq 1\right\},\qquad \zeta\in M^+,$$

which is also a saturated function norm having the Fatou property.

A function $\xi\in L_\rho$ is said to be *of absolutely continuous norm* if $\rho(1_{A_n}|\xi|)\downarrow 0$ for every sequence $\{A_n\}$ in $\mathcal{A}$ such that $A_n\downarrow\emptyset$. The space L_ρ^a of all $\xi\in L_\rho$ of absolutely continuous norm is a closed order ideal of L_ρ, that is, L_ρ^a is a closed subspace of L_ρ such that $\zeta\in L_\rho^a$ and $|\xi(\omega)|\leqq|\zeta(\omega)|$ a.e. imply $\xi\in L_\rho^a$. Then the dominated convergence theorem holds as follows: If $\xi_n(\omega)\to\xi(\omega)$ a.e. and $|\xi_n(\omega)|\leqq\zeta(\omega)$ a.e. with $\zeta\in L_\rho^a$, then $\rho(|\xi_n-\xi|)\to 0$. We shall always assume that ρ is an *absolutely continuous norm*, i.e., $L_\rho^a=L_\rho$. It is well known that $L_\rho^a=L_\rho$ when $L_\rho=L_p(1\leqq p<\infty)$ or more generally when L_ρ is an Orlicz space with a Young's function obeying Δ_2-condition. After all, it will be assumed in this paper that ρ is a saturated absolutely continuous norm. Therefore the dual space $L_\rho{}^*$ of L_ρ is isometrically isomorphic to the Banach function space $L_{\rho'}$ with the associate norm ρ' under the bilinear form $\langle\xi,\zeta\rangle=\int_\Omega\xi\zeta\,d\mu$ of $\xi\in L_\rho$ and $\zeta\in L_{\rho'}$. For detailed arguments on normed Köthe spaces, see [22, Chap. 15]. The proofs of above stated facts can be found there.

It is worth while remarking that even when ρ is not absolutely continuous, the representation theorems in §5 hold for additive functionals restricted on $L_\rho^a(X)=\{f\in L_\rho(X):\|f\|\in L_\rho^a\}$. However, for the uniqueness of kernel functions, it must be assumed that the carrier of L_ρ^a (cf. [22, p. 481]) is the whole set Ω. See also Remark 1 to Theorem 5.3.

§3. Decomposable subsets in $L_\rho(X)\times L_1$.

For a set-valued function $F:\Omega\to 2^X$ where 2^X is the collection of all subsets of X, let $D(F)=\{\omega\in\Omega: F(\omega)\neq\emptyset\}$ and $G(F)=\{(\omega,x)\in\Omega\times X: x\in F(\omega)\}$. The inverse image $F^{-1}(U)$ of $U\subset X$ is defined by $F^{-1}(U)=\{\omega\in\Omega: F(\omega)\cap U\neq\emptyset\}$. As to the following conditions for $F:\Omega\to 2^X$ such that $F(\omega)$ is closed for every $\omega\in\Omega$, the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) hold, and moreover if $(\Omega,\mathcal{A},\mu)$ is complete, then all the conditions (1)-(4) are equivalent:

(1) $F^{-1}(C)\in\mathcal{A}$ for every closed subset C of X;

(2) $F^{-1}(O)\in\mathcal{A}$ for every open subset O of X;

(3) $D(F)\in\mathcal{A}$ and there exists a sequence $\{f_n\}$ of measurable functions $f_n: D(F)\to X$ such that $F(\omega)=\mathrm{cl}\{f_n(\omega)\}$ for all $\omega\in D(F)$;

(4) $G(F)\in\mathcal{A}\otimes\mathcal{B}_X$ where $\mathcal{B}_X$ is the Borel σ-field of X.

A set-valued function $F:\Omega\to 2^X$ is called *measurable* (resp. *weakly measurable*) if F satisfies the above condition (1) (resp. (2)). We shall denote by $\mathcal{M}[\Omega; X]$ the collection of all weakly measurable set-valued functions $F:\Omega\to 2^X$ such that $F(\omega)$ is nonempty and closed for every $\omega\in\Omega$. We observe that if $G(F)\in\mathcal{A}\otimes\mathcal{B}_X$ and $F(\omega)$ is nonempty and closed for every $\omega\in\Omega$, then there exists an $F'\in\mathcal{M}[\Omega: X]$ such that $F'(\omega)=F(\omega)$ a.e. Indeed, since there exists a sequence $\{f_n\}$ of $\bar{\mathcal{A}}$-measurable functions such that $F(\omega)=\mathrm{cl}\{f_n(\omega)\}$ for all $\omega\in\Omega$, we obtain a desired $F'\in\mathcal{M}[\Omega; X]$ by taking $\mathcal{A}$-measurable functions f_n' with $f_n'(\omega)=f_n(\omega)$ a.e. and defining $F'(\omega)=\mathrm{cl}\{f_n'(\omega)\}$. For more complete discussions of measurability of set-valued functions whose values are closed subsets in a separable metric spaces, see [9] and [20].

Let M be a set of measurable functions $f: \Omega \to X$. We call M *decomposable* if $1_A f + 1_{\Omega \setminus A} g \in M$ for each $f, g \in M$ and $A \in \mathcal{A}$. It is clear that if M is decomposable, then $\sum_{i=1}^{n} 1_{A_i} f_i \in M$ for each finite measurable partition $\{A_1, \cdots, A_n\}$ of Ω and $\{f_1, \cdots, f_n\} \subset M$. We showed in [8, Theorem 3.1] that any closed decomposable subset of $L_p(X)$, $1 \leqq p < \infty$, is characterized as a set of the form $S_p(F) = \{f \in L_p(X) : f(\omega) \in F(\omega) \text{ a.e.}\}$ with $F \in \mathcal{M}[\Omega; X]$. In this section, we obtain an analogous result for subsets of $L_\rho(X) \times L_1$ which will play an important role in the proof of Theorem 5.1. The product space $L_\rho(X) \times L_1$ is equipped with the norm $\rho(\|f\|) + \|\xi\|_1$ for $f \in L_\rho(X)$ and $\xi \in L_1$ where $\|\xi\|_1$ is the L_1-norm. A subset M of $L_\rho(X) \times L_1$ is decomposable if and only if $(1_A f + 1_{\Omega \setminus A} g, 1_A \xi + 1_{\Omega \setminus A} \zeta) \in M$ for each $(f, \xi), (g, \zeta) \in M$ and $A \in \mathcal{A}$. For given $F \in \mathcal{M}[\Omega; X \times R]$, we define the subset $S_{\rho,1}(F)$ of $L_\rho(X) \times L_1$ by

$$S_{\rho,1}(F) = \{(f, \xi) \in L_\rho(X) \times L_1 : (f(\omega), \xi(\omega)) \in F(\omega) \text{ a.e.}\}.$$

We first give some properties of subsets $S_{\rho,1}(F)$ in the following lemmas.

LEMMA 3.1. *If $F \in \mathcal{M}[\Omega; X \times R]$, then $S_{\rho,1}(F)$ is closed in $L_\rho(X) \times L_1$.*

Proof. Let $\{(f_n, \xi_n)\}$ be a sequence in $S_{\rho,1}(F)$ convergent to $(f, \xi) \in L_\rho(X) \times L_1$. Passing to a subsequence, we may assume that $\rho(\|f_n - f\|) < 1/2^n$ for all n and $\xi_n(\omega) \to \xi(\omega)$ a.e. To prove $(f, \xi) \in S_{\rho,1}(F)$, it now suffices to show that $\|f_n(\omega) - f(\omega)\| \to 0$ a.e. Taking a ρ-admissible sequence, we may assume in addition that $1_\Omega \in L_\rho$. For each $k > 0$, let $A_n = \{\omega \in \Omega : \|f_n(\omega) - f(\omega)\| \geqq 1/k\}$ and $A_\infty = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$. Since $\rho(1_{A_n}) \leqq \rho(k\|f_n - f\|) < k/2^n$, we have

$$\rho(1_{A_\infty}) \leqq \sum_{n=m}^{j} \rho(1_{A_n}) + \rho(1_{\bigcup_{n>j} A_n}) < k/2^{m-1} + \rho(1_{\bigcup_{n>j} A_n})$$

for each $j \geqq m \geqq 1$. Since ρ is absolutely continuous, it follows that $\rho(1_{\bigcup_{n>j} A_n}) \downarrow 0$ as $j \to \infty$, so that $\rho(1_{A_\infty}) = 0$ and hence $\mu(A_\infty) = 0$. Letting $k = 1, 2, \cdots$, we obtain

$$\mu\Big(\bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega \in \Omega : \|f_n(\omega) - f(\omega)\| \geqq 1/k\}\Big) = 0,$$

which shows that $\|f_n(\omega) - f(\omega)\| \to 0$ a.e. Thus the lemma is proved.

LEMMA 3.2. *If $F \in \mathcal{M}[\Omega; X \times R]$ and $S_{\rho,1}(F)$ is nonempty, then there exists a sequence $\{(f_n, \xi_n)\}$ in $S_{\rho,1}(F)$ such that $F(\omega) = \text{cl}\{(f_n(\omega), \xi_n(\omega))\}$ for all $\omega \in \Omega$.*

Proof. There exists a sequence $\{(g_k, \zeta_k)\}$ of measurable functions $g_k : \Omega \to X$ and $\zeta_k : \Omega \to R$ such that $F(\omega) = \text{cl}\{(g_k(\omega), \zeta_k(\omega))\}$ for all $\omega \in \Omega$ (see the above condition (3)). Since $S_{\rho,1}(F) \neq \emptyset$, we can select an element $(f, \xi) \in S_{\rho,1}(F)$ such that $(f(\omega), \xi(\omega)) \in F(\omega)$ for all $\omega \in \Omega$. Taking a ρ-admissible sequence $\{\Omega_j\}$, we define

$$A_{jmk}=\{\omega\in\Omega_j:\ m-1\leqq\|g_k(\omega)\|+|\zeta_k(w)|<\omega\},$$

$$f_{jmk}=1_{A_{jmk}}g_k+1_{\Omega\setminus A_{jmk}}f,\qquad \xi_{jmk}=1_{A_{jmk}}\zeta_k+1_{\Omega\setminus A_{jmk}}\xi,$$

$$j,\ m,\ k\geqq 1.$$

Then it is easy to see that $\{(f_{jmk},\ \xi_{jmk})\}\subset S_{\rho,1}(F)$ and $F(\omega)=\text{cl}\{(f_{jmk}(\omega),\ \xi_{jmk}(\omega))\}$ for all $\omega\in\Omega$, completing the proof.

LEMMA 3.3. *If $F\in\mathcal{M}[\Omega;\ X\times R]$ and $S_{\rho,1}(F)$ is nonempty and convex, then $F(\omega)$ is convex for a.e. $\omega\in\Omega$.*

Proof. By Lemma 3.2, there exists a sequence $\{(f_n,\ \xi_n)\}$ in $S_{\rho,1}(F)$ such that $F(\omega)=\text{cl}\{(f_n(\omega),\ \xi_n(\omega))\}$ for all $\omega\in\Omega$. Since $((f_i+f_j)/2,\ (\xi_i+\xi_j)/2)\in S_{\rho,1}(F)$, we can take an $N\in\mathcal{A}$ with $\mu(N)=0$ such that

$$((f_i(\omega)+f_j(\omega))/2,\ (\xi_i(\omega)+\xi_j(\omega))/2)\in F(\omega),\qquad i,\ j\geqq 1,\ \omega\in\Omega\setminus N.$$

This shows that $F(\omega)$ is convex for every $\omega\in\Omega\setminus N$, and the lemma is proved.

THEOREM 3.4. *Let M be a nonempty subset of $L_\rho(X)\times L_1$. Then there exists an $F\in\mathcal{M}[\Omega;\ X\times R]$ such that $M=S_{\rho,1}(F)$ if and only if M is closed and decomposable in $L_\rho(X)\times L_1$.*

Proof. If there exists an $F\in\mathcal{M}[\Omega;\ X\times R]$ such that $M=S_{\rho,1}(F)$, then M is closed by Lemma 3.1 and clearly decomposable.

To prove the converse, let M be a nonempty closed and decomposable subset of $L_\rho(X)\times L_1$. Take an element $(f_0,\ \xi_0)\in M$ and let $M_0=\{(f-f_0,\ \xi-\xi_0):(f,\ \xi)\in M\}$. Then M_0 is a closed decomposable subset of $L_\rho(X)\times L_1$ containing $(0,\ 0)$. If there exists an $F_0\in\mathcal{M}[\Omega;\ X\times R]$ such that $M_0=S_{\rho,1}(F_0)$, then defining $F(\omega)=F_0(\omega)+(f_0(\omega),\ \xi_0(\omega))$ we obtain $F\in\mathcal{M}[\Omega;\ X\times R]$ and $M=S_{\rho,1}(F)$. Thus we may assume that M contains $(0,\ 0)$. Now let $M_1=M\cap(L_1(X)\times L_1)$ and M_2 the closure of M_1 in $L_1(X)\times L_1$. Then it follows that M_2 is a nonempty closed and decomposable subset of $L_1(X)\times L_1$. Noting $L_1(X)\times L_1=L_1(X\times R)$ where the norm of $X\times R$ is taken by $\|(x,\ \alpha)\|=\|x\|+|\alpha|$, we obtain, by [8, Theorem 3.1], an $F\in\mathcal{M}[\Omega;\ X\times R]$ such that

$$M_2=\{(f,\ \xi)\in L_1(X)\times L_1:\ (f(\omega),\ \xi(\omega))\in F(\omega)\ \text{a.e.}\}.$$

We shall then prove that $M=S_{\rho,1}(F)$. For each $(f,\ \xi)\in L_\rho(X)\times L_1$, taking a ρ-admissible sequence $\{\Omega_n\}$ we put $A_n=\{\omega\in\Omega_n:\ \|f(\omega)\|\leqq n\}$ for $n\geqq 1$. Then $(1_{A_n}f,\ 1_{A_n}\xi)\in L_1(X)\times L_1$ for all n and it follows from $A_n\uparrow\Omega$ that

$$\rho(\|1_{A_n}f-f\|)+\|1_{A_n}\xi-\xi\|_1=\rho(1_{\Omega\setminus A_n}\|f\|)+\|1_{\Omega\setminus A_n}\xi\|_1\downarrow 0.$$

Thus we deduce in view of $(0,\ 0)\in M$ that M_1 and $S_{\rho,1}(F)\cap(L_1(X)\times L_1)=S_{\rho,1}(F)\cap M_2$ are dense in M and $S_{\rho,1}(F)$, respectively. Since both M and $S_{\rho,1}(F)$ are closed, it remains to show that $M_1\subset S_{\rho,1}(F)$ and $S_{\rho,1}(F)\cap M_2\subset M$. The first inclusion is obvious. To see the second inclusion, let $(f, \xi)\in S_{\rho,1}(F)\cap M_2$. Then there exists a sequence $\{(f_k, \xi_k)\}$ in M_1 convergent in $L_1(X)\times L_1$ to (f, ξ). It can be assumed that $\|f_k(\omega)-f(\omega)\|\to 0$ a.e. Taking a ρ-admissible sequence $\{\Omega_n\}$, we put $B_{nk}=\{\omega\in\Omega_n : \|f_k(\omega)\|\leqq\|f(\omega)\|+1\}$ for $n, k\geqq 1$. As $k\to\infty$ for each fixed n, it follows from $\mu(\Omega_n\setminus B_{nk})\to 0$ that

$$\|1_{B_{nk}}\xi_k-1_{\Omega_n}\xi\|_1\leqq\|\xi_k-\xi\|_1+\|1_{\Omega_n\setminus B_{nk}}\xi\|_1\to 0\,.$$

Moreover, since

$$2\|f\|+1_{\Omega_n}\geqq\|1_{B_{nk}}f_n(\omega)-1_{\Omega_n}f(\omega)\|\to 0 \text{ a.e.},$$

we obtain $\rho(\|1_{B_{nk}}f_k-1_{\Omega_n}f\|)\to 0$ by the dominated convergence theorem. Since $(1_{B_{nk}}f_k, 1_{B_{nk}}\xi_k)\in M$ by $(0, 0)\in M$, it follows that $(1_{\Omega_n}f, 1_{\Omega_n}\xi)\in M$ for all n, so that $(f, \xi)\in M$. Thus $M=S_{\rho,1}(F)$ is proved.

4. Additive functionals and integral functionals.

A functional $\phi: V\to\bar{R}$ on a topological vector space V is called *proper* if $\phi(x)>-\infty$ for all $x\in V$ and $\phi\not\equiv\infty$. The *epigraph* Epi ϕ of ϕ is defined by Epi ϕ $=\{(x, \alpha)\in V\times R : \phi(x)\leqq\alpha\}$. A functional $\phi: V\to\bar{R}$ is lower semicontinuous (resp. convex) if and only if Epi ϕ is closed (resp. convex) in $V\times R$. Let ϕ: $\Omega\times X\to\bar{R}$ be an $\mathcal{A}\otimes\mathcal{B}_X$-measurable function. For a measurable function f: $\Omega\to X$, since the function $\phi(\omega, f(\omega))$ is measurable, we define $I_\phi(f)=\int_\Omega\phi(\omega, f(\omega))d\mu$ if the integral exists permitting $\pm\infty$. We call I_ϕ the *integral functional* associated with the kernel function ϕ. A function $\phi: \Omega\times X\to\bar{R}$ is called *normal* if ϕ is $\mathcal{A}\otimes\mathcal{B}_X$-measurable and $\phi(\omega, \cdot)$ is lower semicontinuous for every $\omega\in\Omega$. Let Epi $\phi: \Omega\to 2^{X\times R}$ be defined by (Epi ϕ)(ω)=Epi $\phi(\omega, \cdot)$. By way of the measurability of the function $(\omega, x, \alpha)\mapsto\phi(\omega, x)-\alpha$ with respect to $\mathcal{A}\otimes\mathcal{B}_{X\times R}=\mathcal{A}\otimes\mathcal{B}_X\otimes\mathcal{B}_R$, it is seen that ϕ is normal if and only if G(Epi ϕ) $\in\mathcal{A}\otimes\mathcal{B}_{X\times R}$ and (Epi ϕ)(ω) is closed for every $\omega\in\Omega$. Thus ϕ is normal if Epi ϕ $\in\mathcal{M}[\Omega; X\times R]$, and vice versa when $(\Omega, \mathcal{A}, \mu)$ is complete.

For a measurable function $f: \Omega\to X$, let Supp $f=\{\omega\in\Omega : f(\omega)\neq 0\}$. A functional $\Phi: L_\rho(X)\to\bar{R}$ is called to be *additive* if $\Phi(f+g)=\Phi(f)+\Phi(g)$, where the addition $\infty+(-\infty)$ is not permitted, for each $f, g\in L_\rho(X)$ such that $\mu(\text{Supp} f\cap\text{Supp} g)=0$. The additivity of Φ means that for each $f\in L_\rho(X)$ the set function $A\mapsto\Phi(1_A f)$ is finitely additive on $\mathcal{A}$. If $\Phi: L_\rho(X)\to\bar{R}$ is additive and proper, then $\Phi(0)=0$ is readily verified. The integral functional I_ϕ with $\phi(\omega, 0)=0$ a.e. is obviously additive on $L_\rho(X)$, if it is defined on $L_\rho(X)$. In the remainder of this section, we provide lemmas which will be needed in the next section.

LEMMA 4.1. *If $\Phi: L_\rho(X)\to\bar{R}$ is an additive lower semicontinuous proper functional, then for each $f\in L_\rho(X)$ the set function $A\mapsto\Phi(1_A f)$ is countably additive on $\mathcal{A}$.*

Proof. Let $f\in L_\rho(X)$ and $A=\bigcup_{n=1}^{\infty} A_n$ with disjoint $A_n\in\mathcal{A}$. Then we have

$$\Phi(1_A f)=\sum_{i=1}^{n} \Phi(1_{A_i} f)+\Phi(1_{B_n} f), \qquad n\geqq 1,$$

where $B_n=\bigcup_{i>n} A_i$. Since $\liminf_{n\to\infty} \Phi(1_{B_n} f)\geqq\Phi(0)=0$ by $\rho(1_{B_n}\|f\|)\downarrow 0$, it follows that

$$\Phi(1_A f)\geqq \limsup_{n\to\infty} \sum_{i=1}^{n} \Phi(1_{A_i} f)+\liminf_{n\to\infty} \Phi(1_{B_n} f)$$

$$\geqq \limsup_{n\to\infty} \sum_{i=1}^{n} \Phi(1_{A_i} f).$$

On the other hand, since $\rho(\|\sum_{i=1}^{n} 1_{A_i} f-1_A f\|)=\rho(1_{B_n}\|f\|)\downarrow 0$, we have

$$\Phi(1_A f)\leqq\liminf_{n\to\infty} \Phi(\sum_{i=1}^{n} 1_{A_i} f)=\liminf_{n\to\infty} \sum_{i=1}^{n} \Phi(1_{A_i} f).$$

Thus $\Phi(1_A f)=\sum_{i=1}^{\infty} \Phi(1_{A_i} f)$ is obtained.

The following three lemmas are concerned with the relationship between integral functionals and their kernel functions.

LEMMA 4.2. *Let $\phi_1, \phi_2: \Omega\times X\to \bar{R}$ be two $\mathcal{A}\otimes\mathcal{B}_X$-measurable functions with $\phi_1(\omega, 0)=\phi_2(\omega, 0)=0$ a.e. such that $I_{\phi_1}(f)\leqq I_{\phi_2}(f)$ (resp. $I_{\phi_1}(f)=I_{\phi_2}(f)$) for each $f\in L_\rho(X)$ whenever both $I_{\phi_1}(f)$ and $I_{\phi_2}(f)$ are defined. Then there exists an $N\in\mathcal{A}$ with $\mu(N)=0$ such that $\phi_1(\omega, x)\leqq\phi_2(\omega, x)$ (resp. $\phi_1(\omega, x)=\phi_2(\omega, x)$) for all $\omega\in\Omega\backslash N$ and $x\in X$.*

Proof. Taking $\mathrm{Epi}\,\phi_1$, $\mathrm{Epi}\,\phi_2: \Omega\to 2^{X\times R}$, we define $H: \Omega\to 2^{X\times R}$ by $H(\omega)=(\mathrm{Epi}\,\phi_2)(\omega)\backslash(\mathrm{Epi}\,\phi_1)(\omega)$. Since $G(\mathrm{Epi}\,\phi_1)$, $G(\mathrm{Epi}\,\phi_2)\in\mathcal{A}\otimes\mathcal{B}_{X\times R}$, it follows that $G(H)=G(\mathrm{Epi}\,\phi_2)\backslash G(\mathrm{Epi}\,\phi_1)$ is $\mathcal{A}\otimes\mathcal{B}_{X\times R}$-measurable. Thus it follows (cf. [17, Theorem 4]) that $D(H)\in\bar{\mathcal{A}}$. To prove the lemma, it suffices to show that $D(H)$ is μ-null. Now suppose the contrary. By von Neumann-Aumann's selection theorem (cf. [9, Theorem 5.2], [17, Theorem 3]), there exists an $\bar{\mathcal{A}}$-measurable function $(g, \zeta): \Omega\to X\times R$ such that $(g(\omega), \zeta(\omega))\in H(\omega)$ for all $\omega\in D(H)$. Taking an $\mathcal{A}$-measurable function $(f, \xi): \Omega\to X\times R$ with $(f(\omega), \xi(\omega))=(g(\omega), \zeta(\omega))$ a.e., we can choose an $A\in\mathcal{A}$ with $\mu(A)>0$ such that $(f(\omega), \xi(\omega))\in H(\omega)$ for a.e. $\omega\in A$ and moreover $(1_A f, 1_A\xi)\in L_\rho(X)\times L_1$. Since $\phi_1(\omega, f(\omega))>\xi(\omega)\geqq\phi_2(\omega, f(\omega))$ a.e. on A, it is seen that both $I_{\phi_1}(1_A f)$ and $I_{\phi_2}(1_A f)$ are defined, and hence we have

$$I_{\phi_1}(1_A f)=\int_A \phi_1(\omega, f(\omega))\,d\mu>\int_A \xi\, d\mu$$

$$\geqq \int_A \phi_2(\omega, f(\omega))d\mu = I_{\phi_2}(1_A f),$$

a contradiction. This completes the proof.

LEMMA 4.3. *Let* $\phi: \Omega \times X \to \bar{R}$ *be a normal function with* $\phi(\omega, 0)=0$ *a.e. such that* I_ϕ *is defined on* $L_\rho(X)$. *If* I_ϕ *is convex on* $L_\rho(X)$, *then* $\phi(\omega, \cdot)$ *is convex on* X *for a.e.* $\omega \in \Omega$.

Proof. Since $G(\mathrm{Epi}\,\phi) \in \mathcal{A} \otimes \mathcal{B}_{X \times R}$ and $(\mathrm{Epi}\,\phi)(\omega)$ is closed for every $\omega \in \Omega$, we can take, as observed in § 3, an $F \in \mathcal{M}[\Omega; X \times R]$ such that $F(\omega)=(\mathrm{Epi}\,\phi)(\omega)$ a.e. To prove the lemma, it suffices by Lemma 3.3 to show that $S_{\rho,1}(F)$ is nonempty and convex. It is immediate that $(0, 0) \in S_{\rho,1}(F)$. The convexity assumption of I_ϕ means that $\mathrm{Epi}\, I_\phi$ is convex in $L_\rho(X) \times R$. Thus the convexity of $S_{\rho,1}(F)$ follows from the following observation: For each $(f, \xi) \in L_\rho(X) \times L_1$, $(f, \xi) \in S_{\rho,1}(F)$ if and only if $(1_A f, \int_A \xi\, d\mu) \in \mathrm{Epi}\, I_\phi$ for all $A \in \mathcal{A}$. Indeed, $(f, \xi) \in S_{\rho,1}(F)$ if and only if $(f(\omega), \xi(\omega)) \in (\mathrm{Epi}\,\phi)(\omega)$ a.e., i.e., $\phi(\omega, f(\omega)) \leqq \xi(\omega)$ a.e. which is equivalent to $\int_A \phi(\omega, f(\omega))d\mu \leqq \int_A \xi\, d\mu$ for all $A \in \mathcal{A}$. This means in view of $\phi(\omega, 0)=0$ a.e. that $(1_A f, \int_A \xi\, d\mu) \in \mathrm{Epi}\, I_\phi$ for all $A \in \mathcal{A}$. Thus the lemma is proved.

LEMMA 4.4. *Let* ϕ *be as in Lemma* 4.3. *If there is an* $\alpha \in R$ *such that* $I_\phi(f) \geqq \alpha$ *for all* $f \in L_\rho(X)$, *then there exists a* $\xi \in L_1$ *such that* $\phi(\omega, x) \geqq \xi(\omega)$ *on* X *for a.e.* $\omega \in \Omega$.

Proof. Take an $F \in \mathcal{M}[\Omega; X \times R]$ as in the proof of Lemma 4.3. Since $(0, 0) \in S_{\rho,1}(F)$, there exists, by Lemma 3.2, a sequence $\{(f_n, \xi_n)\}$ in $S_{\rho,1}(F)$ such that $F(\omega)=\mathrm{cl}\{(f_n(\omega), \xi_n(\omega))\}$ for all $\omega \in \Omega$. Then it is easy to see that

$$\inf_{x \in X} \phi(\omega, x) = \inf_{n \geqq 1} \xi_n(\omega) \text{ a.e.}$$

Let $\zeta(\omega)=\inf_n \xi_n(\omega)$. Since $\zeta(\omega) \leqq \phi(\omega, 0)=0$ a.e., it now suffices to show that $\int_\Omega \zeta\, d\mu \geqq \alpha$. Suppose $\int_\Omega \zeta\, d\mu < \alpha$. Then a $\zeta' \in L_1$ can be chosen so that $\zeta(\omega) < \zeta'(\omega)$ a.e. and $\int_\Omega \zeta'\, d\mu < \alpha$. It follows that there exists a countable measurable partition $\{A_n\}$ of Ω such that $\xi_n(\omega) < \zeta'(\omega)$ a.e. on A_n for $n \geqq 1$. Taking an integer k such that $\int_{\bigcup_{n=1}^k A_n} \zeta'\, d\mu < \alpha$ and defining $g = \sum_{n=1}^k 1_{A_n} f_n \in L_\rho(X)$, we have

$$I_\phi(g) = \sum_{n=1}^k \int_{A_n} \phi(\omega, f_n(\omega))d\mu \leqq \sum_{n=1}^k \int_{A_n} \xi_n\, d\mu$$

$$\leqq \int_{\bigcup_{n=1}^k A_n} \zeta'\, d\mu < \alpha,$$

a contradiction, which completes the proof.

§ 5. Representation theorems.

We now present integral representation theorems for several types of additive functionals on $L_\rho(X)$.

THEOREM 5.1. *Let* $\Phi: L_\rho(X)\to\bar{R}$ *be an additive lower semicontinuous proper functional. Then there exists a normal function* $\phi: \Omega\times X\to\bar{R}$ *with* $\phi(\omega, 0)=0$ *a.e. such that* $\phi(\omega, \cdot)$ *is proper for every* $\omega\in\Omega$ *and* $\Phi=I_\phi$ *on* $L_\rho(X)$. *Moreover such a normal function* ϕ *is unique up to sets of the form* $N\times X$ *with* $\mu(N)=0$.

Proof. The final uniqueness assertion follows immediately from Lemma 4.2. Since Φ is additive and proper, we get $\Phi(0)=0$. Define a subset M of $L_\rho(X)\times L_1$ by

$$M=\{(f, \xi)\in L_\rho(X)\times L_1: \Phi(1_A f)\leqq\int_A \xi\, d\mu \text{ for all } A\in\mathcal{A}\}.$$

Let $\{(f_n, \xi_n)\}$ be a sequence in M convergent to $(f, \xi)\in L_\rho(X)\times L_1$. Then we have

$$\Phi(1_A f)\leqq\liminf_{n\to\infty}\Phi(1_A f_n)\leqq\lim_{n\to\infty}\int_A\xi_n\, d\mu=\int_A\xi\, d\mu, \qquad A\in\mathcal{A},$$

and hence $(f, \xi)\in M$. Thus M is closed in $L_\rho(X)\times L_1$. For each $(f, \xi), (g, \zeta)\in M$ and $B\in\mathcal{A}$, we have

$$\begin{aligned}\Phi(1_A(1_B f+1_{\Omega\setminus B}g))&=\Phi(1_{A\cap B}f)+\Phi(1_{A\setminus B}g)\\&\leqq\int_{A\cap B}\xi\, d\mu+\int_{A\setminus B}\zeta\, d\mu=\int_A(1_B\xi+1_{\Omega\setminus B}\zeta)d\mu, \qquad A\in\mathcal{A},\end{aligned}$$

and hence $(1_B f+1_{\Omega\setminus B}g, 1_B\xi+1_{\Omega\setminus B}\zeta)\in M$. Thus M is decomposable. Moreover M is nonempty since $(0, 0)\in M$. Thus, by Theorem 3.4, there exists an $F\in\mathcal{M}[\Omega; X\times R]$ such that $M=S_{\rho,1}(F)$. We can choose, by Lemma 3.2, a sequence $\{(f_i, \xi_i)\}$ in $S_{\rho,1}(F)$ such that $F(\omega)=\mathrm{cl}\{(f_i(\omega), \xi_i(\omega))\}$ for all $\omega\in\Omega$, and a sequence $\{\zeta_j\}$ in L_1 such that $\{\zeta_j(\omega)\}$ is dense in $[0, \infty)$ for every $\omega\in\Omega$. Since $(f_i, \xi_i+\zeta_j)\in M$ for all $i, j\geqq 1$, we obtain

$$F(\omega)=\mathrm{cl}\{(f_i(\omega), \xi_i(\omega)+\zeta_j(\omega)): i, j\geqq 1\} \text{ a.e.},$$

which shows that there exists an $N\in\mathcal{A}$ with $\mu(N)=0$ such that $(x, \alpha)\in F(\omega)$ implies $\{x\}\times[\alpha, \infty)\subset F(\omega)$ for each $\omega\in\Omega\setminus N$. Now define $\phi: \Omega\times X\to\bar{R}$ by

$$\phi(\omega, x)=\begin{cases}\inf\{\alpha: (x, \alpha)\in F(\omega)\} & \text{if } \omega\in\Omega\setminus N\\ 0 & \text{if } \omega\in N.\end{cases}$$

Then we have

$$(\mathrm{Epi}\,\phi)(\omega)=\begin{cases} F(\omega) & \text{if } \omega\in\Omega\backslash N \\ X\times[0,\infty) & \text{if } \omega\in N, \end{cases}$$

and hence $\mathrm{Epi}\,\phi\in\mathcal{M}[\Omega;X\times R]$ which implies that ϕ is normal. We shall then prove that $\Phi=I_\phi$ on $L_\rho(X)$ in the following three parts:

(I) Let $f\in L_\rho(X)$ and $\Phi(f)<\infty$. We show that $I_\phi(f)$ is defined and $I_\phi(f)\leqq\Phi(f)$. In view of Lemma 4.1, the set function $A\mapsto\Phi(1_A f)$ is a μ-continuous bounded signed measure on $\mathcal{A}$, and hence it has a Radon-Nikodym derivative $\xi\in L_1$ with respect to μ. Then we have $(f,\xi)\in M$ and hence $(f(\omega),\xi(\omega))\in F(\omega)$ a.e., so that $\phi(\omega,f(\omega))\leqq\xi(\omega)$ a.e. This implies that $I_\phi(f)$ is defined and $I_\phi(f)\leqq\int_\Omega\xi\,d\mu=\Phi(f)$.

(II) Let $f\in L_\rho(X)$ and assume that $I_\phi(f)$ is defined. We show that $\Phi(f)\leqq I_\phi(f)$. Assuming $I_\phi(f)<\infty$, we can select a sequence $\{\xi_n\}$ in L_1 such that $\xi_n(\omega)\downarrow\phi(\omega,f(\omega))$ a.e. Since $(f(\omega),\xi_n(\omega))\in(\mathrm{Epi}\,\phi)(\omega)=F(\omega)$ a.e., we get $(f,\xi_n)\in M$ for all n, and hence $\Phi(f)\leqq\int_\Omega\xi_n\,d\mu\downarrow I_\phi(f)$ by the monotone convergence theorem. Thus $\Phi(f)\leqq I_\phi(f)$.

(III) We now deduce that $I_\phi(f)$ is defined for every $f\in L_\rho(X)$. To see this, suppose that $I_\phi(f)$ is not defined, and let $A=\{\omega\in\Omega:\phi(\omega,f(\omega))<0\}$. Then it follows that $\int_A\phi(\omega,f(\omega))d\mu=-\infty$. By part (I), we obtain $I_\phi(0)\leqq\Phi(0)=0$ and so $\int_{\Omega\backslash A}\phi(\omega,0)d\mu<\infty$. Hence we have

$$I_\phi(1_A f)=\int_A\phi(\omega,f(\omega))d\mu+\int_{\Omega\backslash A}\phi(\omega,0)d\mu=-\infty,$$

so that by part (II) we have $\Phi(1_A f)=-\infty$ contradicting the assumption of Φ being proper.

The above three parts (I)-(III) yield that $\Phi=I_\phi$ on $L_\rho(X)$. We shall finally show that ϕ can be modified so as to satisfy the conditions in the theorem. Define $H:\Omega\to 2^X$ by $H(\omega)=\{x\in X:\phi(\omega,x)=-\infty\}$. Since $G(H)\in\mathcal{A}\otimes\mathcal{B}_X$, $D(H)\in\overline{\mathcal{A}}$ and there exists an $\overline{\mathcal{A}}$-measurable function $g:\Omega\to X$ such that $g(\omega)\in H(\omega)$ for all $\omega\in D(H)$. Suppose that $D(H)$ is not μ-null. Taking an $\mathcal{A}$-measurable function $f:\Omega\to X$ with $f(\omega)=g(\omega)$ a.e., we can choose an $A\in\mathcal{A}$ with $\mu(A)>0$ such that $f(\omega)\in H(\omega)$ for a.e. $\omega\in A$ and moreover $1_A f\in L_\rho(X)$. Then we have $\Phi(1_A f)=-\infty$, a contradiction, which implies that $D(H)$ is μ-null. Since ϕ may be modified appropriately on a set $N\times X$ with $\mu(N)=0$, ϕ can be taken so that $\phi(\omega,\cdot)$ is proper for every $\omega\in\Omega$. Furthermore, in view of $\Phi(0)=0$, replacing $\phi(\omega,\cdot)$ by $\phi(\omega,\cdot)-\phi(\omega,0)$ for $\omega\in\Omega$ with $\phi(\omega,0)<\infty$, we can let $\phi(\omega,0)=0$ a.e. Thus the proof is completed.

We call a function $\phi:\Omega\times X\to R$ to be *of Carathéodory type* if ϕ satisfies the following two conditions:

(i) $\phi(\cdot,x):\Omega\to R$ is measurable for each $x\in X$,

(ii) $\phi(\omega, \cdot): X \to R$ is continuous for each $\omega \in \Omega$.
It is known (cf. [9, Theorem 6.1]) that a function of Carathéodory type as above is $\mathcal{A} \otimes \mathcal{B}_X$-measurable. In the usual definition of Carathéodory function, the condition (ii) is weakened so that $\phi(\omega, \cdot)$ is continuous for a.e. $\omega \in \Omega$. Whenever a function $\phi: \Omega \times X \to R$ is considered as an integral kernel function, we may modify ϕ appropriately on a set $N \times X$ with $\mu(N)=0$. Hence we adopt here the above definition. Let $\mathrm{Car}_\rho(\Omega; X)$ denote the collection of all functions $\phi: \Phi \times X \to R$ of Carathéodory type such that for each $f \in L_\rho(X)$ the function $\phi(\omega, f(\omega))$ is in L_1.

THEOREM 5.2. *If $\Phi: L_\rho(X) \to R$ is an additive continuous functional, then there exists a $\phi \in \mathrm{Car}_\rho(\Omega; X)$ with $\phi(\omega, 0)=0$ a.e. such that $\Phi = I_\phi$ on $L_\rho(X)$. Moreover such a function ϕ is unique up to sets of the form $N \times X$ with $\mu(N)=0$.*

Proof. By Theorem 5.1, there exist two normal functions $\phi, \psi: \Omega \times X \to \overline{R}$ with $\phi(\omega, 0)=\psi(\omega, 0)=0$ a.e. such that $\Phi = I_\phi = -I_\psi$ on $L_\rho(X)$. Then, applying Lemma 4.2, we can take an $N \in \mathcal{A}$ with $\mu(N)=0$ such that $\phi(\omega, x) = -\psi(\omega, x)$ for all $\omega \in \Omega \backslash N$ and $x \in X$. Redefining $\phi(\omega, x)=0$ on $N \times X$, we obtain a desired $\phi \in \mathrm{Car}_\rho(\Omega; X)$.

REMARK. When $L_\rho(X)$ is a Banach space (for example, when ρ has the weak Fatou property), it can be shown as in [10, pp. 22-25] that if $\phi \in \mathrm{Car}_\rho(\Omega; X)$, then the operator $T: L_\rho(X) \to L_1$ defined by $Tf(\omega)=\phi(\omega, f(\omega))$ is continuous. Thus, in this situation, the converse of Theorem 5.2 holds: If $\phi \in \mathrm{Car}_\rho(\Omega; X)$ and $\phi(\omega, 0)=0$ a.e., then the integral functional I_ϕ is additive and continuous on $L_\rho(X)$.

We denote by $\mathcal{L}_{\rho'}(X^*)$ the space of all functions $f^*: \Omega \to X^*$ satisfying the following two conditions:

(1) $<x, f^*(\cdot)>: \Omega \to R$ is measurable for each $x \in X$,

(2) the function $\|f^*\| = \|f^*(\cdot)\|$ is in $L_{\rho'}$.

Note that the condition (1) implies the measurability of $\|f^*(\cdot)\|$. Under the usual identification of μ-almost everywhere equal functions, $\mathcal{L}_{\rho'}(X^*)$ is a normed linear space (in fact, a Banach space) with the norm $\rho'(\|f^*\|)$.

THEOREM 5.3. *The dual space $L_\rho(X)^*$ of $L_\rho(X)$ is isometrically isomorphic to $\mathcal{L}_{\rho'}(X^*)$ under the bilinear form $\langle f, f^* \rangle = \int_\Omega \langle f(\omega), f^*(\omega) \rangle d\mu$ of $f \in L_\rho(X)$ and $f^* \in \mathcal{L}_{\rho'}(X^*)$.*

Proof. Let $f^* \in \mathcal{L}_{\rho'}(X^*)$. For each $f \in L_\rho(X)$, it follows that the function $\langle f(\omega), f^*(\omega) \rangle$ is measurable and

$$\int_\Omega |\langle f(\omega), f^*(\omega) \rangle| \, d\mu \leqq \int_\Omega \|f(\omega)\| \|f^*(\omega)\| \, d\mu \leqq \rho(\|f\|) \rho'(\|f^*\|) < \infty .$$

Thus the linear functional $\Phi(f) = \langle f, f^* \rangle$ is well-defined on $L_\rho(X)$ and we get $\|\Phi\| \leqq \rho'(\|f^*\|)$.

Conversely let $\Phi \in L_\rho(X)^*$. By Theorem 5.2, there exists a $\phi \in \mathrm{Car}_\rho(\Omega; X)$ with $\phi(\omega, 0)=0$ a.e. such that $\Phi = I_\phi$ on $L_\rho(X)$. For each $f, g \in L_\rho(X)$ and each $\alpha, \beta \in R$, since

$$\begin{aligned}\int_A \phi(\omega, \alpha f(\omega)+\beta g(\omega))d\mu &= \Phi(1_A(\alpha f+\beta g)) \\ &= \alpha\Phi(1_A f)+\beta\Phi(1_A g) \\ &= \int_A \{\alpha\phi(\omega, f(\omega))+\beta\phi(\omega, g(\omega))\}\, d\mu, \qquad A \in \mathcal{A},\end{aligned}$$

it follows that $\phi(\omega, \alpha f(\omega)+\beta g(\omega))=\alpha\phi(\omega, f(\omega))+\beta\phi(\omega, g(\omega))$ a.e. There exists, as in Lemma 3.2, a sequence $\{f_n\}$ in $L_\rho(X)$ such that $\{f_n(\omega)\}$ is dense in X for every $\omega \in \Omega$. We can now take an $N \in \mathcal{A}$ with $\mu(N)=0$ such that

$$\phi(\omega, \alpha f_i(\omega)+\beta f_j(\omega))=\alpha\phi(\omega, f_i(\omega))+\beta\phi(\omega, f_j(\omega)), \qquad \omega \in \Omega \backslash N,$$

for each $i, j \geqq 1$ and each rational numbers α, β. This shows that $\phi(\omega, \cdot) \in X^*$ for every $\omega \in \Omega \backslash N$. Define

$$f^*(\omega)=\begin{cases} \phi(\omega, \cdot) & \text{if } \omega \in \Omega \backslash N \\ 0 & \text{if } \omega \in N. \end{cases}$$

Then it is clear that f^* satisfies the above condition (1). It remains to show that $\rho'(\|f^*\|) \leqq \|\Phi\|$. Since ρ' is a saturated function norm having the Fatou property, for any given $\varepsilon>0$ there exists a strictly positive $\eta \in M^+$ with $\rho'(\eta) < \varepsilon$. Then we can select a measurable function $u: \Omega \to X$ such that $\|u(\omega)\| \leqq 1$ and $\langle u(\omega), f^*(\omega)\rangle \geqq \max(0, \|f^*(\omega)\|-\eta(\omega))$ for all $\omega \in \Omega$. Putting $\zeta(\omega)=\langle u(\omega), f^*(\omega)\rangle$, we have $\zeta \in M^+$ and $\|f^*\| \leqq \zeta+\eta$. For each $\xi \in M^+$ with $\rho(\xi) \leqq 1$, it follows that

$$\begin{aligned}\int_\Omega \xi\zeta\, d\mu &= \int_\Omega \langle \xi(\omega)u(\omega), f^*(\omega)\rangle d\mu \\ &= \Phi(\xi u) \leqq \|\Phi\|\rho(\|\xi u\|) \leqq \|\Phi\|,\end{aligned}$$

which shows $\rho'(\zeta) \leqq \|\Phi\|$ and so $\rho'(\|f^*\|) \leqq \rho'(\zeta)+\rho'(\eta) < \|\Phi\|+\varepsilon$. Thus we have the desired conclusion.

REMARK 1. When ρ is not necessarily absolutely continuous, Theorem 5.3 is extended as follows: If the carrier of L_ρ^a is the whole set Ω, then $L_\rho^a(X)^*$ is isometrically isomorphic to $\mathcal{L}_{\rho'}(X^*)$ in the manner as in Theorem 5.3.

REMARK 2. If X^* is separable, or equivalently if X^* has the Radon-Nikodym property (cf. [18]), then Theorem 5.3 asserts that $L_\rho(X)^*$ is isometrically isomorphic to $L_{\rho'}(X^*)$. This conclusion is a special case of [7, Theorem 3.2], but ρ is as- sumed in [7] to have the weak Fatou property.

For the case of lower semicontinuous convex functionals, we give a representation theorem in a somewhat detailed form.

THEOREM 5.4. *For each proper functional* $\Phi: L_\rho(X) \to \overline{R}$, Φ *is additive lower* *semicontinuous and convex if and only if there exists a normal function* $\phi: \Omega\times X\to\bar{R}$ *with* $\phi(\omega, 0)=0$ *a.e. such that*

(i) $\phi(\omega, \cdot)$ *is proper and convex for every* $\omega\in\Omega$,

(ii) *there exists an* $f^*\in\mathcal{L}_{\rho'}(X^*)$ *and a* $\xi\in L_1$ *satisfying* $\phi(\omega, x)\geqq\langle x, f^*(\omega)\rangle+\xi(\omega)$ *on* X *for a.e.* $\omega\in\Omega$,

(iii) $\Phi=I_\phi$ *on* $L_\rho(X)$.

Proof. Let $\Phi: L_\rho(X)\to\bar{R}$ be additive, lower semicontinuous, proper, and convex. By Theorem 5.1 and Lemma 4.3, there exists a normal function $\phi: \Omega\times X\to\bar{R}$ with $\phi(\omega, 0)=0$ a.e. for which the conditions (i) and (iii) are satisfied. Since Epi Φ is closed and convex in $L_\rho(X)\times R$ and $(0, -1)\notin\text{Epi}\,\Phi$, the separation theorem gives, in view of Theorem 5.3, an $f^*\in\mathcal{L}_{\rho'}(X^*)$ and a $\beta\in R$ such that $\langle f, f^*\rangle+\alpha\beta<-\beta$ for all $(f, \alpha)\in\text{Epi}\,\Phi$. Then $\beta<0$ follows from $(0, 0)\in\text{Epi}\,\Phi$, and hence we can let $\beta=-1$. We now have

$$\int_\Omega\{\phi(\omega, f(\omega))-\langle f(\omega), f^*(\omega)\rangle\}\,d\mu$$
$$=\Phi(f)-\langle f, f^*\rangle>-1, \qquad f\in L_\rho(X),$$

which implies the condition (ii) by Lemma 4.4.

Conversely let ϕ be a normal function with $\phi(\omega, 0)=0$ a.e. satisfying (i)-(iii). It is immediate that $\Phi=I_\phi$ is additive and convex. To show the lower semicontinuity, let $\{f_n\}\subset L_\rho(X)$, $f\in L_\rho(X)$, and $\rho(\|f_n-f\|)\to 0$. As is seen from the proof of Lemma 3.1, we can select a subsequence $\{g_k\}$ of $\{f_n\}$ such that $\|g_k(\omega)-f(\omega)\|\to 0$ a.e. and $\Phi(g_k)\to\liminf_{n\to\infty}\Phi(f_n)$. Then, using Fatou's lemma, we have

$$\Phi(f)-\langle f, f^*\rangle-\int_\Omega\xi\,d\mu$$
$$=\int_\Omega\{\phi(\omega, f(\omega))-\langle f(\omega), f^*(\omega)\rangle-\xi(\omega)\}\,d\mu$$
$$\leqq\int_\Omega\liminf_{k\to\infty}\{\phi(\omega, g_k(\omega))-\langle g_k(\omega), f^*(\omega)\rangle-\xi(\omega)\}\,d\mu$$
$$\leqq\lim_{k\to\infty}\{\Phi(g_k)-\langle g_k, f^*\rangle-\int_\Omega\xi\,d\mu\}$$
$$=\liminf_{n\to\infty}\Phi(f_n)-\langle f, f^*\rangle-\int_\Omega\xi\,d\mu,$$

and hence $\Phi(f)\leqq\liminf_{n\to\infty}\Phi(f_n)$. The proof is now completed.

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