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A CERTAIN PROPERTY OF GEODESICS OF THE FAMILY OF RIEMANNIAN MANIFOLDS O_n^2 (II)

Dedicated to Professor Hitoshi HONBU on his 70th birthday

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§0. Introduction.

This is a continuation of the part (I) with the same title written by the present author. We shall use the same notation in it.

The period T of any non-constant solution $x(t)$ of the non-linear differential equation of order 2:

$$\text{(E)} \qquad nx(1-x^2)\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+(1-x^2)(nx^2-1)=0$$

with a constant $n>1$ such that $x^2+x'^2<1$ is given by the integral:

$$\text{(0.1)} \qquad T=\sqrt{nc}\int_{x_0}^{x_1}\frac{dx}{x\sqrt{(n-x)\{x(n-x)^{n-1}-c\}}},$$

where $0<x_0<1<x_1<n$ and $c=x_0(n-x_0)^{n-1}=x_1(n-x_1)^{n-1}$.

At the early stage of this work, the author imagined that T as a function of x_0 and n is monotone decreasing with respect to $n(\geqq 2)$, in order to imply the inequality:

$$\text{(U)} \qquad T<\sqrt{2}\pi,$$

which can be easily proved in the case of $n=2$. This inequality was proved in [8] and [9]. But this supposition is not true as is shown in the table of the values of T for $x_0=1/2$, $1/4$ and $n=2, 4, 8$ in Remark 2 in §4 of the part (I) ([11]).

On the other hand, he obtained also certain negative facts for the supposition. By (1.8), (1.9) and Proposition 1 of [11], we have the formula:

$$\text{(0.2)} \qquad \frac{\partial T(x_0, n)}{\partial n}=-\frac{1}{2b^2}\sqrt{\frac{c}{n}}\int_{x_0}^{x_1}\frac{M(x, x_0)dx}{\sqrt{(n-x)^3}\{x(n-x)^{n-1}-c\}},$$

where

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$$M(x, x_0):=\frac{\{4n-1-(2n+1)x\}\mu(x)-n(n-x)^{n-1}}{n(n-x)^{n-1}}\cdot F(x, x_0) \tag{0.3}$$
$$+2n(x-1)(n-x)\mu(x)\{\lambda(x)-\lambda(x_0)\},$$

$$\lambda(x):=\log(n-x)+\frac{n-1}{n-x}, \tag{0.4}$$

$$F(x, x_0):=x(n-x)^{n-1}-x_0(n-x_0)^{n-1}+n(n-x)^n\{\lambda(x)-\lambda(x_0)\}, \tag{0.5}$$

$$\mu(x):=\begin{cases} \dfrac{B-x(n-x)^{n-1}}{(x-1)^2} & \text{for } 0<x<n,\ x\neq 1, \\ \dfrac{n(n-1)^{n-2}}{2} & \text{for } x=1, \end{cases} \tag{0.6}$$

$$c=x_0(n-x_0)^{n-1} \quad \text{and} \quad B=(n-1)^{n-1}. \tag{0.7}$$

The function $M(x, x_0)$ is real analytic for $0<x<n$, positive in $(x_0, 1)$, negative in $(1, x^*)$ and $M(x_0, x_0)=M(1, x_0)=0$ by Proposition 2 in [11], where x^* is the value such that $\lambda(x_0)=\lambda(x^*)$ and $1<x^*<n$. Now, we define a function $X=X_n(x)$ $(0\leqq x\leqq 1)$ by

$$x(n-x)^{n-1}=X(n-X)^{n-1}, \qquad 1\leqq X\leqq n, \tag{0.8}$$

and then we have

$$\frac{dX}{dx}=\frac{1-x}{x(n-x)}\cdot\frac{X(n-X)}{1-X}. \tag{0.9}$$

Using this function, we have

$$\int_{x_0}^{x_1}\frac{M(x, x_0)dx}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}}}=\int_{x_0}^{1}\frac{1-x}{x(n-x)\sqrt{x(n-x)^{n-1}-c}}\cdot K(x, x_0)dx, \tag{0.10}$$

where

$$K(x, x_0):=\begin{cases} \dfrac{xM(x, x_0)}{(1-x)\sqrt{n-x}}-\dfrac{XM(X, x_0)}{(1-X)\sqrt{n-X}} & \text{for } 0<x<1, \\ 0 \quad \text{for } x=1. \end{cases}$$

We can prove that $K(x, x_0)$ is continuous for $0\leqq x_0\leqq x\leqq 1$ except $x=x_0=0$, $K(x_0, x_0)>0$ for $0<x_0<1$. If $K(x, x_0)\geqq 0$ for $0<x_0\leqq x\leqq 1$, we could obtain the inequality $\dfrac{\partial T(x_0, n)}{\partial n}<0$ for $0<x_0<1$ and $n>2$. But, being contrary to this expectation, we can prove the following facts:

$$\lim_{x_0\to 0} K(x_0, x_0)=+\infty \quad \text{and} \quad \lim_{x\to 0} K(x, 0)=-\infty.$$

Through these experiments and others, the author sets the following conjecture.

CONJECTURE A. *The period function $T(x_0, n)$ of the solutions of the differential equation* (E) *is monotone increasing with respect to $n(\geqq 2)$ for any fixed x_0* $(0<x_0<1)$.

In this paper, he will try to prove this conjecture by the fundamental principle as follows: To prove

$$\text{(i)}\qquad \frac{\partial}{\partial x_0}\int_{x_0}^{x_1}\frac{M(x, x_0)\,dx}{\sqrt{(n-x)^3}\{x(n-x)^{n-1}-c\}}>0$$

and

$$\text{(ii)}\qquad \lim_{x_0\to 1}\int_{x_0}^{x_1}\frac{M(x, x_0)\,dx}{\sqrt{(n-x)^3}\{x(n-x)^{n-1}-c\}}=0,$$

from which we shall obtain

$$\int_{x_0}^{x_1}\frac{M(x, x_0)\,dx}{\sqrt{(n-x)^3}\{x(n-x)^{n-1}-c\}}<0.$$

In § 1, we shall treat the fundamental formulas (1.9) and (1.10) related with (i) and several auxiliary functions $f_0(x)$, $F_0(x)$, $f_1(x)$, $F_1(x)$ and $U(x, x_0)$ appeared in it. In § 2, we shall study properties of $f_0(x)$ and $F_0(x)$. In § 3, we shall study the function $F_2(x)$ which is the principal factor of $F_0'(x)$. In § 4 and § 5, we shall study properties of $f_1(x)$ and $F_1(x)$ according to the same method as is used in § 2 and § 3 for $f_0(x)$ and $F_0(x)$. In § 6, we shall prove the positiveness of $U(x, x_0)$. In § 7, we shall prove an inequality on the function $\tilde{M}(x, x_0)$ defined by (1.7) and the above equality (ii). In this work, we could not succeed disappointedly in proving this conjecture (see Appendix) and need further study of a function of x and x_0 made from the quantity in the brackets of the right hand side of (1.9), in the same way as $K(x, x_0)$ is made from $M(x, x_0)$.

However, the main purpose of the series of the present papers with the same title is to prove the following Conjecture B or Conjecture C which implies the inequality (U), and the first one of them is supported numerically and partially by means of the data obtained by M. Urabe for the integers $n=2, 3, \cdots, 10, 30, 50, 100$ (See Fig. 9 in [6]).

CONJECTURE B. *The period function T as a function of $\sigma=(\sqrt{x_1}-1)/(\sqrt{n}-1)$ and n is monotone decreasing with respect to $n(\geqq 2)$ for any fixed $\sigma(0<\sigma<1)$.*

CONJECTURE C. *The period function T as a function of $\tau=(x_1-1)/(n-1)$ and n is monotone decreasing with respect to $n(\geqq 2)$ for any fixed $\tau(0<\tau<1)$.*

The facts obtained in this paper will be also useful in proving these conjectures. In Appendix, we shall give a new proof for the fact (iv) in § 0 of Part (I), which was proved by a complex analysis on a Riemann surface in [10], as an application of some inequalities of these facts.

§ 1. Fundamental formulas and auxiliary functions $f_0(x)$ and $f_1(x)$.

Replacing the real variable x in $M(x, x_0)$, $\lambda(x)$, $F(x)$ and $\mu(x)$ by the complex variable z, we obtain the corresponding complex valued functions to them. Then, $M(z, x_0)$ is complex regular on the segment $0<x<n$ of the real axis. Setting

$$\psi(z):=z(n-z)^{n-1}, \tag{1.1}$$

we have

$$M(z, x_0)=\left[\frac{\{4n-1-(2n+1)z\}\{B-\psi(z)\}}{n(1-z)^2(n-z)^{n-1}}-1\right]$$
$$\times[\psi(z)-\psi(x_0)+n(n-z)^n\{\lambda(z)-\lambda(x_0)\}]$$
$$-\frac{2n(n-z)}{1-z}\cdot\{B-\psi(z)\}\{\lambda(z)-\lambda(x_0)\}$$

$$=\frac{n-z}{(1-z)^2}\cdot[(2n-1-z)B-(n-z)^{n-1}\{n-z+(n-1)z^2\}]\{\lambda(z)-\lambda(x_0)\}$$
$$+\frac{1}{n(1-z)^2(n-z)^{n-1}}[\{4n-1-(2n+1)z\}B-(n-z)^{n-1}\{n+(2n-1)z-(n+1)z^2\}]$$
$$\times\{\psi(z)-\psi(x_0)\}.$$

Hence, setting

$$f_0(z):=(2n-1-z)B-(n-z)^{n-1}\{n-z+(n-1)z^2\}, \tag{1.1}$$

$$f_1(z):=\{4n-1-(2n+1)z\}B-(n-z)^{n-1}\{n+(2n-1)z-(n+1)z^2\}, \tag{1.2}$$

we obtain another expression of $M(z, x_0)$ as

$$M(z, x_0)=\frac{n-z}{(1-z)^2}f_0(z)\{\lambda(z)-\lambda(x_0)\} \tag{1.3}$$
$$+\frac{1}{n(1-z)^2(n-z)^{n-1}}f_1(z)\{\psi(z)-\psi(x_0)\}.$$

Here, we notice that $f_0(z)$ and $f_1(z)$ do not depend on x_0.

Now, using a closed curve γ on the Riemann surface $\mathcal{F}$: $z(n-z)^{n-1}-w^2=c$ as in [11], we have easily

$$\int_{x_0}^{x_1}\frac{M(x, x_0)\,dx}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-c\}}}=-\frac{1}{2}\int_\gamma\frac{M(z, x_0))\,dz}{\sqrt{(n-z)^3\{z(n-z)^{n-1}-c\}}}$$

and so

$$\frac{\partial}{\partial x_0}\int_{x_0}^{x_1}\frac{M(x, x_0)\,dx}{\sqrt{(n-x)^3\{x(n-x)^{n-1}\quad x_0(n\quad x_0)^{n-1}\}}}$$

$$=-\frac{1}{2}\frac{\partial}{\partial x_0}\int_r \frac{M(z,\ x_0)\,dz}{\sqrt{(n-z)^3}\{z(n-z)^{n-1}-x_0(n-x_0)^{n-1}\}}$$

$$=-\frac{1}{2}\int_r \frac{1}{\sqrt{(n-z)^3}\{\phi(z)-\phi(x_0)\}^3}\Big[\{\phi(z)-\phi(x_0)\}\frac{\partial M(z,\ x_0)}{\partial x_0}$$

$$+\frac{1}{2}M(z,\ x_0)\phi'(x_0)\Big]dz\,.$$

Since we have

$$\phi'(z)=n(1-z)(n-z)^{n-2} \tag{1.4}$$

and

$$\lambda'(z)=-\frac{1-z}{(n-z)^2}\,, \tag{1.5}$$

we obtain from (1.3)

$$\{\phi(z)-\phi(x_0)\}\frac{\partial M(z,\ x_0)}{\partial x_0}+\frac{1}{2}M(z,\ x_0)\phi'(x_0)$$

$$=\{\phi(z)-\phi(x_0)\}\Big[\frac{(n-z)f_0(z)}{(1-z)^2}\cdot\frac{1-x_0}{(n-x_0)^2}-\frac{f_1(z)}{(1-z)^2(n-z)^{n-1}}\cdot(1-x_0)(n-x_0)^{n-2}\Big]$$

$$+\frac{n(1-x_0)(n-x_0)^{n-2}}{2}\Big[\frac{(n-z)f_0(z)}{(1-z)^2}\{\lambda(z)-\lambda(x_0)\}$$

$$+\frac{f_1(z)}{n(1-z)^2(n-z)^{n-1}}\cdot\{\phi(z)-\phi(x_0)\}\Big]$$

$$=\frac{1-x_0}{2(n-x_0)^2}\cdot\frac{(n-z)f_0(z)}{(1-z)^2}\cdot[2\{\phi(z)-\phi(x_0)\}+n(n-x_0)^n\{\lambda(z)-\lambda(x_0)\}]$$

$$-\frac{(1-x_0)(n-x_0)^{n-2}}{2}\cdot\frac{f_1(z)}{(1-z)^2(n-z)^{n-1}}\cdot\{\phi(z)-\phi(x_0)\}\,.$$

Hence, setting

$$U(z,\ x_0):=2\{\phi(z)-\phi(x_0)\}+n(n-x_0)^n\{\lambda(z)-\lambda(x_0)\} \tag{1.6}$$

and

$$\tilde{M}(z,\ x_0):=\frac{1-x_0}{2(n-x_0)^2}\cdot\frac{(n-z)f_0(z)}{(1-z)^2}\cdot U(z,\ x_0) \tag{1.7}$$

$$-\frac{(1-x_0)(n-x_0)^{n-2}}{2}\cdot\frac{f_1(z)}{(1-z)^2(n-z)^{n-1}}\cdot\{\phi(z)-\phi(x_0)\}\,.$$

We obtain the following formula

$$\frac{\partial}{\partial x_0}\int_{x_0}^{x_1}\frac{M(x,\ x_0)\,dx}{\sqrt{(n-x)^3\{x(n-x)^{n-1}-x_0(n-x_0)^{n-1}\}}} \tag{1.8}$$
$$=-\frac{1}{2}\int_\gamma\frac{\tilde{M}(z,\ x_0)\,dz}{\sqrt{(n-z)^3\{\psi(z)-\psi(x_0)\}^3}}\,.$$

Since we have

$$\tilde{M}(z,\ x_0)=\frac{(1-x_0)}{(n-x_0)^2}\cdot\left[\frac{(n-z)f_0(z)}{(1-z)^2}-\frac{(n-x_0)^n f_1(z)}{2(n-z)^{n-1}(1-z)^2}\right]\cdot\{\psi(z)-\psi(x_0)\}$$
$$+\frac{n(1-x_0)(n-x_0)^{n-2}}{2}\cdot\frac{(n-z)f_0(z)}{(1-z)^2}\cdot\{\lambda(z)-\lambda(x_0)\}\ ,$$

we have

$$-\frac{1}{2}\int_\gamma\frac{\tilde{M}(z,\ x_0)\,dz}{\sqrt{(n-z)^3\{\psi(z)-\psi(x_0)\}^3}}$$
$$=\frac{1-x_0}{(n-x_0)^2}\int_{x_0}^{x_1}\frac{1}{\sqrt{(n-x)\{\psi(x)-c\}}}\left[\frac{f_0(x)}{(1-x)^2}-\frac{(n-x_0)^n}{2(n-x)^n}\cdot\frac{f_1(x)}{(1-x)^2}\right]dx \tag{1.8'}$$
$$-\frac{n}{4}(1-x_0)(n-x_0)^{n-2}\int_\gamma\frac{1}{\sqrt{(n-z)\{\psi(z)-c\}^3}}\cdot\frac{f_0(z)}{(1-z)^2}\cdot\{\lambda(z)-\lambda(x_0)\}\,dz\,.$$

On the other hand, we have along γ the equality

$$\frac{d}{dz}\left\{\frac{2f_0(z)}{n(z-1)^3(n-z)^{n-3/2}}\cdot\frac{\lambda(z)-\lambda(x_0)}{\sqrt{\psi(z)-c}}\right\}$$
$$=\frac{1}{\sqrt{(n-z)\{\psi(z)-c\}^3}}\cdot\frac{f_0(z)}{(1-z)^2}\cdot\{\lambda(z)-\lambda(x_0)\}$$
$$+\frac{2}{n\sqrt{\psi(z)-c}}\cdot\frac{d}{dz}\left\{\frac{f_0(z)}{(z-1)^3(n-z)^{n-3/2}}\cdot(\lambda(z)-\lambda(x_0))\right\}.$$

Using the fact that the function $\frac{f_0(z)}{(z-1)^3(n-z)^{n-3/2}}\cdot(\lambda(z)-\lambda(x_0))$ is regular analytic in a small neighborhood of $z=x$, $0\leqq x<n$, on $\mathfrak{F}$, which will be proved in Lemma 2.2, we obtain

$$\int_\gamma\frac{1}{\sqrt{(n-z)\{\psi(z)-c\}^3}}\cdot\frac{f_0(z)}{(1-z)^2}\cdot\{\lambda(z)-\lambda(x_0)\}\,dz$$
$$=-\frac{2}{n}\int_\gamma\frac{1}{\sqrt{\psi(z)-c}}\cdot\left\{\frac{f_0(z)}{(z-1)^3(n-z)^{n-3/2}}\cdot(\lambda(z)-\lambda(x_0))\right\}'dz$$
$$=\frac{4}{n}\int_{x_0}^{x_1}\frac{1}{\sqrt{\psi(x)-c}}\cdot\left\{\frac{f_0(x)}{(x-1)^3(n-x)^{n-3/2}}\cdot(\lambda(x)-\lambda(x_0))\right\}'dx\,.$$

Thus, we obtain the formula

$$-\frac{1}{2}\int_{r}\frac{\tilde{M}(z,\, x_0)\,dz}{\sqrt{(n-z)^3\{\phi(z)-\phi(x_0)\}^3}}$$

$$(1.9)\qquad =\frac{1-x_0}{(n-x_0)^2}\int_{x_0}^{x_1}\frac{1}{\sqrt{(n-x)\{\phi(x)-\phi(x_0)\}}}\cdot\Big[\frac{f_0(x)}{(x-1)^2}-\frac{(n-x_0)^n}{2(n-x)^n}\frac{f_1(x)}{(1-x)^2}$$
$$-(n-x_0)^n\sqrt{n-x}\Big\{\frac{f_0(x)}{(n-x)^{n-3/2}(x-1)^3}\cdot(\lambda(x)-\lambda(x_0))\Big\}'\Big]dx\,.$$

Finally, we write more exactly the function $K(x,\, x_0)$ in §0. By (1.3) and (1.6) we obtain

$$K(x,\, x_0)=\frac{xM(x,\, x_0)}{(1-x)\sqrt{n-x}}-\frac{XM(X,\, x_0)}{(1-X)\sqrt{n-X}}$$
$$=-\Big[\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\cdot\{\lambda(x)-\lambda(x_0)\}-\frac{X\sqrt{n-X}\,f_0(X)}{(X-1)^3}\cdot\{\lambda(X)-\lambda(x_0)\}\Big]$$
$$+\frac{1}{n}\Big[\frac{x^2}{\sqrt{n-x}}\frac{f_1(x)}{(1-x)^3}-\frac{X^2}{\sqrt{n-X}}\frac{f_1(X)}{(1-X)^3}\Big]\cdot\frac{\phi(x)-\phi(x_0)}{\phi(x)}$$

and

$$\frac{x\sqrt{n-x}}{(x-1)^3}\cdot\{\lambda(x)-\lambda(x_0)\}=\frac{1}{n(n-x_0)^n}\cdot\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}U(x,\, x_0)$$
$$-\frac{2}{n(n-x_0)^n}\cdot\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\cdot\{\phi(x)-\phi(x_0)\}\,,$$

where $X=X_n(x)$ in §0. Therefore, we have

$$K(x,\, x_0)$$
$$=-\frac{1}{n(n-x_0)^n}\Big[\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\cdot U(x,\, x_0)-\frac{X\sqrt{n-X}\,f_0(X)}{(X-1)^3}\cdot U(X,\, x_0)\Big]$$
$$(1.10)\qquad +\frac{2}{n(n-x_0)^n}\Big[\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}-\frac{X\sqrt{n-X}\,f_0(X)}{(X-1)^3}\Big]\cdot\{\phi(x)-\phi(x_0)\}$$
$$+\frac{1}{n}\Big[\frac{x^2}{\sqrt{n-x}}\cdot\frac{f_1(x)}{(1-x)^3}-\frac{X^2}{\sqrt{n-X}}\cdot\frac{f_1(X)}{(X-1)^3}\Big]\cdot\frac{\phi(x)-\phi(x_0)}{\phi(x)}\,.$$

REMARK. The three quantities in the pairs of brackets of (1.10) are all negative as will be shown in Proposition 6, Proposition 2 and Proposition 4, respectively. We obtain easily from (1.10)

$$K(x_0,\, x_0)=\frac{x_1\sqrt{n-x_1}\,f_0(x_1)}{(x_1-1)^3}\cdot\{\lambda(x_1)-\lambda(x_0)\}>0$$

by Lemma 2.2 in Part (II) and Lemma 2.2 in Part (I).

§ 2. Certain properties of $f_0(x)$ and $F_0(x)$.

LEMMA 2.1. *$f_0(x)<0$ for $0<x<1$ and $f_0(x)>0$ for $1<x<n$, when $n>1$.*

Proof. We have easily

$$f_0(1)=2B(n-1)-(n-1)^{n-1}\cdot 2(n-1)=0, \qquad f_0(n)=(n-1)^n>0$$

and

$$f_0(x)=(2n-1-x)\left[B-\frac{(n-x)^{n-1}\{n-x+(n-1)x^2\}}{2n-1-x}\right].$$

Noticing $n-x+(n-1)x^2>0$ in $[0, 2n-1]$, we have

$$\frac{d}{dx}\left(\log\frac{(n-x)^{n-1}\{n-x+(n-1)x^2\}}{2n-1-x}\right)$$

$$=\frac{n(n-1)\{x^3-2(n+1)x^2+(4n+1)x-2n\}}{(n-x)(2n-1-x)\{n-x+(n-1)x^2\}}.$$

Since we have

$$x^3-2(n+1)x^2+(4n+1)x-2n=(x-1)^2(x-2n)<0$$

for $0<x<n$, $x\neq 1$, it must be

$$\frac{d}{dx}\frac{(n-x)^{n-1}\{n-x+(n-1)x^2\}}{2n-1-x}<0 \qquad \text{for} \quad 0<x<n,\ x\neq 1.$$

On the other hand, since we have

$$\left[\frac{(n-x)^{n-1}\{n-x+(n-1)x^2\}}{2n-1-x}\right]_{x=1}=B,$$

it must be therefore

$$B-\frac{(n-x)\{n-x+(n-1)x^2\}}{2n-1-x}\begin{cases} <0 & \text{for} \quad 0<x<1, \\ >0 & \text{for} \quad 1<x<n, \end{cases}$$

which implies the inequalities for $f_0(x)$ in this lemma. Q.E.D.

LEMMA 2.2. *The function $\Phi_0(x)$ defined by*

$$\Phi_0(x):=\begin{cases} \dfrac{f_0(x)}{(x-1)^3} & \text{for} \quad 0\leqq x\leqq n,\ x\neq 1, \\ \dfrac{n(2n-1)B}{6(n-1)} & \text{for} \quad x=1 \end{cases} \tag{2.1}$$

is positive and real analytic for $0\leqq x<n$, when $n>1$.

Proof. It is clear that the statement is true for $0\leqq x<n$ and $x\neq 1$. From the computation in the proof of Lemma 2.1, we obtain easily

$$\text{(2.2)}\qquad \frac{d}{dx}\,\frac{(n-x)^{n-1}\{n-x+(n-1)x^2\}}{2n-1-x}=-\frac{n(n-1)(x-1)^2(2n-x)(n-x)^{n-2}}{(2n-1-x)^2}\,.$$

Hence we have

$$\lim_{x\to1}\Phi_0(x)=2(n-1)\cdot\lim_{x\to1}\frac{B-\dfrac{(n-x)^{n-1}\{n-x+(n-1)x^2\}}{2n-1-x}}{(x-1)^3}$$

$$=2(n-1)\cdot\lim_{x\to1}\frac{\dfrac{n(n-1)(x-1)^2(2n-x)(n-x)^{n-2}}{(2n-1-x)^2}}{3(x-1)^2}$$

$$=\frac{2n(n-1)^2}{3}\cdot\lim_{x\to1}\frac{(2n-x)(n-x)^{n-2}}{(2n-1-x)^2}$$

$$=\frac{2n(n-1)^2}{3}\cdot\frac{(2n-1)(n-1)^{n-2}}{4(n-1)^2}=\frac{n(2n-1)B}{6(n-1)}\,.$$

Since $f_0(z)$ as a function of the complex variable z has its singular point only at $z=n$, the regularity of $\Phi_0(x)$ at $x=1$ is evident. Q. E. D.

LEMMA 2.3. *The function $F_0(x)$ defined by*

$$\text{(2.3)}\qquad F_0(x):=\begin{cases}\dfrac{1}{(n-x)^{n-3/2}}\cdot\dfrac{f_0(x)}{(x-1)^3} & \textit{for}\quad 0\leqq x<n,\ x\neq1\,,\\[2ex] \dfrac{n(2n-1)}{6\sqrt{n-1}} & \textit{for}\quad x=1\end{cases}$$

is positive and real analytic for $0\leqq x<n$, and

$$\text{(2.4)}\qquad \begin{cases}F_0(0)=\dfrac{\sqrt{n}}{e_{n-1}}\{n(e_{n-1}-2)+1\}<F_0(1)\,,\\[2ex] \lim\limits_{x\to n}F_0(x)=+\infty\,,\end{cases}$$

where $e_m:=(1+1/m)^m$, when $n\geqq2$.

Proof. By means of Lemma 2.2, it is clear that $F_0(x)$ is positive and real analytic for $0\leqq x<n$.

We have from (1.1) and (2.1)

$$F_0(0)=\frac{1}{n^{n-3/2}}\cdot\{n^n-(2n-1)(n-1)^{n-1}\}$$

$$=\sqrt{n}\left\{n\left(\frac{n}{n-1}\right)^{n-1}-2n+1\right\}\Big/\left(\frac{n}{n-1}\right)^{n-1}=\frac{\sqrt{n}}{e_{n-1}}\{n(e_{n-1}-2)+1\}$$

and

$$F_0(1)=\frac{1}{(n-1)^{n-3/2}}\cdot\frac{n(2n-1)}{6(n-1)}\cdot(n-1)^{n-1}=\frac{n(2n-1)}{6\sqrt{n-1}}\,.$$

Hence we obtain

$$\frac{F_0(0)}{F_0(1)}=\frac{6}{2n-1}\cdot\sqrt{\frac{n-1}{n}}\cdot\frac{n(e_{n-1}-2)+1}{e_{n-1}}=\frac{6}{2-\frac{1}{n}}\cdot\sqrt{1-\frac{1}{n}}\cdot\frac{e_{n-1}-2+\frac{1}{n}}{e_{n-1}}.$$

Since the function $\frac{\sqrt{1-t}}{2-t}$ of t is decreasing in the interval (0, 1), we have

$$\frac{1}{2-\frac{1}{n}}\sqrt{1-\frac{1}{n}}<\frac{1}{2}$$

and so

$$\frac{F_0(0)}{F_0(1)}<3\left(\frac{e_{n-1}-2}{e_{n-1}}+\frac{1}{ne_{n-1}}\right), \qquad \text{when} \quad n>1.$$

We shall show

$$3\left(\frac{e_{n-1}-2}{e_{n-1}}+\frac{1}{ne_{n-1}}\right)<1 \qquad \text{for} \quad n\geqq 2,$$

which is equivalent to

$$e_{n-1}<3-\frac{3}{2n} \qquad \text{for} \quad n\geqq 2. \tag{2.5}$$

In order to prove (2.5), we consider the function

$$\left(1+\frac{1}{x}\right)^x\Big/\left(3-\frac{3}{2(x+1)}\right) \qquad \text{for} \quad x\geqq 1.$$

Its logarithmic derivative is

$$\begin{aligned}&\left\{x\log\left(1+\frac{1}{x}\right)-\log\left(1-\frac{1}{2(x+1)}\right)\right\}'\\&=\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}-\frac{1}{(x+1)(2x+1)}\\&=\log(1+t)-\frac{t}{1+t}-\frac{t^2}{(1+t)(2+t)},\end{aligned}$$

where $t=\frac{1}{x}$. Furthermore, we have

$$\begin{aligned}&\frac{d}{dt}\left\{\log(1+t)-\frac{t}{1+t}-\frac{t^2}{(1+t)(2+t)}\right\}\\&=\frac{1}{1+t}-\frac{1}{(1+t)^2}-\frac{t(4+3t)}{(1+t)^2(2+t)^2}=\frac{t^2}{(1+t)(2+t)^2}>0\end{aligned}$$

and

$$\left[\log(1+t)-\frac{t}{1+t}-\frac{t^2}{(1+t)(2+t)}\right]_{t=0}=0,$$

which imply

$$\log(1+t)-\frac{t}{1+t}-\frac{t^2}{(1+t)(2+t)}>0 \qquad \text{for} \quad t>0.$$

Thus we have shown that the above function of x is monotone increasing for $x\geqq 1$ and so

$$\left(1+\frac{1}{x}\right)^x\Big/\left(3-\frac{3}{2(x+1)}\right)<\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\Big/\left(3-\frac{3}{2(x+1)}\right)=\frac{e}{3}<1.$$

Therefore (2.5) is true.

Finally we see easily that

$$\lim_{x\to n} F_0(x)=+\infty. \qquad \text{Q.E.D.}$$

Now, we shall compute the derivative of the positive function $F_0(x)$ $(0\leqq x<n)$. From (1.1) and (2.3), we obtain

$$\begin{aligned}(\log F_0(x))'&=\frac{n-\dfrac{3}{2}}{n-x}-\frac{3}{x-1}+\frac{f_0'}{f_0}\\&=\frac{\{(2n+3)x-(8n-3)\}f_0+2(x-1)(n-x)f_0'}{2(n-x)(x-1)f_0},\end{aligned}$$

whose denominator is positive for $0<x<n$, $x\neq 1$, by Lemma 2.1 and numerator becomes

$$\begin{aligned}&\{(2n+3)x-(8n-3)\}[(2n-1-x)B-(n-x)^{n-1}\{(n-1)x^2-x+n\}]\\&\quad+2(x-1)(n-x)[-B+(n-x)^{n-2}\{(n^2-1)x^2-n(2n-1)x+n^2\}]\\&=-\{(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3\}B\\&\quad+(n-x)^{n-1}\{-(n-1)x^3+(2n^2-7n+8)x^2+(n-3)(4n-1)x+3n(2n-1)\}.\end{aligned}$$

Hence setting

$$\begin{aligned}(2.6)\qquad F_2(x):=&-\{(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3\}B\\&+(n-x)^{n-1}\{-(n-1)x^3+(2n^2-7n+8)x^2+(n-3)(4n-1)x\\&+3n(2n-1)\},\end{aligned}$$

we obtain

$$(2.7)\qquad F_0'(x)=\frac{F_2(x)}{2(x-1)^4(n-x)^{n-1/2}},$$

which shows that $F_2(x)$ is real analytic for $0\leqq x<n$ and has a zero point of

order at least 4 at $x=1$. Our pressing purpose of the argument in the following is to show that $F_2(x)$ is positive for $0<x<n$, $x\neq 1$.

§ 3. Positiveness of $F_2(x)$.

LEMMA 3.1. *When* $n\geqq 2$, *we have*

$$-(n-1)x^3+(2n^2-7n+8)x^2+(n-3)(4n-1)x+3n(2n-1)>0 \qquad \textit{for} \quad 0\leqq x\leqq n\,.$$

Proof. Since the above polynomial of order 3 in x takes the positive values $3n(2n-1)$ at $x=0$ and $n^2(n-1)^2$ at $x=n$, it suffices to prove the following inequality

$$P(t):=3n(2n-1)t^3+(n-3)(4n-1)t^2+(2n^2-7n+8)t-(n-1)>0 \qquad \text{for} \quad t>\frac{1}{n}\,.$$

The discriminant of the polynomial of order 2 in t

$$P'(t)=9n(2n-1)t^2+2(n-3)(4n-1)t+2n^2-7n+8$$

is

$$4(n-3)^2(4n-1)^2-36n(2n-1)(2n^2-7n+8)$$
$$=-4(20n^4-40n^3+14n^2+6n-9)<0 \qquad \text{for} \quad n\geqq 2\,.$$

Hence $P'(t)>0$ for any t and so $P(t)$ is monotone increasing. Thus we see that $P(t)>0$ for $t>\dfrac{1}{n}$. Q. E. D.

The coefficient of B in (2.6) regarding as a quadratic polynomial in x has its symmetric axis at

$$x=\frac{2n^2+5n-4}{2n+1}=n+\frac{4(n-1)}{2n+1}>n$$

and

$$[(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3]_{x=1}=12(n-1)^2>0$$

when $n>1$, and

$$[(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3]_{x=n}=-(n-1)^2(2n-3)<0$$

when $n>3/2$. Supposing $n\geqq 2$, we denote the root of the quadratic equation:

$$(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3=0 \tag{3.1}$$

in the interval $1<x<n$ by γ_0. From the above facts and Lemma 3.1 we obtain easily the following lemmas.

LEMMA 3.2. $F_2(x)>0$ *for* $\gamma_0\leqq x\leqq n$.

LEMMA 3.3. *When $n \geqq 2$, we have*

$$1 < \frac{8(n-1)}{2n+1} < \gamma_0 < \min\left\{n, \frac{8n-5}{2n+1}\right\}.$$

Proof. We have easily

$$\begin{aligned}&[(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3]_{x=\frac{8(n-1)}{2n+1}}\\ &=\frac{1}{2n+1}\{64(n-1)^2-16(n-1)(2n^2+5n-4)+(2n+1)(16n^2-16n+3)\}\\ &=3>0\end{aligned}$$

and

$$\begin{aligned}&[(2n+1)n^2-2(2n^2+5n-4)x+16n^2-16n+3]_{x=\frac{8n-5}{2n+1}}\\ &=\frac{1}{2n+1}\{(8n-5)^2-2(8n-5)(2n^2+5n-4)+(2n+1)(16n^2-16n+3)\}\\ &=-\frac{12(n-1)^2}{2n+1}<0.\end{aligned}$$

Hence it must be $\dfrac{8(n-1)}{2n+1} < \gamma_0 < \dfrac{8n-5}{2n+1}$. Q. E. D.

Now, setting

$$P_2(x) := (2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3, \tag{3.2}$$

$$P_3(x) := -(n-1)x^3+(2n^2-7n+8)x^2+(4n^2-13n+3)x+3n(2n-1), \tag{3.3}$$

the sign of $F_2(x)$ is the same as the one of

$$\frac{(n-x)^{n-1}P_3(x)}{P_2(x)}-B \qquad \text{for} \quad 0 \leqq x < \gamma_0.$$

We have easily

$$\left[\frac{(n-x)^{n-1}P_3(x)}{P_2(x)}\right]_{x=1}=B \tag{3.4}$$

and

$$\begin{aligned}&\left\{\frac{(n-x)^{n-1}P_3(x)}{P_2(x)}\right\}'\\ &\qquad=\frac{P_2\{(n-x)^{n-1}P_3'-(n-1)(n-x)^{n-2}P_3\}-(n-x)^{n-1}P_3P_2'}{(P_2)^2}.\end{aligned}$$

We obtain also

$$\begin{aligned}&(n-x)^{n-1}P_3'(x)-(n-1)(n-x)^{n-2}P_3(x)\\ &\quad=(n-x)^{n-2}[(n-x)\{-3(n-1)x^2+2(2n^2-7n+8)x+4n^2-13n+3\}\end{aligned}$$

$$-(n-1)\{-(n-1)x^3+(2n^2-7n+8)x^2+(4n^2-13n+3)x+3n(2n-1)\}]$$
$$=-(n-x)^{n-2}Q_3(x),$$

where

$$Q_3(x):=2n^2(n+2)+n(n-13)x+2(n^3-n^2-n+4)x^2-(n-1)(n+2)x^3. \tag{3.5}$$

Hence, we have

$$\left\{\frac{(n-x)^{n-1}P_3(x)}{P_2(x)}\right\}'=\frac{(n-x)^{n-2}}{(P_2)^2}\cdot[-(n-x)P_2'P_3-P_2Q_3].$$

We shall compute the quantity in the above brackets. Since

$$(n-x)P_3(x)=(n-x)\{3n(2n-1)+(4n^2-13n+3)x+(2n^2-7n+8)x^2$$
$$-(n-1)x^3\}$$
$$=3n^2(2n-1)+n(4n^2-19n+6)x+(2n^3-11n^2+21n-3)x^2$$
$$-(3n^2-8n+8)x^3+(n-1)x^4,$$

we have

$$-(n-x)P_2'(x)P_3(x)-P_2(x)Q_3(x)$$
$$=\{2(2n^2+5n-4)-2(2n+1)x\}$$
$$\times\{3n^2(2n-1)+n(4n^2-19n+6)x+(2n^3-11n^2+21n-3)x^2$$
$$-(3n^2-8n+8)x^3+(n-1)x^4\}$$
$$-\{16n^2-16n+3-2(2n^2+5n-4)x+(2n+1)x^2\}$$
$$\times\{2n^2(n+2)+n(n-13)x+2(n^3-n^2-n+4)x^2-(n-1)(n+2)x^3\}$$
$$=-4n^2(n-1)(2n^2-2n+3)+n(n-1)(24n^3-16n^2+34n+9)x$$
$$-4n(n-1)(6n^3+8n+7)x^2+2n(n-1)(4n^3+8n^2+6n+15)x^3$$
$$-4n(n-1)(2n^2+n+3)x^4+n(n-1)(2n+1)x^5$$
$$=n(n-1)\{-4n(2n^2-2n+3)+(24n^3-16n^2+34n+9)x-4(6n^3+8n+7)x^2$$
$$+2(4n^3+8n^2+6n+15)x^3-4(2n^2+n+3)x^4+(2n+1)x^5\}$$
$$=n(n-1)(x-1)^3\{4n(2n^2-2n+3)-(8n^2-2n+9)x+(2n+1)x^2\}.$$

Thus, setting

$$Q_2(x):=(2n+1)x^2-(8n^2-2n+9)x+4n(2n^2-2n+3), \tag{3.6}$$

we obtain finally the following formula:

$$\left\{\frac{(n-x)^{n-1}P_3(x)}{P_2(x)}\right\}'=\frac{n(n-1)}{(P_2(x))^2}\cdot(n-x)^{n-2}(x-1)^3Q_2(x). \tag{3.7}$$

LEMMA 3.4. *When* $n\geqq 2$, $Q_2(x)>0$ *for* $x\leqq\gamma_0$.

Proof. On the graphs of the quadratic polynomials $P_2(x)$ and $Q_2(x)$, their axes of symmetry are

$$x=\frac{2n^2+5n-4}{2n+1} \quad \text{and} \quad x=\frac{8n^2-2n+9}{2(2n+1)}$$

respectively. Since we have

$$\frac{8n^2-2n+9}{2(2n+1)}-\frac{2n^2+5n-4}{2n+1}=\frac{4n^2-12n+17}{2(2n+1)}>0,$$

it must be $Q_2(x)$ for $0\leqq x\leqq\gamma_0$, if we can show

$$Q_2(\gamma_0)>0.$$

Now, noticing the expressions of $P_2(x)$ and $Q_2(x)$, we have

$$\begin{aligned}Q_2(\gamma_0)&=Q_2(\gamma_0)-P_2(\gamma_0)\\&=-(4n^2-12n+17)\gamma_0+8n^3-24n^2+28n-3.\end{aligned}$$

Since we have

$$4n^2-12n+17>0 \quad \text{and} \quad \gamma_0<\frac{8n-5}{2n+1}$$

by Lemma 3.3, and so it must be

$$\begin{aligned}Q_2(\gamma_0)&>-(4n^2-12n+17)\frac{8n-5}{2n+1}+8n^3-24n^2+28n-3\\&=\frac{1}{2n+1}\cdot\{(2n+1)(8n^3-24n^2+28n-3)-(8n-5)(4n^2-12n+17)\}\\&=\frac{2}{2n+1}\cdot[8n^4-36n^3+74n^2-87n+41].\end{aligned}$$

On the other hand, setting $n=t+2$ in the polynomial of n in the above brackets, we obtain

$$\begin{aligned}&8n^4-36n^3+74n^2-87n+41\\&\quad=8t^4+28t^3+50t^2+33t+3>0 \qquad \text{for} \quad t\geqq0.\end{aligned}$$

Thus we have proved that $Q_2(x)>0$ for $x\leqq\gamma_0$ when $n\geqq2$. Q. E. D.

Remark. If we substitute directly $x=\frac{8n-5}{2n+1}$ in the polynomial $Q_2(x)$, we obtain

$$\begin{aligned}Q_2\left(\frac{8n-5}{2n+1}\right)&=\frac{1}{2n+1}\cdot\{(8n-5)^2-(8n^2-2n+9)(8n-5)+4n(2n+1)(2n^2-2n+3)\}\\&=\frac{2}{2n+1}\cdot[8n^4-36n^3+68n^2-75n+35].\end{aligned}$$

Substituting $n=t+2$ in the polynomial of n in the above brackets, we obtain

$$8n^4-36n^3+68n^2-75n+35$$
$$=8t^4+28t^3+44t^2+21t-3,$$

which is not always positive for $t\geqq 0$.

LEMMA 3.5. *When* $n\geqq 2$, *we have*

$$F_2(x)>0 \qquad \text{for} \quad 0\leqq x<1 \quad \text{and} \quad 1<x\leqq\gamma_0.$$

Proof. By means of (3.4), (3.7) and Lemma 3.4, we see that the function

$$\frac{(n-x)^{n-1}P_3(x)}{P_2(x)}-B$$

and hence $F_2(x)$ is positive for $0\leqq x<1$ and $1<x\leqq\gamma_0$. Q. E. D.

Noticing $F_2(1)=0$, we obtain from Lemma 3.2 and Lemma 3.5 the following

PROPOSITION 1. *When* $n\geqq 2$, *we have*

$$F_2(x)>0 \qquad \text{for} \quad 0\leqq x<1 \quad \text{and} \quad 1<x\leqq n,$$

and

$$F_2(1)=0.$$

PROPOSITION 2. *When* $n\geqq 2$, *we have*

$$\frac{X\sqrt{n-X}}{(X-1)^3}f_0(X)>\frac{x\sqrt{n-x}}{(x-1)^3}f_0(x) \qquad \text{for} \quad 0<x<1,$$

where $X=X_n(x)$ *defined by* (0.8).

Proof. By means of Proposition 1 and (2.7), $F_0(x)$ must be monotone increasing in the interval $0<x<n$, hence we obtain

$$F_0(X)>F_0(x) \qquad \text{for} \quad 0<x<1,$$

i. e.

$$\frac{1}{(n-X)^{n-3/2}}\cdot\frac{f_0(X)}{(X-1)^3}>\frac{1}{(n-x)^{n-3/2}}\cdot\frac{f_0(x)}{(x-1)^3} \qquad \text{for} \quad 0<x<1.$$

Since we have $X(n-X)^{n-1}=x(n-x)^{n-1}$, the last inequality is equivalent to

$$\frac{X\sqrt{n-X}}{(X-1)^3}f_0(X)>\frac{x\sqrt{n-x}}{(x-1)^3}f_0(x) \qquad \text{for} \quad 0<x<1. \qquad \text{Q. E. D.}$$

§4. Certain properties of $f_1(x)$.

In Lemma 4.2 in [11], we introduced the function

$$h(x):=\{4n-1-(2n+1)x\}\mu(x)-n(n-x)^{n-1}, \tag{4.1}$$

which is positive in (0, 1) and negative in (1, n).

LEMMA 4.1. $$f_1(x)=(1-x)^2h(x)$$
and $f_1(x)>0$ *for* $0<x<1$ *and* $f_1(x)<0$ *for* $1<x<n$.

Proof. We obtain easily

$$\begin{aligned}(1-x)^2h(x)&=\{4n-1-(2n+1)x\}\{B-x(n-x)^{n-1}\}-n(n-x)^{n-1}(1-x)^2\\&=\{4n-1-(2n+1)x\}B-(n-x)^{n-1}\{(4n-1)x-(2n+1)x^2+n(1-x)^2\}\\&=\{4n-1-(2n+1)x\}B-(n-x)^{n-1}\{n+(2n-1)x-(n+1)x^2\}\\&=f_1(x).\end{aligned}$$

The signs of $f_1(x)$ in $(0, 1)$ and $(1, n)$ are evident from this equality and Lemma 4.2 in [11]. Q. E. D.

LEMMA 4.2. *The function* $\Phi_1(x)$ *defined by*

$$\Phi_1(x):=\begin{cases}\dfrac{f_1(x)}{(1-x)^3} & \text{for } 0\leqq x\leqq n, \quad x\neq 1\\[2mm] \dfrac{n(4n+1)B}{6(n-1)} & \text{for } x=1\end{cases} \tag{4.2}$$

is positive and real analytic for $0\leqq x<n$, *when* $n>1$.

Proof. It is clear that the statement is true for $0\leqq x<n$ and $x\neq 1$. From (1.2), we get easily

$$f_1(1)=2(n-1)B-(n-1)^{n-1}\cdot 2(n-1)=0.$$

Next we have

$$\left\{\frac{(n-x)^{n-1}\{n+(2n-1)x-(n+1)x^2\}}{4n-1-(2n+1)x}\right\}'=\frac{n(n-x)^{n-2}(1-x)^2\{6n^2-(n+1)(2n+1)x\}}{\{4n-1-(2n+1)x\}^2}. \tag{4.3}$$

Hence we obtain

$$\begin{aligned}\lim_{x\to 1}\Phi_1(x)&=2(n-1)\cdot\lim_{x\to 1}\frac{B-\dfrac{(n-x)^{n-1}\{n+(2n-1)x-(n+1)x^2\}}{4n-1-(2n+1)x}}{(1-x)^3}\\&=2(n-1)\cdot\lim_{x\to 1}\frac{\dfrac{n(n-x)^{n-2}(1-x)^2\{6n^2-(n+1)(2n+1)x\}}{\{4n-1-(2n+1)x\}^2}}{3(1-x)^2}\\&=\frac{2n(n-1)}{3}\cdot\lim_{x\to 1}\frac{(n-x)^{n-2}\{6n^2-(n+1)(2n+1)x\}}{\{4n-1-(2n+1)x\}^2}\\&=\frac{2n(n-1)}{3}\cdot\frac{(n-1)^{n-2}(4n^2-3n-1)}{4(n-1)^2}=\frac{n(4n+1)B}{6(n-1)}.\end{aligned}$$

Since $f_1(z)$ is complex regular except $z=n$, the above computation implies that $\Phi_1(x)$ is regular analytic on $0\leqq x<n$. Q. E. D.

LEMMA 4.3. *The function $F_1(x)$ defined by*

$$F_1(x):=\begin{cases}\dfrac{1}{(n-x)^{2n-3/2}}\cdot\dfrac{f_1(x)}{(1-x)^3} & \textit{for}\quad 0\leqq x<n\,,\quad x\neq 1\\[2mm] \dfrac{n(4n+1)B}{6(n-1)^{2n-1/2}}\Big(=\dfrac{n(4n+1)}{6(n-1)^{n+1/2}}\Big) & \textit{for}\quad x=1\end{cases}\tag{4.4}$$

is positive and real analytic for $0\leqq x<n$, and

$$\begin{cases}F_1(0)=\dfrac{1}{n^{2n-3/2}}\cdot(4n-1-ne_{n-1})B<F_1(1)\,,\\ \lim\limits_{x\to n}F_1(x)=+\infty\,,\end{cases}\tag{4.5}$$

when $n\geqq 2$.

Proof. By means of Lemma 4.2 it is clear that $F_1(x)$ is positive and real analytic for $0\leqq x<n$. The values of $F_1(x)$ at $x=0$, 1, n are easily calculated. Since we have

$$\frac{F_1(0)}{F_1(1)}=6\Big(\frac{n-1}{n}\Big)^{2n-1/2}\cdot\frac{4n-1-ne_{n-1}}{4n+1}\,,$$

the inequality $F_1(0)<F_1(1)$ is equivalent to the following inequality:

$$\frac{6(4n-1-ne_{n-1})}{4n+1}<\Big(1+\frac{1}{n-1}\Big)^{2n-1/2}.\tag{4.6}$$

Regarding the right hand side of (4.6), we consider the function of t:

$$\Big(1+\frac{1}{t}\Big)^{2t+3/2}\qquad\text{for}\quad t>0\,.$$

We shall show this function is monotone decreasing. In fact, we have

$$\frac{d}{dt}\Big(1+\frac{1}{t}\Big)^{2t+3/2}=\Big(1+\frac{1}{t}\Big)^{2t+3/2}\Big\{2\log\frac{1+t}{t}+\Big(2t+\frac{3}{2}\Big)\Big(\frac{1}{1+t}-\frac{1}{t}\Big)\Big\}$$

$$=2\Big(1+\frac{1}{t}\Big)^{2t+3/2}\Big\{\log\Big(1+\frac{1}{t}\Big)-\frac{1}{t(1+t)}\cdot\Big(t+\frac{3}{4}\Big)\Big\}\,.$$

Putting $\frac{1}{t}=u$, we obtain

$$\log\Big(1+\frac{1}{t}\Big)-\frac{1}{t(1+t)}\cdot\Big(t+\frac{3}{4}\Big)$$

$$=\log(1+u)-\frac{u}{1+u}\cdot\Big(1+\frac{3}{4}u\Big)\qquad(u>0)\,.$$

Both functions $\log(1+u)$ and $\frac{u}{1+u}\cdot\left(1+\frac{3}{4}u\right)$ of u take the same value 0 at $u=0$ and their derivatives with respect to u are

$$\frac{1}{1+u} \quad \text{and} \quad \frac{1+\frac{3}{2}u+\frac{3}{4}u^2}{(1+u)^2}$$

respectively. Since we have

$$\frac{1+\frac{3}{2}u+\frac{3}{4}u^2}{(1+u)^2}-\frac{1}{1+u}=\frac{u(2+3u)}{4(1+u)^2}>0 \qquad \text{for} \quad u>0,$$

it must be

$$\log(1+u)-\frac{u}{1+u}\left(1+\frac{3}{4}u\right)<0 \qquad \text{for} \quad u>0.$$

Hence, we obtain

$$\frac{d}{dt}\left(1+\frac{1}{t}\right)^{2t+3/2}<0 \qquad \text{for} \quad t>0,$$

which implies

$$\left(1+\frac{1}{t}\right)^{2t+3/2}>\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{2t+3/2}=e^2,$$

i. e.

$$(4.7) \qquad \left(1+\frac{1}{n-1}\right)^{2n-1/2}>e^2 \qquad \text{for} \quad n>1.$$

On the other hand, supposing $n\geqq 2$, we have

$$\frac{6(4n-1-ne_{n-1})}{4n+1}\leqq\frac{6(4n-1-2n)}{4n+1}=\frac{6(2n-1)}{4n+1}<3<e^2.$$

Therefore (4.5) is true when $n\geqq 2$. Q. E. D.

Now, we shall compute the derivative of the positive function $F_1(x)$ $(0\leqq x<n)$. From (1.2) and (4.4), we obtain

$$(\log F_1(x))'=\frac{2n-\frac{3}{2}}{n-x}+\frac{3}{1-x}+\frac{f_1'}{f_1}$$
$$=\frac{\{10n-3-(4n+3)x\}f_1+2(1-x)(n-x)f_1'}{2(n-x)(1-x)f_1},$$

whose denominator is positive for $0<x<n$, $x\neq 1$ by Lemma 4.1 and numerator becomes

$$\{10n-3-(4n+3)x\}[\{4n-1-(2n+1)x\}B-(n-x)^{n-1}\{n+(2n-1)x-(n+1)x^2\}]$$
$$-2(1-x)(n-x)[(2n+1)B+(n-x)^{n-2}\{n^2-n(4n+1)x+(n+1)^2x^2\}]$$

$$=\{3(2n-1)(6n-1)-2(16n^2+3n-4)x+(2n+1)(4n+1)x^2\}B$$
$$-(n-x)^{n-1}\{3n(4n-1)+3(2n^2-7n+1)x-(8n^2+3n-8)x^2+(n+1)(2n+1)x^3\}.$$

Hence, setting

$$\begin{aligned}F_3(x):=&\{3(2n-1)(6n-1)-2(16n^2+3n-4)x+(2n+1)(4n+1)x^2\}B\\ &-(n-x)^{n-1}\{3n(4n-1)+3(2n^2-7n+1)x-(8n^2+3n-8)x^2\\ &+(n+1)(2n+1)x^3\},\end{aligned} \tag{4.8}$$

we obtain

$$F_1'(x)=\frac{F_3(x)}{2(1-x)^4(n-x)^{2n-1/2}}, \tag{4.9}$$

which shows that $F_3(x)$ is real analytic for $0\leqq x<n$ and has a zero point of order at least 4 at $x=1$. Our pressing purpose of the argument in the following is to show that $F_3(x)$ is positive for $0<x<n$, $x\neq 1$.

§ 5. Positiveness of $F_3(x)$.

LEMMA 5.1. *When $n>1$, we have*

$$3(2n-1)(6n-1)-2(16n^2+3n-4)x+(2n+1)(4n+1)x^2>0$$
$$\textit{for} \quad -\infty<x<\infty.$$

Proof. The discriminant of the quadratic polynomial of x of the left hand side of the above inequality is

$$\begin{aligned}&4(16n^2+3n-4)^2-12(2n-1)(6n-1)(2n+1)(4n+1)\\ &=4(256n^4+96n^3-119n^2-24n+16)\\ &\quad -12(96n^4+8n^3-28n^2-2n+1)\\ &=-4(32n^4-72n^3+35n^2+18n-13)\\ &=-4(n-1)^2(32n^2-8n-13).\end{aligned}$$

Since $32n^2-8n-13>0$ for $n>1$, this discriminant is negative. Hence, the statement of this lemma is true. Q.E.D.

Now, setting

$$\tilde{P}_2(x):=3(2n-1)(6n-1)-2(16n^2+3n-4)x+(2n+1)(4n+1)x^2, \tag{5.1}$$

$$\tilde{P}_3(x):=3n(4n-1)+3(2n^2-7n+1)x-(8n^2+3n-8)x^2+(n+1)(2n+1)x^3, \tag{5.2}$$

the sign of $F_3(x)$ is the same as the one of

$$B-\frac{(n-x)^{n-1}\tilde{P}_3(x)}{\tilde{P}_2(x)}$$

by means of Lemma 5.1. We have easily

$$\left[\frac{(n-x)^{n-1}\tilde{P}_3(x)}{\tilde{P}_2(x)}\right]_{x=1}=B, \tag{5.3}$$

since $\tilde{P}_2(1)=\tilde{P}_3(1)=12(n-1)^2$, and

$$\left\{\frac{(n-x)^{n-1}\tilde{P}_3(x)}{\tilde{P}_2(x)}\right\}'$$
$$=\frac{\tilde{P}_2\{(n-x)^{n-1}\tilde{P}_3{}'-(n-1)(n-x)^{n-2}\tilde{P}_3\}-(n-x)^{n-1}\tilde{P}_3\tilde{P}_2{}'}{(\tilde{P}_2)^2}.$$

We obtain also

$$\begin{aligned}&(n-x)^{n-1}\tilde{P}_3{}'(x)-(n-1)(n-x)^{n-2}\tilde{P}_3(x)\\&=(n-x)^{n-2}[(n-x)\{3(2n^2-7n+1)-2(8n^2+3n-8)x+3(n+1)(2n+1)x^2\}\\&\quad-(n-1)\{3n(4n-1)+3(2n^2-7n+1)x-(8n^2+3n-8)x^2+(n+1)(2n+1)x^3\}]\\&=-(n-x)^{n-2}\tilde{Q}_3(x),\end{aligned}$$

where

$$\begin{aligned}\tilde{Q}_3(x):=&6n^2(n+1)+n(22n^2-15n-13)x-2(n+1)^2(7n-4)x^2\\&+(n+1)(n+2)(2n+1)x^3.\end{aligned} \tag{5.4}$$

Hence, we have

$$\left\{\frac{(n-x)^{n-1}\tilde{P}_3(x)}{\tilde{P}_2(x)}\right\}'=\frac{(n-x)^{n-2}}{(\tilde{P}_2)^2}\cdot[-(n-x)\tilde{P}_2{}'\tilde{P}_3-\tilde{P}_2\tilde{Q}_3].$$

We shall compute the quantity in the above brackets. Since

$$\begin{aligned}(n-x)\tilde{P}_3(x)=&(n-x)\{3n(4n-1)+3(2n^2-7n+1)x-(8n^2+3n-8)x^2\\&+(n+1)(2n+1)x^3\}\\=&3n^2(4n-1)+3n(2n^2-11n+2)x-(8n^3+9n^2-29n+3)x^2\\&+(2n^3+11n^2+4n-8)x^3-(n+1)(2n+1)x^4,\end{aligned}$$

we have

$$\begin{aligned}&-(n-x)\tilde{P}_2{}'(x)\tilde{P}_3(x)-\tilde{P}_2(x)\tilde{Q}_3(x)\\&=\{2(16n^2+3n-4)-2(2n+1)(4n+1)x\}\\&\quad\times\{3n^2(4n-1)+3n(2n^2-11n+2)x-(8n^3+9n^2-29n+3)x^2\\&\quad+(2n^3+11n^2+4n-8)x^3-(n+1)(2n+1)x^4\}\\&\quad-\{3(2n-1)(6n-1)-2(16n^2+3n-4)x+(2n+1)(4n+1)x^2\}\\&\quad\times\{6n^2(n+1)+n(22n^2-15n-13)x-2(n+1)^2(7n-4)x^2\\&\quad+(n+1)(n+2)(2n+1)x^3\}\end{aligned}$$

$$=6n^2(28n^3-16n^2+2n+1)-3n(200n^4-60n^3+4n^2+3n+3)x$$
$$+4n(202n^4+18n^3-2n^2+7)x^2-6n(84n^4+56n^3+4n^2+n+5)x^3$$
$$+6n(2n+1)(12n^3+12n^2-n+2)x^4-n(n+1)(2n+1)^2(4n+1)x^5$$
$$=-n(x-1)^3\{8(n-1)^2(11n^2+n-1)-(n-1)(2n+1)(32n^2+34n-7)(x-1)$$
$$+(n+1)(2n+1)^2(4n+1)(x-1)^2\}$$
$$=n(1-x)^3\{6n(28n^3-16n^2+2n+1)-3(2n+1)(16n^3+10n^2-9n+3)x$$
$$+(n+1)(2n+1)^2(4n+1)x^2\}.$$

Thus, setting

$$\tilde{Q}_2(x):=6n(28n^3-16n^2+2n+1)-3(2n+1)(16n^3+10n^2-9n+3)x \tag{5.5}$$
$$+(n+1)(2n+1)^2(4n+1)x^2,$$

we obtain finally the following formula:

$$\left\{\frac{(n-x)^{n-1}\tilde{P}_3(x)}{\tilde{P}_2(x)}\right\}'=\frac{n}{(\tilde{P}_2(x))^2}\cdot(n-x)^{n-2}(1-x)^3\tilde{Q}_2(x). \tag{5.6}$$

LEMMA 5.2. *When* $n>1$, $\tilde{Q}_2(x)>0$ *for* $-\infty<x<+\infty$.

Proof. The discriminant of $\tilde{Q}_2(x)$ is given by

$$9(2n+1)^2(16n^3+10n^2-9n+3)^2$$
$$-24n(28n^3-16n^2+2n+1)(n+1)(2n+1)^2(4n+1)$$
$$=-3(2n+1)^2[-3(16n^3+10n^2-9n+3)^2$$
$$+8n(n+1)(4n+1)(28n^3-16n^2+2n+1)]$$
$$=-3(2n+1)^2[-3(256n^6+320n^5-188n^4-84n^3+141n^2-54n+9)$$
$$+8(112n^6+76n^5-44n^4-2n^3+7n^2+n]$$
$$=-3(2n+1)^2[128n^6-352n^5+212n^4+236n^3-367n^2+170n-27].$$

Substituting $n=t+1$ in the polynomial of n in the last brackets, we obtain

$$128n^6-352n^5+212n^4+236n^3-367n^2+170n-27$$
$$=128t^6+416t^5+372t^4+124t^3+13t^2+27,$$

which is always positive for $t>0$. This fact implies that

$$\tilde{Q}_2(x)>0 \quad \text{for} \quad -\infty<x<+\infty. \qquad \text{Q.E.D.}$$

PROPOSITION 3. *When* $n>1$, *we have*

$$F_3(x)>0 \quad \textit{for} \quad 0\leqq x<1 \quad \textit{and} \quad 1<x<n,$$

and

$$F_3(1)=0.$$

Proof. By means of (5.3), (5.6) and Lemma 5.2, we obtain

$$\frac{(n-x)^{n-1}\tilde{P}_3(x)}{\tilde{P}_2(x)}<B \qquad \text{for} \quad 0\leqq x<n\,,\ x\neq 1\,,$$

which implies

$$F_3(x)=\tilde{P}_2(x)B-(n-x)^{n-1}\tilde{P}_1(x)>0 \qquad \text{for} \quad 0\leqq x<n\,,\ x\neq 1$$

by means of Lemma 5.1. Q. E. D.

PROPOSITION 4. *When $n>1$, we have*

$$\frac{X^2}{(1-X)^3\sqrt{n-X}}f_1(X)>\frac{x^2}{(1-x)^3\sqrt{n-x}}f_1(x) \qquad \textit{for} \quad 0<x<1\,,$$

where $X=X_n(x)$ defined by (0.8).

Proof. By means of Proposition 3 and (4.9), $F_1(x)$ must be monotone increasing in the interval $0<x<n$, hence we obtain

$$F_1(X)>F_1(x) \qquad \text{for} \quad 0<x<1\,,$$

i. e.

$$\frac{1}{(n-X)^{2n-3/2}}\cdot\frac{f_1(X)}{(1-X)^3}>\frac{1}{(n-x)^{2n-3/2}}\cdot\frac{f_1(x)}{(1-x)^3} \qquad \text{for} \quad 0<x<1\,.$$

Since we have $X(n-X)^{n-1}=x(n-x)^{n-1}$, the last inequality is equivalent to

$$\frac{X^2}{(1-X)^3\sqrt{n-X}}f_1(X)>\frac{x^2}{(1-x)^3\sqrt{n-x}}f_1(x) \qquad \text{for} \quad 0<x<1\,. \qquad \text{Q. E. D.}$$

§ 6. Positiveness of $U(x, x_0)$.

In this section we shall investigate the function

(6.1) $$U(x,\ x_0)=2\{\psi(x)-\psi(x_0)\}+n(n-x_0)^n\{\lambda(x)-\lambda(x_0)\}\,.$$

PROPOSITION 5. *When $n\geqq 2$, we have*

$$U(x,\ x_0)>0 \qquad \textit{for} \quad 0<x_0<x\leqq x_1=X(x_0)\,.$$

Proof. From (6.1) we obtain easily

$$\frac{\partial U(x,\ x_0)}{\partial x}=2n(1-x)(n-x)^{n-2}-n(n-x_0)^n\cdot\frac{1-x}{(n-x)^2}$$

$$=\frac{n(1-x)}{(n-x)^2}\cdot\{2(n-x)^n-(n-x_0)^n\}$$

by (1.4) and (1.5). Since $0<x_0<1<x_1<n$, let $\kappa=\kappa(x_0)$ be the constant such that

$$n-\kappa=\frac{1}{\sqrt[n]{2}}(n-x_0)\,.$$

Here, we check the following inequality

$$\left(1-\frac{1}{\sqrt[n]{2}}\right)n<1 \qquad \text{for} \quad n\geqq 2.$$

It is equivalent to the inequality

$$\frac{n}{n-1}>\sqrt[n]{2}$$

or

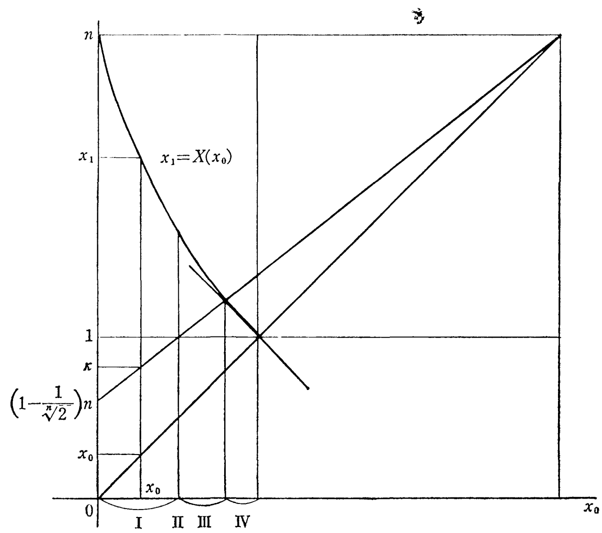
$$\left(1+\frac{1}{n-1}\right)^{n-1}>\frac{2(n-1)}{n},$$

which is clear, because

$$\left(1+\frac{1}{n-1}\right)^{n-1}=e_{n-1}\geqq 2>\frac{2(n-1)}{n} \qquad \text{for} \quad n\geqq 2.$$

We have easily

$$U(x_0,\ x_0)=0.$$



In the following, we shall devide the proof in the four cases shown in the above figure according to the size of κ.

Case I: $x_0<\kappa<1$.

In this case, we see easily that $\dfrac{\partial}{\partial x}U\{x,\ x_0)$ is positive for $x_0\leqq x<\kappa,\ 1<x\leqq x_1$ and negative for $\kappa<x<1$. Since we have

$$\begin{aligned} U(1,\ x_0)&=2\{B-\phi(x_0)\}+n(n-x_0)^n\{\lambda(1)-\lambda(x_0)\}\\ &=2\{B-\phi(x_0)\}+2n(n-\kappa)^n\{\lambda(1)-\lambda(x_0)\}\end{aligned}$$

$$>2\{B-\psi(x_0)\}+2n(n-1)^n\{\lambda(1)-\lambda(x_0)\}$$
$$=2F(1)>0,$$

where $F(x)$ is defined by (0.5) and positive for $x_0<x\leqq x_1$ by Lemma 3.1 in [11]. Hence it must be $U(x, x_0)>0$ for $x_0<x\leqq x_1$.

Case II: $\kappa=1$.

In this case, we see easily that $\frac{\partial}{\partial x}U(x, x_0)$ is positive for $x_0\leqq x<1$ and $1<x\leqq x_1$ and so the claim is evident.

Case III: $1<\kappa<x_1$.

In this case, we see easily that $\frac{\partial}{\partial x}U(x, x_0)$ is positive for $x_0\leqq x<1$ and $\kappa<x\leqq x_1$ and negative for $1<x<\kappa$. Since we have

$$\begin{aligned} U(\kappa, x_0)&=2\{\psi(\kappa)-\psi(x_0)\}+n(n-x_0)^n\{\lambda(\kappa)-\lambda(x_0)\}\\ &=2\{\psi(\kappa)-\psi(x_0)\}+2n(n-\kappa)^n\{\lambda(\kappa)-\lambda(x_0)\}\\ &=2F(\kappa)>0, \end{aligned}$$

it must be $U(x, x_0)>0$ for $x_0<x\leqq x_1$.

Case IV: $x_1\leqq\kappa$.

In this case, we see easily that $\frac{\partial}{\partial x}U(x, x_0)$ is positive for $x_0\leqq x<1$ and negative for $1<x<x_1$. Furthermore, it must be

$$U(x_1, x_0)=n(n-x_0)^n\{\lambda(x_1)-\lambda(x_0)\}>0,$$

since we have the inequality:

$$\lambda(x_1)-\lambda(x_0)>0 \qquad \text{for} \quad 0<x_0<1<x_1=X(x_0) \tag{6.2}$$

by Lemma 2.2 in [11]. Hence we obtain also $U(x, x_0)>0$ for $x_0<x\leqq x_1$.

Q. E. D.

Using Proposition 5, we obtain the following

PROPOSITION 6. *When $n\geqq 2$, we have*

$$\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}U(X, x_0)-\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}U(x, x_0)>0 \qquad \text{for} \quad x_0<x<1.$$

Proof. By means of Propositions 5 and 2, we have

$$\begin{aligned} &\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}U(X, x_0)-\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}U(x, x_0)\\ &>\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}\{U(X, x_0)-U(x, x_0)\}\\ &=n(n-x_0)^n\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}\{\lambda(X)-\lambda(x)\}, \end{aligned}$$

which is positive by (6.2) replaced with $x_0=x$, $x_1=X(x)$. Q. E. D.

§ 7. Certain properties of $\tilde{M}(x, x_0)$ and $M(x, x_0)$.

PROPOSITION 7. *When* $n\geqq 2$, *we have*

$$\frac{x\tilde{M}(x, x_0)}{(1-x)\sqrt{n-x}}-\frac{X\tilde{M}(X, x_0)}{(1-X)\sqrt{n-X}}>0 \qquad for \quad x_0<x<1.$$

Proof. We use the expression of the right hand side of (1.7) for $\tilde{M}(x, x_0)$. Then we have

$$\frac{x\tilde{M}(x, x_0)}{(1-x)\sqrt{n-x}}-\frac{X\tilde{M}(X, x_0)}{(1-X)\sqrt{n-X}}$$

$$(7.1)\qquad =\frac{1-x_0}{2(n-x_0)^2}\left[\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}U(X, x_0)-\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}U(x, x_0)\right]$$
$$+\frac{(1-x_0)(n-x_0)^{n-2}}{2}\left[\frac{Xf_1(X)}{(1-X)^3(n-X)^{n-1/2}}-\frac{xf_1(x)}{(1-x)^3(n-x)^{n-1/2}}\right]$$
$$\times\{\phi(x)-\phi(x_0)\}.$$

By means of Proposition 6, the quantity in the first brackets is positive. We have also the inequality:

$$(7.2)\qquad \frac{Xf_1(X)}{(1-X)^3(n-X)^{n-1/2}}-\frac{xf_1(x)}{(1-x)^3(n-x)^{n-1/2}}>0 \qquad \text{for} \quad 0<x<1,$$

since it is equivalent to the one:

$$\frac{X^2f_1(X)}{(1-X)^3\sqrt{n-X}}-\frac{x^2f_1(x)}{(1-x)^3\sqrt{n-x}}>0 \qquad \text{for} \quad 0<x<1,$$

because $x(n-x)^{n-1}=X(n-X)^{n-1}$, which was proved in Proposition 4. Therefore, the right hand side of (7.1) must be positive since $0<x_0<1<n$ and $\phi(x)-\phi(x_0)>0$ for $x_0<x<1$. Thus we obtain the inequality of this proposition. Q. E. D.

PROPOSITION 8. *When* $n\geqq 2$, *we have*

$$\lim_{x_0\to 1}\int_{x_0}^{X(x_0)}\frac{M(x, x_0)\,dx}{\sqrt{(n-x)^3}\{x(n-x)^{n-1}-x_0(n-x_0)^{n-1}\}}=0.$$

Proof. We have the equalities

$$(7.3)\qquad \frac{M(x, x_0)}{\sqrt{(n-x)^3}\{\phi(x)-\phi(x_0)\}}=\frac{xM(x, x_0)}{n-x}\cdot\frac{1}{x\sqrt{(n-x)}\{\phi(x)-\phi(x_0)\}},$$

where

$$M(x, x_0)=\frac{\{4n-1-(2n+1)x\}\mu(x)-n(n-x)^{n-1}}{n(n-x)^{n-1}}\cdot F(x, x_0)$$
$$+2n(x-1)(n-x)\mu(x)\{\lambda(x)-\lambda(x_0)\},$$

$$F(x,\ x_0)=\psi(x)-\psi(x_0)+n(n-x)^n\{\lambda(x)-\lambda(x_0)\},$$

and

$$\psi'(x)=n(n-x)^{n-2}(1-x),$$

$$\lambda'(x)=-\frac{1}{(n-x)^2}\cdot(1-x)$$

by (0.3), (0.5), (1.4) and (1.5).

First of all, for any small $\varepsilon>0$ we take $\delta_1>0$ such that if $1-\delta_1<x_0<1$, then $X(x_0)-x_0<\varepsilon$. Next, substituting suitably x with 1 in $\frac{xM(x,\ x_0)}{n-x}$ of the right hand side of (7.3) by noticing the above expressions and the mean value theorem, we consider the following constant:

$$\frac{1}{n-1}\cdot\Big[\frac{2(n-1)\mu(1)-n(n-1)^{n-1}}{n(n-1)^{n-1}}\times\Big\{n(n-1)^{n-2}+n(n-1)^n\frac{1}{(n-1)^2}\Big\}$$

$$+2n(n-1)\mu(1)\frac{\varepsilon}{(n-1)^2}\Big]$$

$$=\frac{1}{n-1}\cdot\Big[\Big(\frac{2}{n(n-1)^{n-2}}\mu(1)-1\Big)\cdot 2n(n-1)^{n-2}+\frac{2n\varepsilon}{n-1}\mu(1)\Big]$$

and furthermore using (0.6) this constant becomes $n^2(n-1)^{n-4}\varepsilon$. Then, by the continuity of related functions here we can choose a positive constant $\delta\leqq\delta_1$ such that if

$$1-\delta<x_0\leqq x\leqq x_1=X(x_0),$$

then

$$\left|\frac{xM(x,\ x_0)}{n-x}\right|<\{n^2(n-1)^{n-4}+1\}\,\varepsilon^3.$$

Hence, for such x_0 we obtain the inequalities

$$\left|\int_{x_0}^{x_1}\frac{M(x,\ x_0)\,dx}{\sqrt{(n-x)^3\{\psi(x)-\psi(x_0)\}}}\right|$$

$$<\{n^2(n-1)^{n-4}+1\}\,\varepsilon^3\cdot\int_{x_0}^{x_1}\frac{dx}{x\sqrt{(n-x)\{\psi(x)-\psi(x_0)\}}}$$

$$<\{n^2(n-1)^{n-4}+1\}\,\varepsilon^3\cdot\frac{\sqrt{2}\,\pi}{\sqrt{n\psi(x_0)}}$$

by (0.1) and (U) in § 0, from which we have

$$\lim_{x_0\to 1}\left|\int_{x_0}^{x_1}\frac{M(x,\ x_0)\,dx}{\sqrt{(n-x)^3\{\psi(x)-\psi(x_0)\}}}\right|$$

$$\leqq\frac{\{n^2(n-1)^{n-4}+1\}\sqrt{2}\,\pi\varepsilon^3}{\sqrt{n(n-1)^{n-1}}}$$

and therefore it must be

$$\lim_{x_0\to 1}\int_{x_0}^{x_1}\frac{M(x,\ x_0)\,dx}{\sqrt{(n-x)^3\{\psi(x)-\psi(x_0)\}}}=0. \qquad \text{Q. E. D.}$$

Appendix

In the original manuscripts, the present author made a serious mistake such that from (1.8) he derived the following equality:

$$\frac{\partial}{\partial x_0}\int_{x_0}^{x_1}\frac{M(x,\ x_0)\,dx}{\sqrt{(n-x)^3}\{x(n-x)^{n-1}-x_0(n-x_0)^{n-1}\}}$$
$$=\int_{x_0}^{x_1}\frac{\tilde{M}(x,\ x_0)\,dx}{\sqrt{(n-x)^3}\{x(n-x)^{n-1}-x_0(n-x_0)^{n-1}\}}.$$

The right hand side of this can be expressed as

$$=\int_{x_0}^{1}\frac{1-x}{x(n-x)\sqrt{\{\psi(x)-c\}^3}}\left[\frac{x\tilde{M}(x,\ x_0)}{(1-x)\sqrt{n-x}}-\frac{X\tilde{M}(X,\ x_0)}{(1-X)\sqrt{n-X}}\right]dx,$$

of which the quantity in the brackets is positive by Proposition 7. Thus, he believed at first he succeeded in proving Conjecture A. But this integral becomes $+\infty$.

In the following, we shall show that a large number of the facts obtained in §1∼§7 will be useful in a study of the period function T, by giving a new proof of the following theorem which was proved in [10] by a considerable complicated complex analysis on the Riemann surface $\mathfrak{F}$ given in §1.

THEOREM D. *The period function* $T(x_0,\ n)$ *of the solution of the differential equation* (E) *is monotone increasing with respect to* x_0 $(0<x_0<1)$ *for any fixed* n $(\geqq 2)$.

Proof. By (1.4) in Part (I), we have

$$\frac{\partial T(x_0,\ n)}{\partial x_0}=-n(1-x_0)(n-x_0)^{n-2}\cdot\frac{1}{4}\sqrt{\frac{n}{c}}\int_\gamma\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}},\tag{1}$$

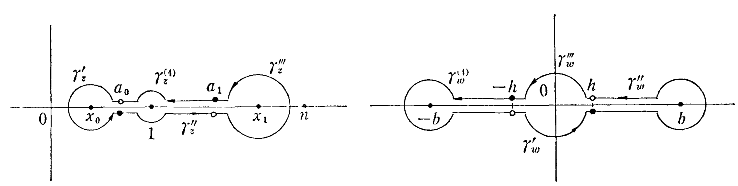
where $c=x_0(n-x_0)^{n-1}$.

Now, setting

$$J_3(\gamma):=\int_\gamma\frac{(n-z)^{n-3/2}dz}{\sqrt{z(n-z)^{n-1}-c}^{\,3}},\tag{2}$$

we divide the closed path γ on the Riemann surface $\mathfrak{F}: z(n-z)^{n-1}-w^2=c$, as is shown in the following figure:

z-plane w-plane



where $\phi(a_0)=\phi(a_1)$, $\sqrt{\phi(a_0)-c}=h$, $x_0<a_0<1<a_1<x_1$ and $\gamma=\gamma'+\gamma''+\gamma'''+\gamma^{(4)}$. Then, we have

$$J_3(\gamma)=\frac{2}{n}\int_\gamma\frac{(n-z)^{1/2}dw}{(1-z)w^2}$$

$$=-\frac{2}{n}\left[\frac{\sqrt{n-z}}{1-z}\cdot\frac{1}{w}\right]_{\partial\gamma}+\frac{2}{n}\int_\gamma\left\{\frac{-1}{2(1-z)\sqrt{n-z}}+\frac{\sqrt{n-z}}{(1-z)^2}\right\}\frac{dz}{w}$$

$$=-\frac{2}{n}\left[\frac{\sqrt{n-z}}{1-z}\cdot\frac{1}{w}\right]_{\partial\gamma}+\frac{1}{n}\int_\gamma\frac{(2n-1-z)\,dz}{(1-z)^2\sqrt{(n-z)\{\phi(z)-c\}}}.$$

Since this equality holds for any path on $\mathcal{F}$, we have

$$J_3(\gamma')=\frac{4}{n}\frac{\sqrt{n-a_0}}{(1-a_0)h}-\frac{2}{n}\int_{x_0}^{a_0}\frac{(2n-1-x)\,dx}{(1-x)^2\sqrt{(n-x)\{\phi(x)-c\}}},$$

$$J_3(\gamma''')=\frac{4}{n}\frac{\sqrt{n-a_1}}{(a_1-1)h}-\frac{2}{n}\int_{a_1}^{x_1}\frac{(2n-1-x)\,dx}{(1-x)^2\sqrt{(n-x)\{\phi(x)-c\}}},$$

and

$$J_3(\gamma''+\gamma^{(4)})=-2\int_{a_0}^{a_1}\frac{(n-x)^{n-3/2}dx}{\sqrt{\{\phi(x)-c\}^3}}.$$

Hence, we obtain

$$J_3(\gamma)=J_3(\gamma')+J_3(\gamma'')+J_3(\gamma''')+J_3(\gamma^{(4)})$$

$$=\frac{4}{n}\cdot\frac{1}{h}\left(\frac{\sqrt{n-a_0}}{1-a_0}+\frac{\sqrt{n-a_1}}{a_1-1}\right)$$

$$-\frac{2}{n}\left\{\int_{x_0}^{a_0}\frac{(2n-1-x)\,dx}{(1-x)^2\sqrt{(n-x)\{\phi(x)-c\}}}+\int_{a_1}^{x_1}\frac{(2n-1-x)\,dx}{(1-x)^2\sqrt{(n-x)\{\phi(x)-c\}}}\right\}$$

$$-2\int_{a_0}^{a_1}\frac{(n-x)^{n-3/2}dx}{\sqrt{\{\phi(x)-c\}^3}}.$$

Since both of the integrals in the braces of the right hand side tend to 0 as $a_0\to x_0$ and $a_1\to x_1$, we obtain

(3)
$$J_3(\gamma)=\frac{2}{n}\lim_{a_0\to x_0}\left[\left(\frac{\sqrt{n-a_0}}{1-a_0}+\frac{\sqrt{n-a_1}}{a_1-1}\right)\frac{2}{\sqrt{\phi(a_0)-c}}\right.$$
$$\left.-n\int_{a_0}^{a_1}\frac{(n-x)^n dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}\right].$$

On the other hand, we have

$$\frac{\sqrt{n-x}}{(1-x)\sqrt{\phi(x)-c}}=\frac{\sqrt{n-x}}{(1-x)\sqrt{\phi(x)-c}}\cdot\frac{\{B-\phi(x)\}+\{\phi(x)-c\}}{B-c}$$

$$=\frac{\sqrt{n-x}\{B-\phi(x)\}}{b^2(1-x)\sqrt{\phi(x)-c}}+\frac{\sqrt{n-x}\sqrt{\phi(x)-c}}{b^2(1-x)},$$

where $b^2=B-c$. The second term of the last side tends to 0 as $x\to x_0$ or $x\to x_1$. Hence we have

$$J_3(\gamma)=\frac{2}{n}\lim_{a_0\to x_0}\Big[\frac{2\sqrt{n-a_0}\{B-\phi(a_0)\}}{b^2(1-a_0)\sqrt{\phi(a_0)-c}}+\frac{2\sqrt{n-a_1}\{B-\phi(a_1)\}}{b^2(a_1-1)\sqrt{\phi(a_1)-c}}$$
$$-n\int_{a_0}^{a_1}\frac{(n-x)^n dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}\Big].$$

Setting

$$V(x):=\frac{\sqrt{n-x}\{B-\phi(x)\}}{x-1}=(x-1)\sqrt{n-x}\,\mu(x),$$

which is real analytic in the interval $(0,n)$, we obtain

$$(4)\qquad J_3(\gamma)=\frac{2}{n}\lim_{a_0\to x_0}\Big[\frac{2}{b^2}\frac{V(a_1)-V(a_0)}{\sqrt{\phi(a_0)-c}}-n\int_{a_0}^{a_1}\frac{(n-x)^n dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}\Big].$$

Now, since we have

$$\Big(\frac{V(x)}{\sqrt{\phi(x)-c}}\Big)'=\frac{2\{\phi(x)-c\}V'(x)-n(1-x)(n-x)^{n-2}V(x)}{2\sqrt{\{\phi(x)-c\}^3}},$$

we get

$$\frac{2}{b^2}\frac{V(a_1)-V(a_0)}{\sqrt{\phi(a_0)-c}}-n\int_{a_0}^{a_1}\frac{(n-x)^n dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}$$
$$=\frac{2}{b^2}\int_{a_0}^{a_1}\Big(\frac{V(x)}{\sqrt{\phi(x)-c}}\Big)'dx-n\int_{a_0}^{a_1}\frac{(n-x)^n dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}$$
$$=\frac{1}{b^2}\int_{a_0}^{a_1}\frac{2\{\phi(x)-c\}V'(x)-n(1-x)(n-x)^{n-2}V(x)-nb^2(n-x)^{n-3/2}}{\sqrt{\{\phi(x)-c\}^3}}dx.$$

The numerator of the integrand of the last integral can be expressed as follows:

$$2\{\phi(x)-c\}\cdot\Big[-\frac{B-\phi(x)}{2(x-1)\sqrt{n-x}}-\frac{\sqrt{n-x}\{B-\phi(x)\}}{(x-1)^2}+n(n-x)^{n-3/2}\Big]$$
$$+n(n-x)^{n-3/2}\{B-\phi(x)\}-nb^2(n-x)^{n-3/2}$$
$$=2\{\phi(x)-c\}\cdot\Big[-\frac{(2n-1-x)\{B-\phi(x)\}}{2(x-1)^2\sqrt{n-x}}+n(n-x)^{n-3/2}\Big]$$
$$-n(n-x)^{n-3/2}\{\phi(x)-c\}$$
$$=\{\phi(x)-c\}\cdot\Big[-\frac{(2n-1-x)\{B-\phi(x)\}}{(x-1)^2\sqrt{n-x}}+n(n-x)^{n-3/2}\Big],$$

hence we have

$$\frac{2}{b^2}\frac{V(a_1)-V(a_0)}{\sqrt{\phi(a_0)-c}}-n\int_{a_0}^{a_1}\frac{(n-x)^n dx}{\sqrt{(n-x)^3\{\phi(x)-c\}^3}}$$

$$=\frac{1}{b^2}\int_{a_0}^{a_1}\frac{1}{\sqrt{\psi(x)-c}}\cdot\left[-\frac{(2n-1-x)\{B-\psi(x)\}}{(x-1)^2\sqrt{n-x}}+n(n-x)^{n-3/2}\right]dx\,.$$

Thus, we obtain the important formula as follows:

$$(5)\qquad J_3(\gamma)=-\frac{2}{nb^2}\int_{x_0}^{x_1}\frac{(2n-1-x)\{B-\psi(x)\}-n(x-1)^2(n-x)^{n-1}}{(x-1)^2\sqrt{(n-x)\{\psi(x)-c\}}}dx$$

$$=-\frac{2}{nb^2}\int_{x_0}^{x_1}\frac{f_0(x)\,dx}{(x-1)^2\sqrt{(n-x)\{\psi(x)-c\}}}\,,$$

because we have

$$(2n-1-x)\{B-\psi(x)\}-n(x-1)^2(n-x)^{n-1}$$
$$=(2n-1-x)B-(n-x)^{n-1}\{x(2n-1-x)+n(x-1)^2\}$$
$$=(2n-1-x)B-(n-x)^{n-1}\{n-x+(n-1)x^2\}=f_0(x)\,.$$

Finally using this formula and (0.9), we have

$$\int_{x_0}^{x_1}\frac{f_0(x)\,dx}{(x-1)^2\sqrt{(n-x)\{\psi(x)-c\}}}$$
$$=\int_{x_0}^{1}\frac{f_0(x)\,dx}{(x-1)^2\sqrt{(n-x)\{\psi(x)-c\}}}+\int_{1}^{x_0}\frac{f_0(X)}{(X-1)^2\sqrt{(n-X)\{\psi(X)-c\}}}$$
$$\cdot\frac{1-x}{x(n-x)}\cdot\frac{X(n-X)}{1-X}dx$$
$$=\int_{x_0}^{1}\frac{1-x}{x(n-x)}\left[\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}\right]\frac{dx}{\sqrt{\psi(x)-c}}\,,$$

that is,

$$(6)\qquad J_3(\gamma)=-\frac{2}{nb^2}\int_{x_0}^{1}\frac{1-x}{x(n-x)\sqrt{\psi(x)-c}}\left[\frac{X\sqrt{n-X}f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}f_0(x)}{(x-1)^3}\right]dx\,.$$

By means of proposition 2, we obtain

$$J_3(\gamma)<0\,,$$

hence

$$\frac{\partial T(x_0,n)}{\partial x_0}>0\,.\qquad\text{Q.E.D.}$$

REMARK. The formula (6) will play an important role in Part (III) to continue to the present part, in which we shall try to make Conjecture B or Conjecture C a theorem.

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