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ON A CHARACTERIZATION OF THE EXPONENTIAL FUNCTION AND THE COSINE FUNCTION BY FACTORIZATION, III

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1. Introduction. This paper is a continuation of our previous one [1] with the same title, in which we proved the following fact.

THEOREM A. *Let $F(z)$ be an entire function, satisfying*

(a) $$F(z)=P_m(f_m(z))$$

with a polynomial P_m of degree m and an entire function f_m for $m=2^j$ (j: natural numbers) and $m=3$. Then

$$F(z)=A\cos\sqrt{H(z)}+B\,,$$

unless $F(z)=Ae^{H(z)}+B$. Here A, B are constants and H is an entire function.

In this paper we shall firstly consider the case that (a) holds for $m=2$, 4 and 3^j, where j runs over all natural numbers. Our theorem is the following.

THEOREM 1. *Let $F(z)$ be an entire function satisfying* (a) *for $m=2$, 4 and 3^j ($j=1, 2, \cdots$). Then*

$$F(z)=A\cos\sqrt{H(z)}+B\,,$$

unless $F(z)=Ae^H+B$. Here A, B and H are the same as in Theorem A.

The method of this paper gives more. Indeed (a) for i) $m=2, 3, 4$ and 5^j, or ii) $m=2, 3, 4, 7^j$ or iii) $m=2, 3, 4$, and 11^j implies the result, respectively.

2. Proof of Theorem 1. The first step, in which the case that

$$F(z)-b=A_2(f_2(z)-w_0)^2$$

has only finitely many zeros was considered in [1], gives the same conclusion, that is, $F(z)=Ae^{H(z)}+B$. Hence from now on we may assume that $F-b$ has infinitely many zeros and hence only infinitely many zeros of even order. The second step. Assuming that

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$$F(z)-b=A_3(f_3(z)-w_1)^3,$$

we have

$$F-b=A_4(f_4-w_1^*)^4.$$

Assume inductively that $F-b=A_{3p}(f_{3p}-\alpha)^{3p}$. Then $F-b$ has only zeros of order $4\cdot 3^p$. We consider

$$F(z)-b=A_{3p+1}\prod_{j=1}^{s}(f_{3p+1}(z)-\alpha_j)^{l_j},$$

$$\sum_{j=1}^{s} l_j=3^{p+1}.$$

Suppose that $\alpha_1\neq\alpha_2$. If l_1 and l_2 are not any divisor of $4\cdot 3^p$, then $f_{3p+1}(z)-\alpha_j$ $(j=1, 2)$ has only zeros of order $4\cdot 3^p$, which is impssible. If l_1 is a divisor of $4\cdot 3^p$ and l_2 is not, then $f_{3p+1}(z)-\alpha_1$ has only zeros of order $4\cdot 3^p/l_1\geqq 2$ and $f_{3p+1}(z)-\alpha_2$ has only zeros of order $4\cdot 3^p$, wihch is again impossible. If l_1 and l_2 are divisors of $4\cdot 3^p$, $f(z)-\alpha_j$ has only zeros of order $4\cdot 3^p/l_j\geqq 2$, which is absurd. Hence $\alpha_1=\alpha_2=\cdots=\alpha_s$, that is,

$$F(z)-b=A_{3p+1}(f_{3p+1}(z)-\alpha_1)^{3p+1}.$$

This implies that $F(z)-b$ has only zeros of order $4\cdot 3^{p+1}$. Thus $F(z)-b$ has only zeros of arbitrarily high order. This is absurd. Hence we may assume that

$$F(z)-b=A_2(f_2-w_0)^2$$
$$=A_3(f_3-w_1)(f_3-w_2)^2.$$

Then as in [1]

$$F(z)-b=A_4(f_4-d_1)^2(f_4-d_2)^2$$
$$=A_{3p}\prod_{\nu=1}^{s}(f_{3p}-e_\nu)^{\mu_\nu},$$
$$\sum_{\nu=1}^{s}\mu_\nu=3^p, \qquad e_i\neq e_j(i\neq j).$$

Hence we can make use of several results in the third and fourth steps in [1]. We summarize them here.

Let us put $f_3-w_1=T^2$. Then

$$T^3-(w_2-w_1)T+C_1=(T-\alpha_{11})(T-\alpha_{21})^2$$
$$=\left(\frac{A_2}{A_3}\right)^{1/2}(f_2-x_1)=M_1^2,$$
$$T^3-(w_2-w_1)T-C_1=(T-\alpha_{12})(T-\alpha_{22})^2$$
$$=\left(\frac{A_2}{A_3}\right)^{1/2}(f_2-x_2)=LM_2^2$$

for

$$C_1=\sqrt{\frac{4}{27}(w_2-w_1)^3},$$

$$x_1=w_0-\left(\frac{A_3}{A_2}\right)^{1/2}C_1, \qquad x_2=w_0+\left(\frac{A_3}{A_2}\right)^{1/2}C_1.$$

Let us put

$$f=-1+a_1(f_2-x_1), \qquad f=1+a_1(f_2-x_2)$$

with $a_1=2/(x_2-x_1)$. Then $f=a_1(f_2-w_0)$ and $f^2-a_1^2LM_1^2M_2^2=1$. Let $\Theta(z)$ be

$$\frac{1}{i}a_1\sqrt{L(z)}\,M_1(z)M_2(z)f(z)-2a_1\frac{1}{i}\int_{\alpha_1}^{z}\sqrt{L}\,M_1M_2f'dz.$$

Then

$$f+a_1\sqrt{L}M_1M_2=e^{i\Theta}$$

and

$$f-a_1\sqrt{L}M_1M_2=e^{-i\Theta},$$

Hence

$$f(z)=\cos\Theta.$$

$\Theta(z)$ depends on paths of integration connecting with α_1 to z.

The fifth step. We now have

$$F(z)-b=A_2(f_2-w_0)^2=A_3(f_3-w_1)(f_3-w_2)^2$$
$$=A_4(f_4-d_1)^2(f_4-d_2)^2$$
$$=A_{3p}\prod_{\nu=1}^{s}(f_{3p}-e_\nu)^{\mu_\nu},$$
$$\sum_{\nu=1}^{s}\mu_\nu=3^p, \qquad e_i\neq e_j (i\neq j).$$

Excepting only one μ_j, say μ_1, all μ_j are even in the above case. We can say more on $\{\mu_\nu\}$ and s.

LEMMA 1.

$$F(z)-b=A_{3p}(f_{3p}-e_1)\prod_{\nu=2}^{(3p+1)/2}(f_{3p}-e_\nu)^2.$$

Proof. We inductively assume that Lemma 1 is true for p. Let us put $f_{3p}-e_1=S_p^2$, $f_{3p+1}-e_1^*=S_{p+1}^2$. Then

$$A_3^{1/2}T(T^2-w_2+w_1)$$
$$=A_{3p}^{1/2}S_p\prod_{\nu=2}^{(3p+1)/2}(S_p^2-e_\nu+e_1)$$
$$=A_{3p+1}^{1/2}S_{p+1}^{\mu_1}\prod_{\nu=2}^{s}(S_{p+1}^2-e_\nu^*+e_1^*)^{\mu_\nu/2}.$$

Since

$$A_3^{1/2}T(T^2-w_2+w_1)+A_3^{1/2}C_1=M_1^2A_3^{1/2},$$

$$A_{3p}^{1/2}S_p\prod_{\nu=2}^{(3p+1)/2}(S_\nu^2-e_\nu+e_1)+A_3^{1/2}C_1$$

$$=A_{3p}^{1/2}\prod_{j=1}^{t}(S_p-\alpha_j)^{l_j}, \qquad \sum_{j=1}^{t}l_j=3^p.$$

Here only one l_j, say l_1, is odd and the others are even. Hence $S_p-\alpha_1=X^2$. Therefore

$$\bar{N}(r, \alpha_1, S_p)\leqq\frac{1}{2}N(r, \alpha_1, S_p)\leqq\frac{1}{2}m(r, S_p).$$

Now we have

$$\begin{aligned}&(1+o(1))3^p m(r, S_p)\\ &\leqq\bar{N}(r, 0, S_p)+\sum_{\nu=2}^{(3p+1)/2}\{\bar{N}(r, \sqrt{e_\nu-e_1}, S_p)+\bar{N}(r, -\sqrt{e_\nu-e_1}, S_p)\}\\ &\quad+\bar{N}(r, \alpha_1, S_p)\\ &\leqq\bar{N}(r, 0, S_{p+1})+\sum_{\nu=2}^{s}\{\bar{N}(r, \sqrt{e_\nu^*-e_1^*}, S_{p+1})+\bar{N}(r, -\sqrt{e_\nu^*-e_1^*}, S_{p+1})\}\\ &\quad+\frac{1}{2}m(r, S_p)\\ &\leqq(2s-1)m(r, S_{p+1})+\frac{1}{2}m(r, S_p).\end{aligned}$$

Evidently $3^p m(r, S_p)\sim 3^{p+1}m(r, S_{p+1})$. Hence

$$3^p\leqq\frac{2s-1}{3}+\frac{1}{2},$$

that is,

$$s\geqq\frac{3^{p+1}+1}{2}-\frac{3}{4}.$$

Hence

$$s\geqq\frac{3^{p+1}+1}{2}.$$

On the other hand

$$\sum_{j=1}^{s}\mu_j=3^{p+1}.$$

Hence

$$2(s-1)\leqq 3^{p+1}-\mu_1\leqq 3^{p+1}-1,$$

that is,

$$s\leqq\frac{3^{p+1}+1}{2}.$$

Therefore

$$s=\frac{3^{p+1}+1}{2},$$

$$\mu_1=1,\ \mu_2=\cdots=\mu_s=2.$$

Thus we have the desired result.

Simultaneously we have

$$A_{3^p}{}^{1/2}\Big\{S_p\prod_{\nu=2}^{(3^p+1)/2}(S_p{}^2-e_\nu+e_1)+\Big(\frac{A_3}{A_{3^p}}\Big)^{1/2}C_1\Big\}$$

$$=A_{3^p}{}^{1/2}(S_p-\alpha_1)^{l_1}\prod_{j=2}^{t}(S_p-\alpha_j)^{2\lambda_j}.$$

Here l_1 is odd. Hence the above expression reduces to

$$A_{3^p}{}^{1/2}(S_p-\alpha_1)(S_p{}^{\frac{3^p+1}{2}-1}+p_1S_p{}^{\frac{3^p+1}{2}-2}+p_2S_p{}^{\frac{3^p+1}{2}-3}+\cdots+p_{(3^p-1)/2})^2.$$

Let us put $p_j=\beta_jp_1{}^j$ for $j=1, 2, \cdots, (3^p-1)/2$ and $S_p=2p_1x$. Then by $\alpha_1=2p_1$ we have

$$(x-1)\Big(x^{\frac{3^p-1}{2}}+\frac{1}{2}x^{\frac{3^p-3}{2}}+\frac{\beta_2}{4}x^{\frac{3^p-5}{2}}+\cdots+\frac{\beta_{(3^p-1)/2}}{2^{(3^p-1)/2}}\Big)^2$$

$$=x\prod_{\nu=2}^{(3^p+1)/2}(x^2-\delta_\nu)+D,$$

$$\delta_\nu=\frac{e_\nu-e_1}{4p_1},$$

$$D=\frac{A_3{}^{1/2}C_1}{A_{3^p}{}^{1/2}2^{3^p}p_1{}^{3^p}}=-\frac{\beta_{(3^p-1)/2}^2}{2^{3^p}-1}\neq 0.$$

Let us put

$$X_n(x)\equiv x\prod_{\nu=2}^{n+1}(x^2-\delta_\nu^*)$$

$$=(x-1)\Big(x^n+\frac{1}{2}x^{n-1}+\cdots+\beta_n^*\Big)^2-D^*$$

$$\equiv(x-1)Q(x)^2-D^*.$$

Evidently

$$-X_n(-x)=X_n(x).$$

Hence

$$X_n(x)=(x+1)Q(-x)^2+D^*$$
$$\equiv(x+1)P(x)^2+D^*.$$

LEMMA 2. *$X_n(x)$ is the Chebyshev polynomial $T_{2n+1}(x)$.*

Proof. The following proof is due to Amemiya. $X_n(x)$ satisfies

$$X_n(x)=(x+1)P(x)^2+D^*$$
$$=(x-1)Q(x)^2-D^*,$$

where $P(x)$, $Q(x)$ are polynomials of degree n, whose leading coefficients are equal to 1, and D^* is a non-zero constant. By differentiation

$$(2(x+1)P'(x)+P(x))P(x)$$
$$=(2(x-1)Q'(x)+Q(x))Q(x).$$

Since $P(x)$ and $Q(x)$ have no common zero,

$$(2n+1)P(x)=Q(x)+2(x-1)Q'(x)$$

and

$$(2n+1)Q(x)=P(x)+2(x+1)P'(x).$$

Suppose that there is another pair $(P_1(x), Q_1(x))$ with the desired condition. Then P_1, Q_1 satisfy the above simultaneous differential equation. Hence by its linearity

$$P(x)-P_1(x), \qquad Q(x)-Q_1(x)$$

satisfy the same equation. Evidently $s=\deg(P-P_1)<n$ and $t=\deg(Q-Q_1)<n$ and $s=t$. Assume that the leading coefficients a_s and b_s of $P-P_1$ and $Q-Q_1$ are not equal to zero. Then we have

$$(2n+1)a_s=(2s+1)b_s,$$
$$(2n+1)b_s=(2s+1)a_s.$$

This is absurd. Hence $P(x)\equiv P_1(x)$ and $Q(x)\equiv Q_1(x)$. The Chebyshev polynomial $T_{2n+1}(x)$ satisfies

$$T_{2n+1}(x)=\frac{1}{2^{2n}}\cos((2n+1)\arccos x)$$
$$=(x-1)\prod_{j=1}^{n}\left(x-\cos\frac{2j\pi}{2n+1}\right)^2+\frac{1}{2^{2n}}$$
$$=(x+1)\prod_{j=1}^{n}\left(x+\cos\frac{2j\pi}{2n+1}\right)^2-\frac{1}{2^{2n}}.$$

By the unicity of the pair (P, Q) $X_n(x)$ coincides with $T_{2n+1}(x)$. Thus we have the desired result.

The above proof implies that $D^*=-2^{-2n}$.

Returning back to the original problem we have

$$D=-\frac{1}{2^{3p-1}}$$

and hence

$$\frac{A_3^{1/2}C_1}{A_{3p}^{1/2}p_1^{3p}}=-2, \qquad \beta_{(3p^2-1)/2}=1.$$

The sixth step. Let us put $T=Bu$, $B^3=4C_1$. Then

$$f=4u^3-3u.$$

Therefore

$$u=\cos\frac{\Theta+2\pi j}{3}, \qquad j=0, 1, 2.$$

By Lemma 1 and Lemma 2 we have

$$\begin{aligned}
&A_2^{1/2}(f_2-w_0)\\
&=A_3^{1/2}T(T^2-w_2+w_1)\\
&=A_{3p}^{1/2}S_p(S_p^2-e_2+e_1)\cdots(S_p^2-e_p+e_1)\\
&=A_{3p}^{1/2}2p_1^{3p}2^{3p-1}T_{3p}(x)\\
&=A_{3p}^{1/2}2p_1^{3p}\cos(3^p\arccos x).
\end{aligned}$$

By $A_3^{1/2}C_1=-2A_{3p}^{1/2}p_1^{3p}$

$$f=4u^3-3u=2^{3p-1}T_{3p}(x).$$

Hence

$$x=\cos\frac{\Theta+2\pi j}{3^p}, \qquad j=0, 1, \cdots, 3^p-1.$$

Now let us consider the Riemann surface defined by $y^2=L$. Let C be a cycle on the surface, along which $\Theta(z)$ has non-zero period $v\pi$. Then $(\Theta(z)+2\pi j)/3^p$ has period $v\pi/3^p$ along C. Therefore x and hence $S_p=2p_1x$ is not one-valued along C. This is absurd. Now by the same reason as in [1]

$$F(z)=A\cos\sqrt{H(z)}+B.$$

3. We shall consider a variant of Theorem 1.

THEOREM 2. *Let $F(z)$ be an entire function satisfying (a) for $m=2, 3, 4$ and 5^j ($j=1, 2, 3, \cdots$). Then*

$$F(z)=A\cos\sqrt{H(z)}+B,$$

unless $F(z)=Ae^H+B$.

Proof. We have to consider Lemma 1 correspondingly.

LEMMA 3. *If $F(z)-b$ satisfies*

$$\begin{aligned}
F(z)-b&=A_2(f_2-w_0)^2=A_3(f_3-w_1)(f_3-w_2)^2\\
&=A_4(f_4-d_1)^2(f_4-d_2)^2
\end{aligned}$$

$$=A_{5^p}(f_{5^p}-e_1)\prod_{\nu=2}^{(5^p+1)/2}(f_{5^p}-e_\nu)^2, \qquad p=1, 2, \cdots, p_0,$$

then

$$F(z)-b=A_{5^{p_0+1}}(f_{5^{p_0+1}}-e_1^*)^{\mu_1}\prod_{\nu=2}^{s}(f_{5^{p_0+1}}-e_\nu^*)^{\mu_\nu}$$

with either $s=(5^{p_0+1}+1)/2$, $\mu_1=1$, $\mu_2=\cdots=\mu_s=2$ *or* $s=(5^{p_0+1}-1)/2$, $\mu_1=3$, $\mu_2=\cdots=\mu_s=2$ *or* $s=(5^{p_0+1}-1)/2$, $\mu_1=1$, $\mu_2=\cdots=\mu_{s-1}=2$, $\mu_s=4$.

Proof of Lemma 3. We abbreviate 5^{p_0} as n_0. Let us put $f_{n_0}-e_1=S^2$ and $f_{5n_0}-e_1^*=V^2$. Then as in the proof of Lemma 1 there is a constant α_1 for which

$$\bar{N}(r, \alpha_1, S)\leqq\frac{1}{2}N(r, \alpha_1, S)\leqq\frac{1}{2}m(r, S),$$

$$\alpha_1\neq 0, \pm\sqrt{e_\nu-e_1} \qquad (\nu=2, 3, \cdots, (n_0+1)/2).$$

Hence

$$\begin{aligned}&(1+o(1))n_0 m(r, S)\\ &\leqq\bar{N}(r, 0, S)+\sum_{\nu=2}^{(n_0+1)/2}\{\bar{N}(r, \sqrt{e_\nu-e_1}, S)+\bar{N}(r, \sqrt{e_\nu-e_1}, S)\}\\ &\qquad\qquad +\bar{N}(r, \alpha_1, S)\\ &\leqq\bar{N}(r, 0, V)+\sum_{\nu=2}^{s}\{\bar{N}(r, \sqrt{e_\nu^*-e_1^*}, V)+\bar{N}(r, -\sqrt{e_\nu^*-e_1^*}, V)\}\\ &\qquad\qquad +\frac{1}{2}m(r, S)\\ &\leqq(2s-1)m(r, V)+\frac{1}{2}m(r, S).\end{aligned}$$

By $n_0 m(r, S)\sim 5n_0 m(r, V)$

$$s\geqq\frac{5n_0-1}{2}-\frac{1}{4}.$$

Thus

$$s\geqq\frac{5n_0-1}{2}.$$

On the other hand we easily have

$$s\leqq\frac{5n_0+1}{2}\equiv s_1.$$

If $s=s_1$, then $\mu_1=1$, $\mu_2=\cdots=\mu_s=2$. If $s=s_1-1$, then either $\mu_1=3$, $\mu_2=\cdots=\mu_s=2$ or $\mu_1=1$, $\mu_2=\cdots=\mu_{s-1}=2$, $\mu_s=4$. This is just the desired result.

Next we shall prove Lemma 1. Assume inductively that Lemma 1 holds for $p-1$. We abbreviate 5^p as n. By Lemma 3 we have

$$A_n^{1/2}\left\{V^{\mu_1}\prod_{\nu=2}^{s}(V^2-e_\nu^*+e_1^*)^{\mu_\nu/2}+\left(\frac{A_3}{A_n}\right)^{1/2}C_1\right\}$$

$$=A_n^{1/2}(V-\alpha_1^*)^{l_1}\prod_{j=2}^{t}(V-\alpha_j^*)^{2\lambda_j}$$

with $f_n-e_1^*=V^2$ and with odd l_1 and $\alpha_j^*\neq 0$, since

$$T(T^2-w_2+w_1)+C_1=M^2.$$

Now we put $V=\alpha_1^*x$. Then

$$X(x)\equiv x^{\mu_1}\prod_{\nu=2}^{s}(x^2-\delta_\nu)^{\mu_\nu/2}$$

$$=(x-1)^{l_1}\prod_{j=2}^{t}(x-\varepsilon_j)^{2\lambda_j}-D$$

$$\equiv(x-1)Q(x)^2-D.$$

Here

$$D=\frac{A_3^{1/2}C_1}{A_n^{1/2}\alpha_1^{*n}}\neq 0.$$

Since $-X(-x)=X(x)$, we have

$$X(x)=(x-1)Q(x)^2-D$$
$$=(x+1)P(x)^2+D.$$

Thus by Lemma 2 $X(x)$ reduces to the Chebyshev polynomial $T_n(x)$. Hence $X(x)$ must have the following form:

$$x\prod_{\nu=2}^{(n+1)/2}(x^2-\delta_\nu)$$

$$\delta_\nu\neq 0, \qquad \delta_\nu\neq\delta_\mu(\nu\neq\mu)$$

and

$$D=-\frac{1}{2^{n-1}}.$$

Now returning back to V and then to f_n we have

$$F(z)-b=A_n(f_n-e_1^*)\prod_{\nu=2}^{(n+1)/2}(f_n-e_\nu^*)^2,$$

$$e_\nu^*\neq e_\mu^* \qquad \text{for} \quad \nu\neq\mu.$$

In order to complete the proof of Lemma 1 we should consider the case f_5. Then

$$F-b=A_2(f_2-w_0)^2=A_3(f_3-w_1)(f_3-w_2)^2$$
$$=A_5(f_5-e_1)^{\mu_1}(f_5-e_2)^{\mu_2}(f_5-e_3)^{\mu_3},$$

$$\sum_{j=1}^{3}\mu_j=5, \qquad \mu_1:\text{ odd}, \quad \mu_2,\ \mu_3:\text{ even}.$$

If $\mu_1=1$, $\mu_2=\mu_3=2$, there is nothing to prove. Hence there remain two cases:

$$\mu_1=3, \quad \mu_2=2, \quad \mu_3=0$$

or

$$\mu_1=1, \quad \mu_2=4, \quad \mu_3=0.$$

Let us put $f_3-w_1=T^2$ and $f_5-e_1=S^2$. Then

$$A_3^{1/2}T(T^2-w_2+w_1) \\ =A_5^{1/2}S^{\mu_1}(S^2-e_2+e_1)^{\mu_2/2}.$$

Since there is a constant $\alpha_{11}(\neq 0, \pm\sqrt{w_2-w_1})$ such that

$$\bar{N}(r, \alpha_{11}, T)\leqq\frac{1}{2}N(r, \alpha_{11}, T)\leqq\frac{1}{2}m(r, T).$$

Hence

$$\begin{aligned}(1+o(1))3m(r, T)&\leqq\bar{N}(r, 0, T)+\bar{N}\{r, \sqrt{w_2-w_1}, T)\\ &\quad+\bar{N}(r, -\sqrt{w_2-w_1}, T)+\bar{N}(r, \alpha_{11}, T)\\ &\leqq\bar{N}(r, 0, S)+\bar{N}(r, \sqrt{e_2-e_1}, S)+\bar{N}(r, -\sqrt{e_2-e_1}, S)\\ &\quad+\frac{1}{2}m(r, T)\\ &\leqq 3m(r, S)+\frac{1}{2}m(r, T)\end{aligned}$$

Evidently $3m(r, T)\sim 5m(r, S)$. Hence

$$3-\frac{1}{2}\leqq\frac{9}{5},$$

which is absurd. Thus by induction Lemma 1 holds.

Now we can proceed similarly as in Theorem 1.

The following cases give the same result as in Theorem 2.

i) $m=2, 3, 4, \quad 7^j(j=1, 2, \cdots)$,

ii) $m=2, 3, 4, \quad 11^j(j=1, 2, \cdots)$,

iii) $m=2^s(s=1, \cdots, p\geqq 2), 3, \quad q^j(j=1, 2, \cdots)$
$$q<3\cdot 2^p, \quad (q, 6)=1,$$

iv) $m=2^s(s=1, \cdots, p\geqq 2), 3, 5, \quad q^j(j=1, 2, \cdots)$
$$q<15\cdot 2^p, \quad (q, 30)=1,$$

v) $m=2^s(s=1, \cdots, p\geqq 2), 3, 7, \quad q^j(j=1, 2, \cdots)$
$$q<21\cdot 2^p, \quad (q, 42)=1,$$

vi) $m=2^s(s=1, \cdots, p\geqq 2), 3, 11, \quad q^j(j=1, 2, \cdots)$
$$q<33\cdot 2^p, \quad (q, 66)=1.$$

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