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# ON MINIMAL $n$-DIMENSIONAL SUBMANIFOLDS OF A SPACE FORM $R^m(k)$, WHICH ARE FOLIATED BY $(n-1)$-DIMENSIONAL TOTALLY GEODESIC SUBMANIFOLDS OF $R^m(k)$

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In this paper we prove that a minimal $n$-dimensional ($n\geqq 3$) submanifold of $R^m(k)$, foliated by $(n-1)$-dimensional totally geodesic submanifolds of $R^m(k)$, is locally contained in an 1-dimensional totally geodesic submanifold of $R^m(k)$ (i. e. in a space form $R^1(k)$), with $1\leqq 2n-1$.

## §1. Preliminaries

We assume throughout that all manifolds, maps, vector fields, etc.... are differentiable of class $C^\infty$.

Let $N$ be an $n$-dimensional submanifold of a Riemannian manifold $R^m$ and let $D$ (resp. $\bar{D}$) be the Riemannian connection of $N$ (resp. $R^m$). If $X$ and $Y$ are tangent vector fields on $N$, then the second fundamental form $V$ is given by

$$\bar{D}_xY=D_xY+V(X, Y).$$

$V(X, Y)$ is a normal vector field on $N$ and is symmetric in $X$ and $Y$.

Let $\xi$ be a normal vector field on $N$, then, by decomposing $\bar{D}_x\xi$ in a tangent and a normal component, we find

$$\bar{D}_x\xi=-A_\xi(X)+D^\perp_x\xi.$$

$D^\perp$ is a metric connection in the normal bundle $N^\perp$ of $N$ in $R^m$ and $A_\xi$ determines at each point $p$ of $N$ a self adjoint linear map $N_p\to N_p$. Moreover, we have

$$\langle V(X, Y), \xi\rangle=\langle A_\xi(X), Y\rangle. \tag{1.1}$$

If $e_1, \cdots, e_n$ is an orthonormal base field of $N$, then the mean curvature vector $H$ is given by

$$H=\frac{1}{n}\sum_{i=1}^{n} V(e_i, e_i).$$

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If $H=0$ at each point of $N$, then $N$ is said to be minimal.

## 2. Minimal $n$-dimensional submanifolds of $R^m(k)$, foliated by $(n-1)$-dimensional totally geodesic submanifolds of $R^m(k)$ $(n\geqq 3)$

A space form $R^m(k)$ is by definition a complete simply connected Riemannian manifold of constant sectional curvature $k$ (see [1]).

Suppose that the $n$-dimensional submanifold $N$ of $R^m(k)$ is a locus of $(n-1)$-dimensional totally geodesic submanifolds of $R^m(k)$. Assume that $\bar{D}$, $D$ and $D'$ are the Riemannian connections of respectively $R^m(k)$, $N$ and the leave $L$ (i.e. the totally geodesic submanifold) through the point $p$ of $N$. Then, if $x$ and $y$ are $L$-vector fields, we find, since $L$ is totally geodesic in $R^m(k)$:

$$\bar{D}_x y = D'_x y .$$

But, if $V$ is the second fundamental form of $N$ in $R^m(k)$, then

$$\bar{D}_x y = D_x y + V(x, y) .$$

Moreover, if $V'$ is the second fundamental form of $L$ in $N$, then

$$D_x y = D'_x y + V'(x, y) .$$

From all this we get

$$V(x, y) + V'(x, y) = 0 ,$$

and thus $V'(x, y)=0$, i.e. $L$ is also totally geodesic in $N$, and $V(x, y)=0$ for each two $L$-vector fields $x$ and $y$. Consider an orthonormal base field $e_1, \cdots, e_n$ of $N$, such that $e_1, \cdots, e_{n-1}$ constitute at each point of the domain of the field an orthonormal base of the tangent space of the leave through that point.

We find, since $V(e_i, e_j)=0 \quad i, j=1, \cdots, n-1$,

$$H = \frac{1}{n} V(e_n, e_n). \tag{2.1}$$

So, $N$ is minimal iff each normal in $N$ at each point of each leave determines an asymptotic direction of $N$.

We define the normal subspace $F_p$ at each point $p$ of $N$ as the subspace of the normal space $N_p^{\perp}$, spanned by the normal vectors

$$\{V(X, Y) \| (X, Y) \in N_p \times N_p\} .$$

So we have a normal subbundle $F$ of the normal bundle $N^{\perp}$. Using the same base field $e_1, \cdots, e_n$ as above, we find, if $X=\sum_{i=1}^{n} a^i (e_i)_p$ and $Y=\sum_{i=1}^{n} b^i (e_i)_p$,

$$V(X, Y) = \sum_{i=1}^{n-1} (a^i b^n + b^i a^n) V((e_i)_p, (e_n)_p) + a^n b^n V((e_n)_p, (e_n)_p) . \tag{2.2}$$

So we have: $0 \leqq \dim F \leqq n$. But if $N$ is minimal, then $V((e_n)_p, (e_n)_p)=0$ and

$0 \leqq \dim F \leqq n-1$ at each point.

THEOREM 1. *If the manifold $N$ is minimal and if $\dim F_p = f$ ($f$ constant; $0 \leqq f \leqq n-1$) at each point $p$ of $N$, then $N$ is (locally) contained in an $(n+f)$-dimensional totally geodesic submanifold of $R^m(k)$.*

*Proof.* If $F_p$ is 0-dimensional at each point $p$ of $N$, then $N$ is totally geodesic in $R^m(k)$.

Suppose that $\dim F_p = f \neq 0$ at each point $p$ of $N$ and take an orthonormal normal base field $\xi_1, \cdots, \xi_{m-n}$ such that $\xi_1, \cdots, \xi_f$ constitute an orthonormal base of $F_p$ at each point $p$ of the domain of the field. Then it is clear from (1.1) that

$$A_{\xi_{f+1}} = \cdots = A_{\xi_{m-n}} = 0 . \tag{2.3}$$

Assume that $X$ and $Y$ are $N$-vector fields and set

$$V(X, Y) = \sum_{i=1}^{m-n} V^i(X, Y)\xi_i ,$$

then we find immediately that

$$V^{f+1} = \cdots = V^{m-n} = 0 \quad \text{at each point}. \tag{2.4}$$

If $\bar{R}$ is the curvature tensor of $R^m(k)$ and if $Z$ is another vector field of $N$, then the Codazzi equation says

$$(\bar{R}(X, Y)Z)^{\perp} = \sum_{j=1}^{m-n} \{(D_X V^j)(Y, Z) - D_Y V^j)(X, Z)\} \xi_j + \sum_{j=1}^{m-n} V^j(Y, Z) D_X^{\perp} \xi_j - \sum_{j=1}^{m-n} V^j(X, Z) D_Y^{\perp} \xi_j = 0 . \tag{2.5}$$

Consider again an orthonormal base field $e_1, \cdots, e_n$ of $N$ such that $e_1, \cdots, e_{n-1}$ constitute at each point $p$ of the domain of the field, a base of the tangent space $L_p$ of the leave $L$ through $p$.

Put

$$D_{e_i}^{\perp} \xi_l = \sum_{h=1}^{f} C_{il}^h \xi_h + \sum_{r=f+1}^{m-n} C_{il}^r \xi_r \quad {\scriptstyle i=1,\cdots,n;\ p=1,\cdots,f} . \tag{2.6}$$

Then, from (2.4) and (2.5), we have

$$(\bar{R}(e_i, e_n)e_j)^{\perp} = \sum_{l=1}^{f} \{\cdots\} \xi_l + \sum_{l=1}^{f} V^l(e_n, e_s) D_{e_i}^{\perp} \xi_l - \sum_{l=1}^{f} V^l(e_i, e_j) D_{e_n}^{\perp} \xi_l = 0 \quad i, j = 1, \cdots, n-1 . \tag{2.7}$$

But $V(e_i, e_j) = 0$ $i, j = 1, \cdots, n-1$ and so we find from (2.6) and (2.7)

$$\sum_{l=1}^{f} V^l(e_n, e_s) C_{il}^r = 0 \quad i, s = 1, \cdots, n-1 ; \ r = f+1, \cdots, m-n . \tag{2.8}$$

Now it is clear from (2.2), that, since $N$ is minimal, $F_p$ is spanned at each point $p$ by the vectors $(V(e_n, e_s))_p$ $s=1, \cdots, n-1$ and so the rank of the matrix

$$[V^l(e_n, e_s)]_{\substack{s=1,\cdots,n-1\\ l=1,\cdots,f}}$$

is at each point (of the domain of the field $e_1, \cdots, e_n$) equal to $f$. So, it is easy to see that (2.8) gives

$$C^r_{il}=0 \quad i=1, \cdots, n-1\,;\ r=f+1, \cdots, m-n\,;\ l=1, \cdots, f. \tag{2.9}$$

We also have

$$(\bar{R}(e_n, e_i)e_n)^\perp=\sum_{l=1}^{f}\{\cdots\}\xi_l+\sum_{l=1}^{f}V^l(e_i, e_n)D^\perp_{e_n}\xi_l - \sum_{l=1}^{f}V^l(e_n, e_n)D^\perp_{e_i}\xi_l=0 \qquad i=1, \cdots, n-1. \tag{2.10}$$

But $H=0$, so $V(e_n, e_n)=0$ and we find from (2.6)

$$\sum_{l=1}^{f}V^l(e_i, e_n)C^r_{nl}=0 \qquad i=1, \cdots, n-1\,;\ r=f+1, \cdots m-n\,.$$

This gives analogously

$$C^r_{nl}=0 \qquad r=f+1, \cdots, m-n\,;\ l=1, \cdots, f. \tag{2.11}$$

From (2.6), (2.9) and (2.11) we see that the subbundle $F$ is parallel in the normal bundle $N^\perp$. This fact together with (2.3) gives that $N$ is (locally) contained in an $(n+f)$-dimensional totally geodesic submanifold of $R^m(k)$, which completes the proof.

Suppose now that the submanifold $N$ is not minimal and consider again the orthonormal base field $e_1, \cdots, e_n$ used in the proof of theorem 1, then we have:

THEOREM 2. *If the mean curvature vector $H\neq 0$ of the manifold $N$ is a vector of the normal subspace $F_p$ spanned by the fields $(V(e_i, e_n))_p$ $i=1, \cdots, n-1$ at each point $p$ of $N$, and if* $\dim F_p=f$ *($f$ constant; $1\leqq f\leqq n-1$) at each point $p$, then $N$ is (locally) contained in an $(n+f)$-dimensional totally geodesic submanifold of $R^m(k)$.*

*Proof.* Take again an orthonormal base field $\xi_1, \cdots \xi_{m-n}$ such that $\xi_1, \cdots, \xi_f$ is a base field of $F$. Then, since $H\in F$, we have again

$$V^{f+1}(X, Y)= \cdots =V^{m-n}(X, Y)=0\,,$$

for each two $N$-vector fields $X$ and $Y$.

Next, if we have (2.6), then we find from (2.7) again (2.9). Moreover, since the vector fields $D^\perp_{e_i}\xi_1$ $i=1, \cdots, n-1$; $l=1, \cdots, f$ have no components in the complementary subbundle $F^\perp$, we find because of (2.10) again (2.11) and this completes the proof.

We try now to formulate theorem 1 and 2 in terms of the sectional curvature of $N$.

If $X$ and $Y$ are vectors of $N_p$, then, from the Gauss equation, we know that the sectional curvature $K(X, Y)$ of $N$ in the two-dimensional direction of $N_p$ spanned by $X$ and $Y$, is given by

$$K(X, Y)=k-\langle V(X, Y), V(X, Y)\rangle+\langle V(X, X), V(Y, Y)\rangle .$$

Consider again the special base field $e_1, \cdots, e_n$ of $N$ (used in the proofs of the preceding theorems). Then we find, since $V(e_i, e_i)=0$ $i=1, \cdots, n-1$,

$$K(e_i, e_n)=k-\langle V(e_i, e_n), V(e_i, e_n)\rangle \qquad i=1, \cdots, n-1 . \tag{2.12}$$

A two-dimensional direction of a tangent space $N_p$ which contain $(e_n)_p$ (a unit normal vector in $N_p$ on $L_p$) is called a normal two-dimensional direction of $N_p$. So, from (2.12) we see that if the dimension of the subbundle $F$, spanned by $V(e_i, e_n)$ $i=1, \cdots, n-1$, is $f$ (constant; $0\leqq f\leqq n-1$), at each point, then we find at each point of $N$ in the tangent space $L_p$ of the leave through $p$, an $(n-f-1)$-dimensional subspace $I_p$, such that for all $x\in I_p$, $x\neq 0$: $K(x, (e_n)_p)=k$. Now we can formulate theorem 1 as follows: If $N$ is minimal and if at each point $p$ of $N$ the tangent space $L_p$ of the leave through $p$ contains an $(n-f-1)$-dimensional subspace $I_p$ ($f$ constant; $0\leqq f\leqq n-1$), such that for each vector $x\in I_p$, $x\neq 0$, the sectional curvature of $N$ at $p$ in the normal two-dimensional direction of $N_p$ determined by $x$, is equal to $k$, then $N$ is (locally) contained in an $(n+f)$-dimensional totally geodesic submanifold of $R^m(k)$.

Theorem 2 can be formulated in a similar way.

THEOREM 3. *If $N$ is minimal and if for every leaf $L$ of $N$ the unit normal vector field on $L$ in $N$ is parallel in the normal bundle of $L$ in $R^m(k)$, then $N$ is totally geodesic in $R^m(k)$.*

*Proof.* The unit normal vector field on $L$ in $M$ is locally denoted by $e_n$ (such as in the proofs of the preceding theorems). We find, if $x$ is any vector field of $L$ and $\bar{D}$ the Riemannian connection of $R^m(k)$, by decomposing $\bar{D}_x e_n$ in a tangent and a normal component

$$\bar{D}_x e_n=-A_{e_n}(x)+D'^{\perp}_x e_n .$$

But we also have, if $D$ is the connection of $N$ and $V$ his second fundamental form

$$\bar{D}_x e_n=D_x e_n+V(x, e_n) .$$

So, it is at once clear (since $D_x e_n\perp e_n$), that

$$D'^{\perp}_x e_n=V(x, e_n) .$$

If $e_n$ is parallel in the normal bundle $L^\perp$ and if $N$ is minimal, then (2.2) says that $V(X, Y)=0$ for each two $N$-vector fields $X$ and $Y$, which completes the proof.

*Remark.* If $N$ is a $n$-dimensional submanifold of the euclidean space $E^m$, foliated by $(n-1)$-dimensional linear subspaces of $E^m$, then $N$ is called a monosystem. The condition $\dim F_p=f$ at each point $p$, which appears in the statements of theorem 1 and 2, means that $N$ is $(n-f-2)$-developable (see [3]).

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