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ANTI-INVARIANT SUBMANIFOLDS OF A SASAKIAN SPACE FORM

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§ 1. Introduction.

In a previous paper [8] we studied anti-invariant submanifolds in a Kähler manifold, especially in a complex space form. In the present paper we shall study anti-invariant submanifolds of a Sasakian manifold, especially those of a Sasakian space form, in the same way as taken in [8].

An $(n+1)$-dimensional Riemannian manifold M isometrically immersed in a $(2m+1)$-dimensional Sasakian manifold $\bar{M}$ with structure tensors $(\phi, \xi, \eta, \bar{g})$ is said to be anti-invariant (resp. invariant) if $\phi T_p(M) \subset T_p(M)^{\perp}$ (resp. $\phi T_p(M) \subset T_p(M)$) for each point p of M, where $T_p(M)$ and $T_p(M)^{\perp}$ denote respectively the tangent and the normal spaces to M at p. Thus in an anti-invariant submanifold ϕX is normal to M for any vector X tangent to M. Since ϕ is necessarily of rank $2m$, we have $n \leqq (2m+1)-(n+1)$ which implies $n \leqq m$. In the present paper, we assume that for any anti-invriant submanifold M we consider the structure vector field ξ of the ambient manifold is tangent to M everywhere.

When for an anti-invariant submanifold M the structure vector field ξ of the ambient manifold $\bar{M}$ is tangent to M, then each of the following assumptions (a), (b), (c) is not meaningful: (a) the second fundamental form is parallel; (b) the mean curvature vector is parallel; (c) the connection induced in the normal bundle is flat. So, in the present paper, we shall replace the assumptions (a), (b), (c) respectively by new but rather weaker assumptions (a′), (b′), (c′) as follows: (a′) the second fundamental form is pseudo-parallel; (b′) the mean curvature vector is pseudo-parallel; (c′) the connection induced in the normal bundle is pseudo-flat (see Lemmas 3.2, 3.3 and 4.1).

§ 2. Sasakian manifolds.

First, we would like to recall definitions and some fundamental properties of Sasakian manifolds. Let $\bar{M}$ be a $(2m+1)$-dimensional differentiable manifold of class C^{∞} and ϕ, ξ, η be a tensor field of type $(1,1)$, a vector field, a 1-form on $\bar{M}$ respectively such that

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(2.1) $$\phi^2=-I+\eta\otimes\xi, \qquad \phi\xi=0, \qquad \eta(\phi X)=0, \qquad \eta(\xi)=1$$

for any vector field X on $\bar{M}$, where I denotes the identity tensor of type (1, 1). Then $\bar{M}$ is said to admit an *almost contact structure* (ϕ, ξ, η) and called an *almost contact manifold*. The almost contact structure is said to be *normal* if

(2.2) $$N+d\eta\otimes\xi=0,$$

where N denotes the Nijenhuis tensor formed with ϕ. If there is given in $\bar{M}$ a Riemannian metric $\bar{g}$ satisfying

(2.3) $$\bar{g}(\phi X, \phi Y)=\bar{g}(X, Y)-\eta(X)\eta(Y), \qquad \eta(X)=\bar{g}(X, \xi)$$

for any vector fields X and Y on $\bar{M}$, then the set $(\phi, \xi, \eta, \bar{g})$ is called a *almost contact metric structure* and $\bar{M}$ an *almost contact metric manifold*. If

(2.4) $$d\eta(X, Y)=\bar{g}(\phi X, Y)$$

for any vector fields X and Y an $\bar{M}$, then the almost contact metric structure is called a *contact metric structure*. If the structure is moreover normal, then the contact metric structure is called a *Sasakian structure* and $\bar{M}$ a *Sasakian manifold*. As is well known, in a Sasakian manifold $\bar{M}$ with structure $(\phi, \xi, \eta, \bar{g})$

(2.5) $$\bar{\nabla}_X\xi=\phi X, \qquad (\bar{\nabla}_X\phi)Y=-\bar{g}(X, Y)\xi+\eta(Y)X$$

are established for any vector fields X and Y on $\bar{M}$, where $\bar{\nabla}$ denotes the operator of covariant differentiantion with respect to $\bar{g}$.

A plane section σ in the tangent space $T_p(\bar{M})$ of a Sasakian manifold $\bar{M}$ at p is called a *ϕ-section* if it is spanned by vectors X and ϕX, where X is assumed to be orthogonal to ξ. The sectional curvature $K(\sigma)$ with respect to a ϕ-section σ is called a *ϕ-sectional curvature*. When the ϕ-sectional curvature $K(\sigma)$ is independent of the ϕ-section σ at each point of $\bar{M}$, as is well known, the function $K(\sigma)$ defined in $\bar{M}$ is necessarily a constant c. A Sasakian manifold $\bar{M}$ is called a *Sasakian space form* and denoted by $\bar{M}(c)$ if it has constant ϕ-sectional curvature c (see [4]). The curvature tensor K of a Sasakian space form $\bar{M}(c)$ is given by

$$\begin{aligned}K(X, Y)Z=\frac{1}{4}(c+3)(\bar{g}(Y, Z)X-\bar{g}(X, Z)Y)-\frac{1}{4}(c-1)(\eta(Y)\eta(Z)X\\ -\eta(X)\eta(Z)Y+\bar{g}(Y, Z)\eta(X)\xi-\bar{g}(X, Z)\eta(Y)\xi\\ -\bar{g}(\phi Y, Z)\phi X+\bar{g}(\phi X, Z)\phi Y+2\bar{g}(\phi X, Y)\phi Z).\end{aligned}$$

EXAMPLE 1. Let S^{2n+1} be a $(2n+1)$-dimensional unit sphere, i.e.,

$$S^{2n+1}=\{z\in C^{n+1}: |z|=1\},$$

where C^{n+1} is a complex $(n+1)$-space. For any point $z\in S^{2n+1}$, we put $\xi=Jz$, J being the complex structure of C^{n+1}. Considering the orthogonal projection

$$\pi : T_z(C^{n+1}) \longrightarrow T_z(S^{2n+1}),$$

at each point z in S^{2n+1} and putting $\phi=\pi\circ J$, we have a Sasakian structure (ϕ, ξ, η, g) on S^{2n+1}, where η is a 1-form dual to ξ and g the standard metric tensor field on S^{2n+1}. Obviously, S^{2n+1} is of constant ϕ-sectional curvature 1.

EXAMPLE 2. Let E^{2n+1} be a Euclidean space with cartesian coordinates $(x^1, \cdots, x^n, y^1, \cdots, y^n, z)$. Then a Sasakian structure on E^{2n+1} is defined by ϕ, ξ, η and g such that

$$\xi=(0, \cdots, 0, 2), \qquad 2\eta=(-y^1, \cdots, -y^n, 0, \cdots, 0, 1),$$

$$(g_{AB})=\begin{pmatrix} \frac{1}{4}(\delta_{ij}+y^i y^j) & 0 & -\frac{1}{4}y^i \\ 0 & \frac{1}{4}\delta_{ij} & 0 \\ -\frac{1}{4}y^i & 0 & \frac{1}{4} \end{pmatrix},$$

$$(\phi^A_B)=\begin{pmatrix} 0 & \delta^i_j & 0 \\ -\delta^i_j & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}.$$

Then E^{2n+1} with such a structure (ϕ, ξ, η, g) is of constant ϕ-sectional curvature -3 and denoted by $E^{2n+1}(-3)$.

§ 3. Fundamental properties of anti-invariant submanifolds.

Let $\bar{M}$ be a Sasakian manifold of dimension $2m+1$ with structure tensors $(\phi, , \xi, \eta, \bar{g})$. An $(n+1)$-dimensional Riemannian manifold M isometrically immersed in $\bar{M}$ is said to be anti-invariant in $\bar{M}$ if $\phi T_p(M)\subset T_p(M)^\perp$ for each point p of M. Throughout the paper, we now restrict ourselves only to anti-invariant submanifolds of a Sasakian manifold such that the structure vector field ξ of the ambient manifold is tangent to the submanifolds.

We choose a local field of orthonormal frames $e_0=\xi, e_1, \cdots, e_n ; e_{n+1}, \cdots, e_m$; $e_{1^*}=\phi e_1, \cdots, e_{n^*}=\phi e_n ; e_{(n+1)^*}=\phi e_{n+1}, \cdots, e_{m^*}=\phi e_m$ in $\bar{M}$ in such a way that $e_0, e_1, \cdots, e_n$ are along M tangent to M. Taking such a field of frames of $\bar{M}$, we denote the dual coframes by $\omega^0=\eta, \omega^1, \cdots, \omega^n ; \omega^{n+1}, \cdots, \omega^m ; \omega^{1^*}, \cdots, \omega^{n^*}$; $\omega^{(n+1)^*}, \cdots, \omega^{m^*}$. Unless otherwise stated, let the ranges of indices be as follows:

$A, B, C, D=0, 1, \cdots, m, 1^*, \cdots m^*,$

$i, j, k, l, s, t=0, 1, \cdots, n,$

$x, y, z, v, w=1, \cdots, n,$

$a, b, c, d=n+1, \cdots, m, 1^*, \cdots m^*,$

$\alpha, \beta, \gamma=n+1, \cdots, m,$

$\lambda, \mu, \nu=n+1, \cdots, m, (n+1)^*, \cdots, m^*,$

and use the so-called summation convention for these systems of indices. Then the structure equations of the Riemannian manifold $\bar{M}$ are given by

$$d\omega^A = -\omega_B^A \wedge \omega^B, \qquad \omega_B^A + \omega_A^B = 0, \tag{3.1}$$

$$d\omega_B^A = -\omega_C^A \wedge \omega_B^C + \Phi_B^A, \qquad \Phi_B^A = \frac{1}{2} K_{BCD}^A \omega^C \wedge \omega^D, \tag{3.2}$$

where K_{BCD}^A are components of the curvature tensor of $\bar{M}$ with respect to $\{e_A\}$ and ω_B^A satisfy

$$\begin{aligned} &\omega_y^x = \omega_{y^*}^{x^*}, \qquad \omega_y^{x^*} = \omega_x^{y^*}, \qquad \omega^x = \omega_0^{x^*}, \qquad \omega^{x^*} = -\omega_0^x, \\ &\omega_\beta^\alpha = \omega_{\beta^*}^{\alpha^*}, \qquad \omega_\beta^{\alpha^*} = \omega_\alpha^{\beta^*}, \qquad \omega^\alpha = \omega_0^{\alpha^*}, \qquad \omega^{\alpha^*} = -\omega_0^\alpha, \\ &\omega_\alpha^x = \omega_{\alpha^*}^{x^*}, \qquad \omega_\alpha^{x^*} = \omega_x^{\alpha^*}. \end{aligned} \tag{3.3}$$

Thus we have along M

$$\omega^a = 0, \tag{3.4}$$

which implies $0 = d\omega^a = -\omega_i^a \wedge \omega^i$ alng M. Thus, by Cartan's lemma, we obtain along M

$$\omega_i^a = h_{ij}^a \omega^j, \qquad h_{ij}^a = h_{ji}^a, \tag{3.5}$$

which imply the following structure equations of the submanifold M:

$$d\omega^i = -\omega_j^i \wedge \omega^j, \qquad \omega_j^i + \omega_i^j = 0, \tag{3.6}$$

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k + \Omega_j^i, \qquad \Omega_j^i = \frac{1}{2} R_{jkl}^i \omega^k \wedge \omega^l, \tag{3.7}$$

$$R_{jkl}^i = K_{jkl}^i + \sum_a (h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a), \tag{3.8}$$

$$d\omega_b^a = -\omega_c^a \wedge \omega_b^c + \Omega_b^a, \qquad \Omega_b^a = \frac{1}{2} R_{bkl}^a \omega^k \wedge \omega^l, \tag{3.9}$$

$$R_{bkl}^a = K_{bkl}^a + \sum_i (h_{ik}^a h_{il}^b - h_{il}^a h_{ik}^b), \tag{3.10}$$

where R_{jkl}^i are components of the curvature tensor of M with respect to $\{e_i\}$ and R_{bkl}^a components of the curvature tensor of the normal bundle with respect to $\{e_i\}$ and $\{e_a\}$. The equations (3.8) and (3.10) are called respectively the equations of Gauss and those of Ricci for the submanifold M. The forms (ω_j^i) define the Riemannian connection of M and the forms (ω_b^a) define the connection induced in the normal bundle of M.

We now state a lemma for later use.

LEMMA 3.1. (*Yano and Kon* [6]) *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian manifold $\bar{M}^{2m+1}$. Then the structure vector field ξ is parallel along M and M is locally a Riemannian direct product $M^n \times M^1$, where*

M^n is an n-dimensional Riemannian manifold and M^1 is a 1-dimensional subspace generated by ξ.

From (3.3), (3.4) and (3.5) we have along M

$$h^x_{yz}=h^y_{xz}=h^z_{xy}\,,\qquad h^\lambda_{0i}=0\,,\qquad h^x_{0i}=\delta_{xi}\,, \tag{3.11}$$

where we denote $h^{x^*}_{ij}$ simply by h^x_{ij}.

The second fundamental form $h^a_{ij}\omega^i\omega^j e_a$ is sometimes denoted by its components h^a_{ij}. When the second fundamental form vanishes identically, i.e., $h^a_{ij}=0$ for all indices, the submanifold is as usual said to be *totally geodesic*. If h^a_{xy} have the form $h^a_{xy}=\frac{1}{n}(\sum_z h^a_{zz})\delta_{xy}$ for a fixed index a, then the submanifold is said to be *contact umbilical* with respect to the normal vector e_a. If the submanifold M is contact umbilical with respect to all e_a, then M is said to be *contact totally umbilical* (see [2]). The vector field $\frac{1}{n+1}(\sum_k h^a_{kk}e_a)$ normal to M is called the *mean curvature* vector of M. The submanifold M is said to be *minimal* if its mean curvature vector vanishes identically, i.e., $\sum_k h^a_{kk}=0$ for all a. We define the covariant derivative h^a_{ijk} of h^a_{ij} by

$$h^a_{ijk}\omega^k=dh^a_{ij}-h^a_{il}\omega^l_j-h^a_{lj}\omega^l_i+h^b_{ij}\omega^a_b\,. \tag{3.12}$$

If $h^a_{ijk}=0$ for all indices, the second fundamental form of M is said to be *parallel*. If the mean curvature vector of M is parallel with respect to the connection D induced in the normal bundle, then the mean curvature vector of M is said to be *parallel*. The Laplacian Δh^a_{ij} of h^a_{ij} is defined as

$$\Delta h^a_{ij}=\sum_k h^a_{ijkk}\,, \tag{3.13}$$

where we have defined h^a_{ijkl} by

$$h^a_{ijkl}\omega^l=dh^a_{ijk}-h^a_{ljk}\omega^l_i-h^a_{ilk}\omega^l_j-h^a_{ijl}\omega^l_k+h^b_{ijk}\omega^a_b\,. \tag{3.14}$$

We shall establish a formula containing the Laplacian of h^a_{ij}. Now, the second fundamental form of M is assumed to satisfy equations of Codazzi type, i.e.,

$$h^a_{ijk}-h^a_{ikj}=0\,. \tag{3.15}$$

Then, from (3.14), we have

$$h^a_{ijkl}-h^a_{ijlk}=h^a_{it}R^t_{jkl}+h^a_{tj}R^t_{ikl}-h^b_{ij}R^a_{bkl}\,. \tag{3.16}$$

On the other hand, (3.13) and (3.15) imply

$$\Delta h^a_{ij}=\sum_k h^a_{ijkk}=\sum_k h^a_{kijk}\,. \tag{3.17}$$

From (3.15), (3.16) and (3.17), we obtain

$$\Delta h_{ij}^{a}=\sum_{k}(h_{kkij}^{a}+h_{kt}^{a}R_{ijk}^{t}+h_{ti}^{a}R_{kjk}^{t}-h_{ki}^{b}R_{bjk}^{a}). \tag{3.18}$$

Therefore for any submanifold M satisfying the equation (3.15) of Codazzi type we have the formula

$$\sum_{a,i,j} h_{ij}^{a}\Delta h_{ij}^{a}=\sum_{a,i,j,k}(h_{ij}^{a}h_{kkij}^{a}+h_{ij}^{a}h_{kt}^{a}R_{ijk}^{t} + h_{ij}^{a}h_{ti}^{a}R_{kjk}^{t}-h_{ij}^{a}h_{ki}^{b}R_{bjk}^{a}). \tag{3.19}$$

We are now going to prove some lemmas for later use. From (3.3), (3.11) and (3.12), we have

$$h_{ij0}^{a}=-h_{ij}^{a^*}, \qquad h_{ij0}^{a^*}=h_{ij}^{a}, \qquad h_{ij0}^{x}=0, \tag{3.20}$$

$$\sum_{a,i,j,k}(h_{ijk}^{a})^2=\sum_{a,x,y,z}(h_{xyz}^{a})^2+3\sum_{\lambda,x,y}(h_{xy}^{\lambda})^2, \tag{3.21}$$

$$\sum_{a,i,j,k} h_{ij}^{a}h_{kkij}^{a}=\sum_{a,x,j,k} h_{xj}^{a}h_{kkxj}^{a}-\sum_{\lambda}(\sum_{k} h_{kk}^{\lambda})^2. \tag{3.22}$$

Thus, we have from (3.20)

LEMMA 3.2. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian manifold $\bar{M}^{2m+1}$. If the second fundamental form of M is parallel, then $h_{ij}^{\lambda}=0$ for all λ.*

Using (3.20), we obtain

$$\sum_{i} h_{ii0}^{a}=-\sum_{i} h_{ii}^{a^*}, \qquad \sum_{i} h_{ii0}^{a^*}=\sum_{i} h_{ii}^{a}, \tag{3.23}$$

which imply

LEMMA 3.3. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian manifold $\bar{M}^{2m+1}$. If the mean curvature vector of M is parallel, then $\sum_{i} h_{ii}^{\lambda}=0$ for all λ.*

When $m>n$, Lemmas 3.2 and 3.3 show that the conditions that the second fundamental form is parallel and that the mean curvature vector is parallel are not meaningful for anti-invariant submanifolds. Therefore we shall now introduce some new concepts as follows. On an anti-invariant submanifold M of a Sasakian manifold $\bar{M}^{2m+1}$, if $h_{xyz}^{a}=0$ for all indices a, x, y and z, then we say that the second fundamental form of M is *pseudo-parallel*. If $\sum_{i} h_{iix}^{a}=0$ for all indices a and x, then the mean curvature vector said to be *pseudo-parallel*.

If the ambient manifold $\bar{M}^{2m+1}$ is of constant ϕ-sectional curvature c, then the Riemannian curvature tensor of $\bar{M}^{2m+1}$ has the form

$$K_{BCD}^{A}=\frac{1}{4}(c+3)(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC})+\frac{1}{4}(c-1)(\eta_{B}\eta_{C}\delta_{AD} \tag{3.24}$$

$$-\eta_B\eta_D\delta_{AC}+\eta_A\eta_D\delta_{BC}-\eta_A\eta_C\delta_{BD}+\phi_{AC}\phi_{BD}$$
$$-\phi_{AD}\phi_{BC}+2\phi_{AB}\phi_{CD}),$$

and the second fundamental form of M satisfies the equation (3.15) of Codazzi type.

§4. The connection D in the normal bundle.

In this section we study the connection D induced in the normal bundle to an $(n+1)$-dimensional anti-invariant submanifold M of a $(2m+1)$-dimensional Sasakian space form $\bar{M}^{2m+1}$ (c) when the structure vector field ξ is tangent to M.

First of all, by (3.24) we obtain

$$K^{\lambda}_{x^*kl}=0, \qquad K^{\lambda}_{jkl}=0, \qquad K^{\lambda}_{\mu kl}=0, \tag{4.1}$$
$$K^{x^*}_{y^*kl}=\frac{1}{4}(c-1)(\delta_{xk}\delta_{yl}-\delta_{xl}\delta_{yk}).$$

From (3.10), (3.11) and (4.1), we have

$$R^{\lambda}_{x^*y0}=h^{\lambda}_{xy}, \qquad R^{x^*}_{y^*z0}=0. \tag{4.2}$$

The connection D induced in the normal bundle to M is said to be *flat* if $R^{a}_{bij}=0$ for all indices. Thus we have from (3.11) and (4.2)

LEMMA 4.1. *Let M be an $(n+1)$-demensional anti-invariant submanifold of a Sassakian space form $\bar{M}^{2m+1}$ (c). If the connection D is flat then $h^{\lambda}_{ij}=0$.*

When $m>n$, Lemma 4.1 shows that the condition that the connection D is flat is not meaningful for anti-invariant submanifolds of a Sasakian space form. Therefore the pseudo-flatness of D will be introduced as follows. On an anti-invariant submanifold M of a Sasakian manifold $\bar{M}^{2m+1}$ the connection D is said to be *pseudo-flat* if $R^{a}_{bxy}=0$ for all indices.

LEMMA 4.2. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian manifold $\bar{M}^{2m+1}$. If the connection D is pseudo-fiat then*

$$R^{x}_{yzw}=\sum_{\lambda}(h^{\lambda}_{xz}h^{\lambda}_{yw}-h^{\lambda}_{xw}h^{\lambda}_{yz}). \tag{4.3}$$

Proof. (3.2) and (3.3) imply

$$K^{x^*}_{y^*zw}=K^{x}_{yzw}-(\delta_{xz}\delta_{yw}-\delta_{xw}\delta_{yz}). \tag{4.4}$$

Moreover, from (3.8), (3.10) and (3.11), we obtain

$$R^{x}_{yzw}=K^{x}_{yzw}+\sum_{a}(h^{a}_{xz}h^{a}_{yw}-h^{a}_{xw}h^{a}_{yz}) \tag{4.5}$$
$$=K^{x}_{yzw}-(\delta_{xz}\delta_{yw}-\delta_{xw}\delta_{yz})+\sum_{t}(h^{x}_{tz}h^{y}_{tw}-h^{x}_{tw}h^{y}_{tz})$$

$$+\sum_{\lambda}(h^{\lambda}_{xz}h^{\lambda}_{yw}-h^{\lambda}_{xw}h^{\lambda}_{yz})$$
$$=R^{x^*}_{y^*zw}+\sum_{\lambda}(h^{\lambda}_{xz}h^{\lambda}_{yw}-h^{\lambda}_{xw}h^{\lambda}_{yz}).$$

This completes the proof.

From (3.3) and (3.8), we have

$$R^{0}_{ijk}=0. \tag{4.6}$$

Thus Lemmas 4.1 and 4.2 imply immediately

COROLLARY 4.3. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$. If the connection D is flat, then M is flat.*

If the connection D is pseudo-flat, then (3.10) and (4.1) imply

$$\sum_{i}(h^{\lambda}_{ix}h^{v}_{iy}-h^{\lambda}_{iy}h^{v}_{ix})=0, \quad \sum_{i}(h^{\lambda}_{ix}h^{\mu}_{iy}-h^{\lambda}_{iy}h^{\mu}_{ix})=0, \tag{4.7}$$

$$\sum_{i}(h^{v}_{ix}h^{w}_{iy}-h^{v}_{iy}h^{w}_{ix})=-\frac{1}{4}(c-1)(\delta_{vx}\delta_{wy}-\delta_{vy}\delta_{wx}). \tag{4.8}$$

Moreover, using (3.11), we have

$$\sum_{z}(h^{\lambda}_{zx}h^{v}_{zy}-h^{\lambda}_{zy}h^{v}_{zx})=0, \quad \sum_{z}(h^{\lambda}_{zx}h^{\mu}_{zy}-h^{\lambda}_{zy}h^{\mu}_{zx})=0, \tag{4.9}$$

$$\sum_{z}(h^{v}_{zx}h^{w}_{zy}-h^{v}_{zy}h^{w}_{zx})=-\frac{1}{4}(c+3)(\delta_{vx}\delta_{wy}-\delta_{vy}\delta_{wx}). \tag{4.10}$$

PROPOSITION 4.4. *Let M be an $(n+1)$-dimensional $(n>1)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ and the connection D induced in the normal bundle to M be pseudo-flat. Then, if M is contact umbilical with respect to e_{v^*} for some index v, then $c=-3$.*

Proof. If M is contact umbilical with respect to e_{v^*}, then h^{v}_{xy} is of the form $h^{v}_{xy}=\frac{1}{n}(\sum_{z}h^{v}_{zz})\delta_{xy}$. Moreover, using (3.11), we have

$$\sum_{i}(h^{v}_{ix}h^{w}_{iy}-h^{v}_{iy}h^{w}_{ix})=\delta_{vx}\delta_{wy}-\delta_{vy}\delta_{wx}.$$

From this and (4.8) we find $c=-3$.

For each index a, the second fundamental form will be represented by a symmetric $(n+1, n+1)$-matrix $A_a=(h^{a}_{ij})$ composed of its components. Following such notations, we have from (3.11)

$$A_{x^*}=\begin{pmatrix} 0 & 0\cdots 0\ \overset{x}{1}\ 0\cdots 0 \\ 0 & \\ \vdots & \\ 0 & H_{x^*} \\ 1\ (x) & \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}, \quad \text{for all } x,$$

$$A_\lambda=\left(\begin{array}{c|c} 0 & 0 \cdots\cdots 0 \\ \hline 0 & \\ \vdots & H_\lambda \\ 0 & \end{array}\right), \qquad \text{for all } \lambda,$$

where $H_a=(h^a_{xy})$ are symmetric (n, n)-matrices.

LEMMA 4.5. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c\neq -3)$. If the connection D is pseudo-flat, then M is contact umbilical with respect to all e_λ.*

Proof. From (4.9) we obtain $H_\lambda H_\mu=H_\mu H_\lambda$ and $H_\lambda H_1=H_1H_\lambda$ for all λ and μ. Therefore we can choose a local field of orthonormal frames with respect to which H_1 and all H_λ are simultaneously diagonal i.e.,

$$A_1=\left(\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ \hline 1 & h^1_{11} & & & \\ 0 & & \cdot & & \\ \vdots & & & \cdot & \\ 0 & & & & h^1_{nn} \end{array}\right), \quad A_\lambda=\left(\begin{array}{c|cccc} 0 & 0 & & \cdots & 0 \\ \hline 0 & h^\lambda_{11} & & & \\ \vdots & & \cdot & & \\ & & & \cdot & \\ 0 & & & & h^\lambda_{nn} \end{array}\right). \tag{4.11}$$

Putting $x=1$ and $v=y$ in the first equation of (4.9) and using (3.11) and (4.11), we find

$$(h^\lambda_{11}-h^\lambda_{yy})h^1_{yy}=0\,. \tag{4.12}$$

On the other hand, putting $v=x=1$ and $w=y\neq 1$ in (4.10) and using (3.11) and (4.11), we have

$$(h^1_{11}-h^1_{yy})h^1_{yy}=-\frac{1}{4}(c+3)\,. \tag{4.13}$$

Since $c\neq -3$, (4.13) implies $h^1_{yy}\neq 0$ $(y=2, \cdots, n)$. From this fact and (4.12) we find that $h^\lambda_{11}=h^\lambda_{yy}$ for all λ. Thus M is contact umbilical with respect to all e_λ. This proves Lemma 4.5.

LEMMA 4.6. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c\neq -3)$. If the connection D is pseudo-flat, then*

$$R^x_{yzw}=\frac{1}{n^2}\sum_\lambda(\mathrm{Tr}\,A_\lambda)^2(\delta_{xz}\delta_{yw}-\delta_{xw}\delta_{yz})\,. \tag{4.14}$$

Proof. Lemma 4.5 implies $h^\lambda_{xy}=\frac{1}{n}(\mathrm{Tr}\,A_\lambda)\delta_{xy}$ for all λ. Therefore (4.3) implies (4.14).

PROPOSITION 4.7. *Let M be an $(n+1)$-dimensional $(n\geqq 3)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c\neq -3)$. If the connection D induced* *in the normal bundle to M is pseudo-flat, then M is locally a Riemannian direct product $M^n \times M^1$, where M^n is of constant curvature, and M^1 is a 1-dimensional subspace generated by ξ.*

Proof. Since $n \geqq 3$, (4.14) implies that $\sum_\lambda (\mathrm{Tr}\, A_\lambda)^2$ is constant. Therefore Proposition 4.7 is proved by means of Lemma 3.1.

If M is minimal, then $\mathrm{Tr}\, A_\lambda = 0$ for all λ. Thus Lemma 4.5 and (4.2) imply immediately

PROPOSITION 4.8. *Let M be an $(n+1)$-dimensional anti-invariant minimal submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c \neq -3)$. Then the connection D induced in the normal bundle to M is flat if and only if the connection D is pseudo-flat.*

Moreover, Corollary 4.3 and Proposition 4.8 imply immediately

COROLLARY 4.9. *Let M be an $(n+1)$-dimensional anti-invariant minimal submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c \neq -3)$. If the connection D induced in the normal bundle to M is pseudo-flat, then M is flat.*

§ 5. Pseudo-parallel mean curvature vector.

Using the results obtain in the previous section, we have

THEOREM 1. *Let M be an $(n+1)$-dimensional $(n \geqq 3)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c \neq -3)$ with pseudo-parallel mean curvature vector. If the connection D induced in the normal bundle to M is pseudo-flat, then there is in $\bar{M}^{2m+1}(c)$ a totally geodesic and invariant submanifold $\bar{M}^{2n+1}(c)$ of dimension $2n+1$ in such a way that M is immersed in $\bar{M}^{2n+1}(c)$ as a flat anti-invariant submanifold.*

Proof. (3.23) implies that $\sum_a (\mathrm{Tr}\, A_a)^2$ is constant because the mean curvature vector is pseudo-parallel. Since $n \geqq 3$, (4.14) implies that $\sum_\lambda (\mathrm{Tr}\, A_\lambda)^2$ is constant. On the other hand, from (3.8), (3.24) and (4.14), we have

$$\frac{n-1}{n} \sum_\lambda (\mathrm{Tr}\, A_\lambda)^2 = \frac{1}{4} n(n-1)(c+3) + \sum_a (\mathrm{Tr}\, A_a)^2 - \sum_{a,x,y} (h_{xy}^a)^2 . \tag{5.1}$$

Moreover, using (3.11), we find that the square of the length of the second fundamental form of M is constant, i.e., $\sum_{a,i,j} (h_{ij}^a)^2$ is constaant. From this we have

$$\sum_{a,i,j,k} (h_{ijk}^a)^2 + \sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a = \frac{1}{2} \Delta \sum_{a,i,j} (h_{ij}^a)^2 = 0 . \tag{5.2}$$

On the other hand the mean curvature vector is pseudo-parallel and the connection D is pseudo-flat. Thus (3.11), (3.19), (3.22) and (4.6) imply

$$(5.3)\qquad \sum_{a,i,j} h^a_{ij}\Delta h^a_{ij} = -\sum_{\lambda}(\mathrm{Tr}\,A_\lambda)^2 + \sum_{a,x,y,z,w} h^a_{yz}h^a_{wx}R^x_{yzw} + \sum_{a,i,x,y,z} h^a_{iz}h^a_{xi}R^x_{yzy} - 2\sum_{a,x,y} h^a_{xy}R^a_{x^*y0}\,.$$

Moreover, substituting (4.14) into (5.3), we obtain by using (3.11) and (5.2),

$$(5.4)\qquad \sum_{a,i,j,k}(h^a_{ijk})^2 = -\frac{1}{n^2}\sum_{\lambda}(\mathrm{Tr}\,A_\lambda)^2\Big(\sum_{a,x,y}(n(h^a_{xy})^2 - h^a_{xx}h^a_{yy}) - n\Big) + 2\sum_{\lambda,x,y}(h^\lambda_{xy})^2\,,$$

from which

$$(5.5)\qquad \sum_{a,x,y,z}(h^a_{xyz})^2 = -\frac{1}{n^2}\sum_{\lambda}(\mathrm{Tr}\,A_\lambda)^2\Big(\sum_{x,y,z}(n(h^z_{xy})^2 - h^z_{xx}h^z_{yy})\Big)$$
$$= -\frac{1}{n^2}\sum_{\lambda}(\mathrm{Tr}\,A_\lambda)^2\sum_{z}\Big(\sum_{x<y}(h^z_{xx}-h^z_{yy})^2 + \sum_{x\neq y}(h^z_{xy})^2\Big)\,,$$

by means of Lemma 4.5 and (3.21).

Since $c\neq -3$ by assumption, Proposition 4.4 implies $\sum_{x<y}(h^z_{xx}-h^z_{yy})^2>0$. Thus (5.5) implies $\mathrm{Tr}\,A_\lambda=0$ for all λ which mean that $A_\lambda=0$ for all λ. And from (5.5) we see that $h^a_{xyz}=0$, i.e., that the second fundamental form of M is pseudo-parallel. Since $A_\lambda=0$ for all λ, (3.20) implies $h^a_{ij0}=0$. Therefore the second fundamental form of M is parallel. Moreover, since $A_\lambda=0$ for all λ, Lemma 4.6 and (4.6) imply that M is flat. From the arguments above, taking account of a fundamental theorem in the theory of submanifolds, we see that M is an anti-invariant submanifold immersed in some totally geodesic and $(2n+1)$-dimensional submanifold $\bar{M}^{2n+1}(c)$ of $\bar{M}^{2m+1}(c)$ (see § 6). And the submanifold $\bar{M}^{2n+1}(c)$ is invariant (see § 6). Thus Theorem 1 is proved.

In Theorem 1, the case where $n=2$, that is, where M is 3-dimensional, is excluded. However, the same conclusions will be established even if $n=2$, provided that M is compact. To establish this fact, we now prove

THEOREM 2. *Let M be a compact $(n+1)$-dimensional $(n>2)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c\neq -3)$ with pseudo-parallel mean curvature vector. If the connection D induced in the normal bundle to M is pseudo-flat, then there is in $\bar{M}^{2m+1}(c)$ a totally geodesic and invariant submanifold $\bar{M}^{2n+1}(c)$ of dimension $2n+1$ in such a way that M is immersed in $\bar{M}^{2n+1}(c)$ as a flat anti-invariant submanifold.*

Proof. Since M is compact, we have

$$\int_M \sum_{a,i,j,k}(h^a_{ijk})^2 *1 = -\int_M \sum_{a,i,j} h^a_{ij}\Delta h^a_{ij} *1\,,$$

where $*1$ denotes the volume element of M. Using this formula, we can prove Theorem 2 by a same way as taken to prove Theorem 1.

In the proof of Theorem 1, the following Corollary 5.1 has already been proved.

COROLLARY 5.1. *Let M be an $(n+1)$-dimensional $(n\geqq 3)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c\neq -3)$ with pseudo-flat normal connection. Then following notions (a), (b), (c), (d) are equivalent to each other: (a) the mean curvature vector is pseudo-parallel, (b) the mean curvature vector is parallel, (c) the second fundamental form is pseudo-parallel, (d) the second fundamental form is parallel.*

Theorem 1 and (4.2) imply immediatelly

COROLLARY 5.2. *Let M be an $(n+1)$-dimensional $(n\geqq 3)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ $(c\neq -3)$ with pseudo-parallel mean curvature vector. Then the connection D induced in the normal bundle to M is flat if and only if the connection D is pseudo-flat.*

Remark. By Theorem 2, if M is compact, Corollaries 5.1 and 5.2 hold even for $n=2$.

We shall now study submanifolds of a Sasakian space form $\bar{M}^{2m+1}(c)$ in the case where $c=-3$.

PROPOSITION 5.3. *Let M be an $(n+1)$-dimensional $(n\geqq 3)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(-3)$ with pseudo-parallel mean curvature vector and with pseudo-flat normal connection. If M is contact umbilical with respect to all e_λ, then M is either a flat anti-invariant submanifold immersed in some totally geodesic and $(2n+1)$-dimensional submanifold $\bar{M}^{2n+1}(-3)$ of $\bar{M}^{2m+1}(-3)$, or a totally contact umbilical submanifold.*

Proof. From the assumption and Lemma 4.2 we have the equation (4.14). Therefore the equation (5.5) holds. Thus we have either $\mathrm{Tr}\, A_\lambda=0$ for all λ, or $\sum_z(\sum_{x<y}(h^z_{xx}-h^z_{yy})^2+n\sum_{x\neq y}(h^z_{xy})^2)=0$. In this step, we can prove the following fact by the same way as taken to prove Theorem 1: M is flat and immersed in some totally geodesic and $(2n+1)$-dimensional submanifold $\bar{M}^{2n+1}(-3)$ of $\bar{M}^{2m+1}(-3)$ as an anti-invariant submanifold, when $\mathrm{Tr}\, A_\lambda=0$ for all λ. When $\sum_z(\sum_{x<y}(h^z_{xx}-h^z_{yy})^2+n\sum_{x\neq y}(h^z_{xy})^2)=0$, $h^z_{xx}=h^z_{yy}$, $h^z_{xy}=0$ $(x\neq y)$ hold and hence M is contact umbilical with respect to each e_{z^*}. Therefore, in this cace, M is a totally contact umbilical. Thus Proposition 5.3 is proved completely.

Remark. If in Proposition 5.3 M is totally contact umbilical, then (3.11) implies $H_z=0$ for all z.

THEOREM 3. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$. If the second fundamental form is parallel, then M is an anti-invariant submanifold immersed in some totally geodesic and $(2n+1)$-dimensional submanifold $\bar{M}^{2n+1}(c)$ of $\bar{M}^{2m+1}(c)$.*

Proof. From Lemma 3.2, (3.3) and (3.5) we have $\omega_{x^*}^{\lambda}=-\omega_x^{\lambda}=-h_{xi}^{\lambda}\omega^i=0$, which and (3.12) imply $h_{ijk}^{\lambda}=0$. From these and a fundamental theorem in the theory of submanifolds, we have our assertion.

Using Lemma 4.1, we have the following Theorem 4 from Corollary 4.3 by the same way as taken in the proof of Theorem 3.

THEOREM 4. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$. If the connection D induced in the normal bundle to M is flat, then M is a flat anti-invariant submanifold immersed in some totally geodesic and $(2n+1)$-dimensional submanifold $\bar{M}^{2n+1}(c)$ of $\bar{M}^{2m+1}(c)$.*

According to a theorem due to Yano and Kon (see [7], p. 100), Theorem 2 and Corollaries 5.1, 5.2 imply

THEOREM 5. *Let M be an $(n+1)$-dimensional $(n>2)$ compact orientable anti-invariant submanifold with pseudo-parallel mean curvature vector of S^{2m+1}. If the connection D induced in the normal bundle to M is pseudo-flat, then*

$$M=S^1(r_1)\times\cdots\times S^1(r_{n+1})$$

immersed in an S^{2n+1} which is totally geodesic in S^{2m+1}, where $r_1^2+\cdots+r_{n+1}^2=1$.

§ 6. Axiom of ϕ-holomorphic planes.

Let M be a $(2n+1)$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g) covered by a system of coordinate neighborhoods $\{U, x^i\}$. (In this section, indices $i, j, k, l, p, q, r, s, t$ run over the range $\{1, \cdots, 2n+1\}$ and the summation convention is used with respect to this system of indices.)

We assume that a Sasakian manifold M admits the axiom of ϕ-holomorphic $(2r+1)$-planes; that is, for each point p of M and any $(2r+1)$-dimensional ϕ-holomorphic subspace S of $T_p(M)$, $1<r<n$, there exists a $(2r+1)$-dimensional totally geodesic submanifold N passing through p and satisfying $T_p(N)=S$, where we mean a ϕ-holomorphic subspace S by a subspace of $T_p(M)$ satisfying $\phi S\subset S$. Since rank ϕ is $2n$, the subspace S contains the structure vector field ξ of the ambient manifold. Thus, by assumption, there is a $(2r+1)$-dimensional totally geodesic submanifold N passing through this point p and being tangent to given subspace S. If we represent such a submanifold N by parametric equations

$$x^i=x^i(y^a), \tag{6.1}$$

where $\{y^a\}$ are local coordinates in N, (In this section, the indices a, b, c, d run over the range $\{1, \cdots, 2r+1\}$ and the summation convention is used with respect to this system of indices.) then the fact that the submanifold N is totally geodesic is represented by the equations

(6.2)
$$\frac{\partial^2 x^i}{\partial y^a \partial y^b} + \left\{ {i \atop j\ k} \right\} \frac{\partial x^j}{\partial y^a} \frac{\partial x^k}{\partial y^b} - \left\{ {c \atop a\ b} \right\} \frac{\partial x^i}{\partial y^c} = 0,$$

where $\left\{ {i \atop j\ k} \right\}$ and $\left\{ {c \atop a\ b} \right\}$ are the Christoffel symbols formed with the Riemannian metric tensor g_{ij} of the ambient manifold M and the naturally induced Riemannian metric tensor $g_{ab} = g_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}$ of the submanifold N respectively.

The integrability condition of the differential equation (6.2) is

(6.3)
$$B_c{}^k B_d{}^l B_b{}^j K^i_{jkl} = B_a{}^i R^a_{bcd},$$

where K^i_{jkl} and R^a_{bcd} are the curvature tensors of the ambient manifold M and the submanifold N respectively, and $B_a{}^i = \partial x^i / \partial y^a$.

We first assume that M satisfies the axiom of ϕ-holomorphic 3-planes. Take arbitrarily a unit vector field v tangent to N. If we put $B_1 = \phi v$, $B_2 = \phi^2 v$, then we have by (6.3)

(6.4)
$$(\phi_s{}^t v^s)(\phi_q{}^r \phi_l{}^q v^l)(\phi_p{}^k \phi_j{}^p v^j) K^i_{ktr} = \alpha \phi_j{}^i v^j + \beta \phi_q{}^i \phi_j{}^q v^j + \gamma \xi^i,$$

from which

(6.5)
$$(\phi_s{}^t K^i_{jtl} - \phi_s{}^i \eta_l \eta_j) v^s v^l v^j = (\alpha \phi_j{}^i - \beta \delta_j{}^i + \beta \xi^i \eta_j) v^j + \gamma \xi^i,$$

or equivalently

(6.5)′
$$(\phi_s{}^t K_{ijtl} - \phi_{si} \eta_l \eta_j) v^s v^l v^j = (\alpha \phi_{ji} - \beta g_{ji} + \beta \eta_i \eta_j) v^j + \gamma \eta_i,$$

where $K_{ijtl} = K^k_{jtl} g_{ik}$ and $\phi_{si} = \phi_s{}^k g_{ik}$. Transvecting (6.5)′ with ξ^i, we get $\gamma = 0$. On the other hand, transvecting (6.5)′ with v^i, we obtain $\beta = 0$, because K_{ijtl} and ϕ_{ij} are skew-symmetric with respect to i and j.

If we put

$$c = \frac{\alpha}{\|\phi v\|^2} = \frac{\alpha}{(g_{sl} - \eta_s \eta_l) v^s v^l},$$

then we have from (6.5)

$$(\phi_s{}^t K^i_{ltl} - \phi_s{}^i \eta_l \eta_j) v^s v^l v^j = c(g_{sl} - \eta_s \eta_l) v^s v^l v^j,$$

from which

$$\begin{aligned}
&\phi_s{}^t K^i_{jtl} + \phi_s{}^t K^i_{ltj} + \phi_l{}^t K^i_{stj} + \phi_l{}^t K^i_{jts} + \phi_j{}^t K^i_{lts} + \phi_j{}^t K^i_{stl} \\
&\quad - 2(\phi_s{}^i \eta_l \eta_j + \phi_l{}^i \eta_j \eta_s + \phi_j{}^i \eta_s \eta_l) \\
&\quad = 2c(g_{sl} \phi_j{}^i + g_{lj} \phi_s{}^i + g_{js} \phi_l{}^i - \eta_s \eta_l \phi_j{}^i - \eta_l \eta_j \phi_s{}^i - \eta_j \eta_s \phi_l{}^i).
\end{aligned}$$

Transvecting this with $\phi_k{}^s$, we obtain

(6.6)
$$-2K^i_{jkl} - 2K^i_{lkj} + \phi_l{}^t \phi_k{}^s K^i_{stj} + \phi_j{}^t \phi_k{}^s K^i_{stl} - \delta_l{}^i g_{kj} - \delta_j{}^i g_{kl}$$

$$+2\delta_k{}^i g_{lj}+2g_{jl}\xi^i\eta_k-\delta_l{}^i\eta_k\eta_j-\delta_j{}^i\eta_k\eta_l+2\delta_k{}^i\eta_l\eta_j$$
$$-2\xi^i\eta_k\eta_l\eta_j+\phi_l{}^i\phi_{kj}+\phi_j{}^i\phi_{kl}$$
$$=2c(\phi_{kl}\phi_j{}^i-g_{lj}\delta_k{}^i+g_{lj}\xi^i\eta_k+\phi_{kj}\phi_l{}^i+\delta_k{}^i\eta_l\eta_j-\xi^i\eta_k\eta_l\eta_j).$$

On the other hand, we get easily by using the Ricci identity

$$\phi_l{}^t\phi_k{}^s K^i_{stj}-\phi_k{}^t\phi_l{}^s K^i_{stj}$$
$$=-K^i_{jkl}+\phi_l{}^i\phi_{kj}-\phi_k{}^i\phi_{lj}-\delta_l{}^i g_{kj}+\delta_k{}^i g_{lj},$$
$$\phi_j{}^t\phi_k{}^s K^i_{stl}-\phi_j{}^t\phi_l{}^s K^i_{stk}$$
$$=-K^i_{jkl}+2(\delta_k{}^i g_{lj}-\delta_l{}^i g_{kj})+\delta_l{}^i\eta_j\eta_k-\delta_k{}^i\eta_j\eta_l+2\phi_{kl}\phi_j{}^i.$$

Using the equations above and taking the skew-symmetric part with respect to the indices k and l in (6.6), we get

$$4K^i_{jkl}=(c+3)(\delta_k{}^i g_{lj}-\delta_l{}^i g_{kj})+(c-1)(\delta_l{}^i\eta_k\eta_j-\delta_k{}^i\eta_l\eta_j$$
$$+g_{kj}\xi^i\eta_l-g_{lj}\xi^i\eta_k-\phi_{kj}\phi_l{}^i+\phi_{lj}\phi_k{}^i-2\phi_{kl}\phi_j{}^i). \qquad (6.7)$$

Conversely, if the curvature tensor has the above form (6.7), it is easily seen that (6.3) is satisfied. Hence we have proved

THEOREM 6. *A Sasakian manifold is of constant ϕ-sectional curvature if and only if the manifold satisfies the axiom of ϕ-holomorphic 3-planes.*

If a Sasakian manifold satisfies the axiom of ϕ-holomorphic $(2r+1)$-planes, then it satisfies the axiom of ϕ-holomorphic 3-planes and hence it is of constant ϕ-sectional curvature. Therefore we obtain

THEOREM 7. *A Sasakian manifold is of constant ϕ-sectional curvature if and only if the manifold satisfies the axiom of ϕ-holomorphic $(2r+1)$-planes.*

In the proof of Theorem 7, we have implicitly used the following Theorem 8.

THEOREM 8. *Let N be a $(2r+1)$-dimensional totally geodesic submanifold of a Sasakian manifold M^{2n+1}. If N is ϕ-holomorphic at one point p, then N is invariant.*

Proof. The van der Waerden-Bortolotti covariant derivative of $\eta_i C_x{}^i$ (see [5]), where $C_x{}^i$ is a unit normal vector field. From (2.5) we have

$$\nabla_a(\eta_i C_x{}^i)=\phi_{ij}B_a{}^i C_x{}^j, \qquad \nabla_b\nabla_a(\eta_i C_x{}^i)=-g_{ba}\eta_i C_x{}^i.$$

Because of the initial condition $(\eta_i C_x{}^i)_p=(\phi_{ij}B_a{}^i C_x{}^j)_p=0$ the relation $\eta_i C_x{}^i=\phi_{ij}B_a{}^i C_x{}^j=0$ holds indentically on N. This equation shows that N is invariant.

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