H. KITAHARA AND S. YOROZU
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ON SOME DIFFERENTIAL GEOMETRIC CHARACTERIZATIONS OF A BUNDLE-LIKE METRIC

BY HARUO KITAHARA AND SHINSUKE YOROZU

§ 1. Introduction.

We have known some interesting theorems about the behaviour of geodesics of a bundle-like metric:

THEOREM A (Y. Muto [6]). *A geodesic of a fibred riemannian manifold tangent to an "allowed curve" at one point is always an "allowed curve" if and only if "the fibres are parallel".*

THEOREM B (B. L. Reinhart [10]). *A geodesic of a bundle-like metric is orthogonal at one point if and only if it is orthogonal at every point.*

The topological obstructions for the existence of the foliation with bundle-like metric were studied by R. Sacksteder [11], J. S. Pasternack [8, 9] and others. But of the conditions for the given riemannian metric of a foliated manifold to be a boundle-like metric very little is definitely known (cf. [5]). Of cource, not all foliations have bundle-like metrics [10]. The completeness of a bundle-like metric was studied by one of the authors [3, 4].

In this note, we will give some differential geometric conditions for a given riemannian metric $\langle\ ,\ \rangle$ on a foliated riemannian manifold of codimension one to be a bundle-like metric in terms of geodesics. Our main theorem is the following:

THEOREM C. *Suppose that a foliated riemannian manifold is of codimension one and that all leaves are totally geodesic with respect to the given riemannian metric* $\langle\ ,\ \rangle$. *Then the metric* $\langle\ ,\ \rangle$ *is a bundle-like metric with respect to the foliation if and only if all geodesics with "angle* α*" to a leaf at one point have "constant angle* α*" to each leaf at every point.*

Furthermore, we can give a proof of the following theorem:

THEOREM D. *The given riemannian metric* $\langle\ ,\ \rangle$ *on a foliated riemannian manifold of codimension one is a bundle-like metric with respect to the foliation if and only if all geodesics orthogonal to a leaf at one point are orthogonal to each leaf at every point.*

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We remark that this theorem was announced by A. M. Naveira [7].

And we will give some typical examples in connection with the above theorems.

§ 2. Notations.

We shall be in C^∞-category. Let M be a connected $(p+q)$-dimensional riemannian manifold with riemannian metric $\langle\ ,\ \rangle$ and riemannian connection ∇ with respect to $\langle\ ,\ \rangle$. A foliated riemannian manifold M of codimension q has, by definition, an integrable subbundle E of fibre dimension p of the tangent bundle TM over M. Then M is covered by the flat coordinate charts $\{(U, x^1, \cdots, x^p, x^{p+1}, \cdots, x^{p+q})\}$ (cf. [10]). Let Q be the quotient bundle TM/E. The restriction of a vector bundle to U and the set of all cross-sections of a vector bundle are denoted by $\cdot|_U$ and $\Gamma(\cdot)$ respectively. The natural projection $\pi: TM \rightarrow Q$ induces a map $\pi: \Gamma(TM) \rightarrow \Gamma(Q)$. $[\ ,\ ]$ denotes the bracket operator. Adopted the following ranges of indices and the corresponding summation convention; $1 \leqq A, B, C, \cdots \leqq p+q$, $1 \leqq i, j, k, \cdots \leqq p$ and $p+1 \leqq \alpha, \beta, \gamma, \cdots \leqq p+q$.

§ 3. Bott partial connection and bundle-like metric.

DEFINITION 3.1. In each flat coordinate chart $(U; x^A)$, a frame $\{X_1, \cdots, X_p, X_{p+1}, \cdots, X_{p+q}\}$ is an *adapted frame to* E if $\{X_1, \cdots, X_p\}$ and $\{\pi(X_{p+1}), \cdots, \pi(X_{p+q})\}$ span $\Gamma(E|_U)$ and $\Gamma(Q|_U)$ respectively.

In each $(U; x^A)$, frames $\{\partial/\partial x^i, \partial/\partial x^\alpha\}$ and $\{\partial/\partial x^i, \partial/\partial x^\alpha - t_\alpha{}^k \partial/\partial x^k\}$ ($t_\alpha{}^k$ are functions on U) are adapted frames to E (cf. [8, 9], [10], [12, 13]).

DEFINITION 3.2. In each flat coordinate chart $(U; x^A)$, an adapted frame $\{E_i, E_\alpha\}$ to E is called a *basic adapted frame to* E if $E_i = \partial/\partial x^i$ and $E_\alpha = \partial/\partial x^\alpha - t_\alpha{}^k \partial/\partial x^k$.

For a basic adapted frame $\{E_i, E_\alpha\}$,

$$\pi([X, E_\alpha]) = 0 \qquad \text{for any} \quad X \in \Gamma(E|_U) \tag{3.1}$$

holds.

In each $(U; x^A)$, by the suitable choice of functions $t_\alpha{}^k$, the given riemannian metric $\langle\ ,\ \rangle$ has a local expression

$$\langle\ ,\ \rangle|_U = g_{ij}(x^k, x^\gamma)\theta^i \otimes \theta^j + g_{\alpha\beta}(x^k, x^\gamma)\theta^\alpha \otimes \theta^\beta, \tag{3.2}$$

where $\{\theta^i, \theta^\alpha\}$ is the dual frame of the adapted frame $\{E_i, E_\alpha\}$ (cf. [10], [12]). We notice that $\det(g_{ij}) \neq 0$ implies $\det(g_{\alpha\beta}) \neq 0$. Then the metric $\langle\ ,\ \rangle$ induces a metric $\langle\!\langle\ ,\ \rangle\!\rangle$ on Q, that is, for any $S_1, S_2 \in \Gamma(Q|_U)$,

$$\langle\!\langle S_1, S_2 \rangle\!\rangle|_U = g_{\alpha\beta}(x^k, x^\gamma)\theta^\alpha(\tilde{S}_1)\theta^\beta(\tilde{S}_2), \tag{3.3}$$

where $\pi(\tilde{S}_1) = S_1$ and $\pi(\tilde{S}_2) = S_2$ (cf. [8]). $\langle\!\langle\ ,\ \rangle\!\rangle$ is well-defined.

DEFINITION 3.3. A map $\hat{\nabla}: \Gamma(E)\times\Gamma(Q)\rightarrow\Gamma(Q)$ is defined by

$$\hat{\nabla}_X S=\pi([X, \tilde{S}]) \tag{3.4}$$

for any $X\in\Gamma(E)$ and $S\in\Gamma(Q)$, where $\tilde{S}\in\Gamma(TM)$ such that $\pi(\tilde{S})=S$. The map $\hat{\nabla}$ is well-defined by the integrability of E. $\hat{\nabla}$ is called the *Bott partial connection on* Q (cf. [1]).

DEFINITION 3.4. The Bott partial connection $\hat{\nabla}$ on Q is *metrical with respect to* $\ll , \gg$ if

$$\ll\hat{\nabla}_X S_1, S_2\gg+\ll S_1, \hat{\nabla}_X S_2\gg=X\ll S_1, S_2\gg \tag{3.5}$$

for any $X\in\Gamma(E|_U)$ and any $S_1, S_2\in\Gamma(Q|_U)$.

DEFINITION 3.5. The riemannian metric $\langle\ ,\ \rangle$ is a *bundlelike metric with respect to* E (or *with respect to the foliation*) if, in its local expression (3.2),

$$\partial g_{\alpha\beta}/\partial x^i=0\,, \tag{3.6}$$

that is, $g_{\alpha\beta}(x^k, x^\gamma)=g_{\alpha\beta}(x^\gamma)$.

Take the basic adapted frame $\{E_i, E_\alpha\}$ to E in each (U, x^A). We notice that (3.6) means

$$E_i\langle E_\alpha, E_\beta\rangle=0\,. \tag{3.7}$$

LEMMA 3.1. *If the riemannian metric $\langle\ ,\ \rangle$ is a bundlelike metric with respect to E, then the Bott partial connection $\hat{\nabla}$ is metrical with respect to the metric $\ll , \gg$.*

Proof. Take the basic adapted frame $\{E_i, E_\alpha\}$ to E. For any $S_1, S_2\in\Gamma(Q|_U)$, suppose that $\tilde{S}_i=f_1^iE_i+f_1^\alpha E_\alpha$ and $\tilde{S}_2=f_2^iE_i+f_2^\alpha E_\alpha$ satisfy $\pi(\tilde{S}_1)=S_1$ and $\pi(\tilde{S}_2)=S_2$ respectively. Then, from (3.3), (3.4) and (3.6), we have

$$\begin{aligned}\ll\hat{\nabla}_{E_j}S_1, S_2\gg&=\ll\pi([E_j, \tilde{S}_1]), S_2\gg\\&=g_{\alpha\beta}(x^\gamma)\theta^\alpha([E_j, \tilde{S}_1])\theta^\beta(\tilde{S}_2)\\&=g_{\alpha\beta}(x^\gamma)E_j(f_1^\alpha)f_2^\beta\,,\end{aligned}$$

and

$$\begin{aligned}&\ll\hat{\nabla}_{E_j}S_1, S_2\gg+\ll S_1, \hat{\nabla}_{E_j}S_2\gg\\&\quad=g_{\alpha\beta}(x^\gamma)E_j(f_1^\alpha)f_2^\beta+g_{\alpha\beta}(x^\gamma)f_1^\alpha E_j(f_2^\beta)\\&\quad=E_j(g_{\alpha\beta}(x^\gamma)f_1^\alpha f_2^\beta)\\&\quad=E_j(g_{\alpha\beta}(x^\gamma)\theta^\alpha(\tilde{S}_1)\theta^\beta(\tilde{S}_2))\\&\quad=E_j\ll S_1, S_2\gg.\end{aligned}$$

Therefore, we have

$$\ll \hat{\nabla}_X S_1, S_2 \gg + \ll S_1, \hat{\nabla}_X S_2 \gg = X \ll S_1, S_2 \gg$$

for any $X \in \Gamma(E|_U)$. q. e. d.

§4. Proofs of THEOREM C, D and examples.

The following lemma is our fundamental tool.

LEMMA 4.1. *The metric $\langle\ ,\ \rangle$ is a bundle-like metric if and only if, for each flat coordinate chart $(U; x^A)$, there exists an orthonormal adapted frame $\{X_i, X_\alpha\}$ to E such that $\hat{\nabla}_X \pi(X_\alpha)=0$ for any $X \in \Gamma(E|_U)$.*

Proof. Suppose that the metric $\langle\ ,\ \rangle$ is a bundle-like metric. We have an orthonormal frame $\{X_i, X_\alpha\}$ from the basic adapted frame $\{E_i, E_\alpha\}$ to E by the Schmidt's orthogonalization process to $\{E_i\}$ and $\{E_\alpha\}$ respectively. Clearly, $\{X_1, \cdots, X_p\}$ and $\{\pi(X_{p+1}), \cdots, \pi(X_{p+q})\}$ span $\Gamma(E|_U)$ and $\Gamma(Q|_U)$ respectively. Then $\{X_i, X_\alpha\}$ is an orthonormal adapted frame to E. By (3.1), $\langle E_\alpha, E_\beta\rangle = g_{\alpha\beta}(x^\gamma)$ implies $\pi([X_i, X_\alpha])=0$. Therefore, we have $\hat{\nabla}_X \pi(X_\alpha)=0$ for any $X \in \Gamma(E|_U)$.

Conversely, let $\{X_i, X_\alpha\}$ be an orthonormal adapted frame to E such that $\hat{\nabla}_X \pi(X_\alpha)=0$ for any $X \in \Gamma(E|_U)$. Since the basic adapted frame $\{E_i, E_\alpha\}$ to E satisfies $\langle E_i, E_\alpha\rangle=0$, we can write E_α of form $E_\alpha = h_\alpha^\beta X_\beta$ where h_α^β are functions on U. Then we have

$$\begin{aligned}\hat{\nabla}_{E_i}\pi(E_\alpha) &= \pi([E_i, E_\alpha]) \\ &= \pi([E_i, h_\alpha^\beta X_\beta]) \\ &= h_\alpha^\beta \pi([E_i, X_\beta]) + E_i(h_\alpha^\beta)\pi(X_\beta) \\ &= h_\alpha^\beta \hat{\nabla}_{E_i}\pi(X_\beta) + E_i(h_\alpha^\beta)\pi(X_\beta) \\ &= E_i(h_\alpha^\beta)\pi(X_\beta)\,.\end{aligned}$$

And, by (3.1), $\hat{\nabla}_{E_i}\pi(E_\alpha)=\pi([E_i, E_\alpha])=0$. Then we have $E_i(h_\alpha^\beta)\pi(X_\beta)=0$, and $E_i(h_\alpha^\beta)=0$ by the linearly independence of $\pi(X_\beta)$. Thus we have

$$\begin{aligned}E_i\langle E_\alpha, E_\beta\rangle &= E_i\langle h_\alpha^\gamma X_\gamma, h_\beta^\tau X_\tau\rangle \\ &= E_i(h_\alpha^\gamma h_\beta^\tau \delta_{\gamma\tau}) \\ &= 0\,.\end{aligned}$$

Therefore, by (3.7), the metric $\langle\ ,\ \rangle$ is a bundle-like metric. q. e. d.

THEOREM 4.1. *The metric $\langle\ ,\ \rangle$ is a bundle-like metric if and only if, for each flat coordinate chart $(U; x^A)$, there exists an orthonormal adapted frame $\{X_i, X_\alpha\}$ to E such that $\langle \nabla_{X_\alpha} X_i, X_\beta\rangle + \langle \nabla_{X_\beta} X_i, X_\alpha\rangle = 0$.*

Proof. Suppose that the metric $\langle\ ,\ \rangle$ is a bundle-like metric. By lemma 3.1, there exists an orthonormal adapted frame $\{X_i, X_\alpha\}$ to E such that $\hat{\nabla}_X \pi(X_\alpha)=0$ for any $X \in \Gamma(E|_U)$. Then we have

$$\langle \nabla_{X_\alpha} X_i, X_\beta \rangle = \langle \nabla_{X_i} X_\alpha, X_\beta \rangle - \langle [X_i, X_\alpha], X_\beta \rangle$$
$$= \langle \nabla_{X_i} X_\alpha, X_\beta \rangle ,$$

since $\hat{\nabla}_{X_i} \pi(X_\alpha)=0$ implies $[X_i, X_\alpha] \in \Gamma(E|_U)$. Therefore, we have

$$\langle \nabla_{X_\alpha} X_i, X_\beta \rangle + \langle \nabla_{X_\beta} X_i, X_\alpha \rangle$$
$$= \langle \nabla_{X_i} X_\alpha, X_\beta \rangle + \langle \nabla_{X_i} X_\beta, X_\alpha \rangle$$
$$= X_i \langle X_\alpha, X_\beta \rangle$$
$$= 0 .$$

Conversely, suppose that there exists an orthonormal adapted frame $\{X_i, X_\alpha\}$ to E such that $\langle \nabla_{X_\alpha} X_i, X_\beta \rangle + \langle \nabla_{X_\beta} X_i, X_\alpha \rangle = 0$. Then we can write the basic adapted frame $\{E_i, E_\alpha\}$ to E of form $E_i = h_i^j X_j$ and $E_\alpha = h_\alpha^\gamma X_\gamma$ (h_i^j and h_α^γ are functions on U). Thus we have

$$E_i \langle E_\alpha, E_\beta \rangle$$
$$= \langle \nabla_{E_i} E_\alpha, E_\beta \rangle + \langle E_\alpha, \nabla_{E_i} E_\beta \rangle$$
$$= h_i^j h_\beta^\tau (X_j(h_\alpha^\gamma) \delta_{\gamma\tau} + h_\alpha^\gamma \langle [X_j, X_\gamma], X_\tau \rangle)$$
$$+ h_i^j h_\alpha^\gamma h_\beta^\tau \langle \nabla_{X_\tau} X_j, X_\gamma \rangle$$
$$+ h_i^j h_\alpha^\gamma (X_j(h_\beta^\tau) \delta_{\gamma\tau} + h_\beta^\tau \langle X_\gamma, [X_j, X_\tau] \rangle)$$
$$+ h_i^j h_\alpha^\gamma h_\beta^\tau \langle X_\gamma, \nabla_{X_\tau} X_j \rangle .$$

Since we have

$$X_j(h_\alpha^\gamma) \delta_{\gamma\tau} + h_\alpha^\gamma \langle [X_j, X_\gamma], X_\tau \rangle$$
$$= X_j(h_\alpha^\gamma) \delta_{\gamma\tau} + \langle [X_j, h_\alpha^\gamma X_\gamma], X_\tau \rangle$$
$$- \langle X_j(h_\alpha^\gamma) X_\gamma, X_\tau \rangle$$
$$= \langle [X_j, E_\alpha], X_\tau \rangle$$
$$= 0 \quad \text{(from (3.1))} ,$$

we have

$$E_i \langle E_\alpha, E_\beta \rangle$$
$$= h_i^j h_\alpha^\gamma h_\beta^\tau (\langle \nabla_{X_\tau} X_j, X_\tau \rangle + \langle \nabla_{X_\tau} X_j, X_\gamma \rangle)$$
$$= 0 .$$

Therefore, the metric $\langle \ , \ \rangle$ is a bundle-like metric. q.e.d.

For each flat coordinate chart $(U;\ x^A)$, let $\{w^i, w^\alpha\}$ be the dual frame of an orthonormal adapted frame $\{X_i, X_\alpha\}$ to E, that is, $w^A(X_B)=\delta^A_B$. Let $\sigma(t)$ and $Z(t)$ be a curve in U and a vector field along $\sigma(t)$ respectively. Then we have

$$w^A(\nabla_{\dot\sigma(t)}Z(t))=\frac{d}{dt}(w^A(Z(t)))+w^B(Z(t))w^A_B(\dot\sigma(t)), \tag{4.1}$$

where $\dot\sigma(t)=d\sigma(t)/dt$ and $dw^A=w^B\wedge w^A_B$ (cf. [14] p. 19 (1.6.12)). If we put $w^A_B=\Gamma^A_{CB}w^C$, then we have $\nabla_{X_C}X_B=\Gamma^A_{CB}X_A$ and

$$\Gamma^A_{CB}+\Gamma^B_{CA}=0. \tag{4.2}$$

(4.1) implies

$$w^A(\nabla_{\dot\sigma(t)}Z(t))=\frac{d}{dt}(w^A(Z(t)))+\Gamma^A_{CB}w^B(Z(t))w^C(\dot\sigma(t)). \tag{4.3}$$

Proof of THEOREM D.

Let $\sigma(t)$ be any geodesic orthogonal to each leaf at every point parametrized by arc-length. By (4.3) we have

$$\begin{aligned} w^i(\nabla_{\dot\sigma(t)}\dot\sigma(t))=&\frac{d}{dt}(w^i(\dot\sigma(t)))\\ &+\Gamma^i_{jk}w^k(\dot\sigma(t))w^j(\dot\sigma(t))\\ &+\Gamma^i_{j\ p+1}w^{p+1}(\dot\sigma(t))w^j(\dot\sigma(t))\\ &+\Gamma^i_{p+1\ j}w^j(\dot\sigma(t))w^{p+1}(\dot\sigma(t))\\ &+\Gamma^i_{p+1\ p+1}w^{p+1}(\dot\sigma(t))w^{p+1}(\dot\sigma(t)). \end{aligned}$$

But $\nabla_{\dot\sigma(t)}\dot\sigma(t)=0$ and $w^i(\dot\sigma(t))=0$ imply

$$\Gamma^i_{p+1\ p+1}w^{p+1}(\dot\sigma(t))w^{p+1}(\dot\sigma(t))=0.$$

Then we have $\Gamma^i_{p+1\ p+1}=0$, and this equality with (4.2) implies

$$\Gamma^{p+1}_{p+1\ i}=0.$$

Then we have

$$\langle\nabla_{X_{p+1}}X_i,\ X_{p+1}\rangle=0.$$

Therefore, by theorem 4.1, the metric $\langle\ ,\ \rangle$ is a bundle-like metric.

The converse assertion is clear from theorem B. q. e. d.

DEFINITION 4.1. Suppose that M is a foliated riemannian manifold of codimension one (if necessarily, M is supposed to be transversally orientable). A geodesic $\sigma(t)$ parametrized by arc-length *has an angle* $\alpha(t_0)$ *to a leaf at* $t=t_0$ if, in a flat coordinate chart $(U; x^A)$ containing the point $\sigma(t_0)$, $\langle\dot\sigma(t), X_{p+1}\rangle|_{t=t_0}$ $=\cos\alpha(t_0)\neq 0$ holds for an orthonormal adapted frame $\{X_1, \cdots, X_p, X_{p+1}\}$ to E. *$\sigma(t)$ has the constant angle α to each leaf at every point* if $\alpha(t)=$constant$=\alpha$, that is, $\langle\dot\sigma(t), X_{p+1}\rangle=$constant$=\cos\alpha\neq 0$.

Proof of THEOREM C.

We have, from (4.3),

$$
\begin{aligned}
(4.4)\qquad & w^{p+1}(\nabla_{\dot\sigma(t)}\dot\sigma(t))\\
&=\frac{d}{dt}(w^{p+1}(\dot\sigma(t)))\\
&\quad+\Gamma^{p+1}_{ij}w^j(\dot\sigma(t))w^i(\dot\sigma(t))\\
&\quad+\Gamma^{p+1}_{p+1\ i}w^i(\dot\sigma(t))w^{p+1}(\dot\sigma(t))\\
&\quad+\Gamma^{p+1}_{i\ p+1}w^{p+1}(\dot\sigma(t))w^i(\dot\sigma(t))\\
&\quad+\Gamma^{p+1}_{p+1\ p+1}w^{p+1}(\dot\sigma(t))w^{p+1}(\dot\sigma(t)).
\end{aligned}
$$

(4.2) implies

$$(4.5)\qquad \Gamma^{p+1}_{i\ p+1}=\Gamma^{p+1}_{p+1\ p+1}=0.$$

Since all leaves are totally geodesic, we have

$$(4.6)\qquad \Gamma^{p+1}_{ij}=0.$$

Suppose that a geodesic $\sigma(t)$ satisfies $\langle\dot\sigma(t), X_{p+1}\rangle=$constant$\neq 0$. Then we have

$$(4.7)\qquad \frac{d}{dt}(w^{p+1}(\dot\sigma(t)))=0 \quad \text{and}$$

$$(4.8)\qquad \nabla_{\dot\sigma(t)}\dot\sigma(t)=0.$$

Then, from (4.4), we have

$$\Gamma^{p+1}_{p+1\ i}w^i(\dot\sigma(t))w^{p+1}(\dot\sigma(t))=0,$$

and

$$\Gamma^{p+1}_{p+1\ i}w^i(\dot\sigma(t))=0 \qquad (\text{by } (4.7)).$$

As, by assumpsion, $w^i(\dot\sigma(t))$ are arbitrary,

$$\Gamma^{p+1}_{p+1\ i}=0,$$

that is, $\langle\nabla_{X_{p+1}}X_i, X_{p+1}\rangle=0$. Therefore, the metric $\langle\ ,\ \rangle$ is a bundle-like metric.

Conversely, let a geodesic $\sigma(t)$ parametrized by arc-length have an angle α to a leaf at $t=t_0$. It is sufficient to prove that $\langle\dot\sigma(t), X_{p+1}\rangle=$constant$=\cos\alpha$ for an orthonormal adapted frame $\{X_1, \cdots, X_p, X_{p+1}\}$ to E on $(U; x^A)$ containing the point $\sigma(t_0)$. (4.4), (4.5), (4.6) and (4.8) imply

$$\frac{d}{dt}(w^{p+1}(\dot{\sigma}(t)))=0,$$

that is, $\langle\dot{\sigma}(t), X_{p+1}\rangle=\text{constant}=\cos\alpha$. Therefore, $\sigma(t)$ has the constant angle α to each leaf at every point. q. e. d.

EXAMPLE 4.1. A family of irrational spirals on a flat torus T^2 of dimension two defines a foliation with a bundle-like metric whose leaves are totally geodesic. Then any geodesic has the constant angle α to each leaf at every point.

EXAMPLE 4.2. Let R^2 be a two dimensional euclidean space and $M=R^2-\{(0, 0)\}$. Then M is a foliated manifold whose leaves are $L_r=\{(x, y)\in M \mid x^2+y^2=r^2, r>0\}$. The canonical metric of R^2 induces a bundle-like metric on M. Then each leaf L_r is not totally geodesic, and, for example, a geodesic $y=c$ (c is a non-zero constant) has distinct angle to each leaf at every point.

EXAMPLE 4.3. Let R^3 be a three dimensional euclidean space and

$$S^2=\{(x, y, z)\in R^3 \mid x^2+y^2+z^2=1\}$$

with canonical metric $\langle\ ,\ \rangle$. If we put $x=\cos\varphi\sin\theta$, $y=\sin\varphi\sin\theta$ and $z=\cos\theta$, then $S^2=\{(\varphi, \theta) \mid 0\leqq\varphi<2\pi, 0\leqq\theta<\pi\}$. For any $\varepsilon>0$, we set $S^2_\varepsilon=\{(\varphi, \theta)\in S^2 \mid \varepsilon<\theta<\pi-\varepsilon\}$. A family of the great circular parts given by $\varphi=\text{constant}$ defines a foliation on S^2_ε whose leaves are totally geodesic with respect to the restricted metric $\langle\ ,\ \rangle|_{S^2_\varepsilon}$ of $\langle\ ,\ \rangle$ on S^2_ε. But the metric $\langle\ ,\ \rangle|_{S^2_\varepsilon}$ is not a bundlelike metric with respect to the foliation. Let $\sigma(t)$ be a great circle (geodesic) on S^2_ε through the two points (φ_0, θ_0) and (φ_1, θ_0) where $\varphi_0+\pi\equiv\varphi_1 \pmod{2\pi}$ and $\theta_0\neq\pi/2$. Then $\sigma(t)$ is not orthogonal to the leaf defined by $\varphi=\varphi_0$, but orthogonal to the leaf defined by $\varphi=(\varphi_0+\varphi_1)/2$.

Remark. In the case of codimension $q\geqq 2$, suppose that the bundle Q (or the orthogonal complement $E^\perp$ of E) is trivial. Let $\{X_i, X_\alpha\}$ be an orthonormal adapted frame to E. And suppose that a geodesic $\sigma(t)$ parametrized by arc-length has the same *normal cosine* $l_\alpha(t)$ at $t=t_0$, that is,

$$l_{p+1}(t_0)=l_{p+2}(t_0)=\cdots=l_{p+q}(t_0)\neq 0,$$

where the normal cosine $l_\alpha(t)$ is defined by

$$\langle\dot{\sigma}(t), X_\alpha\rangle=l_\alpha(t).$$

If the metric $\langle\ ,\ \rangle$ is a bundle-like metric and all leaves are totally geodesic, then, for each t,

$$l_{p+1}(t)=l_{p+2}(t)=\cdots=l_{p+q}(t)=\text{constant}=l_{p+1}(t_0).$$

EXAMPLE 4.4. Let R^n be an n-dimensional euclidean space and $S^{n-1}(r)$ an $(n-1)$-dimensional standard sphere of radius r in R^n. $\boldsymbol{Q}$ and $\boldsymbol{Q}^*$ are the quaternion number field and the pure quaternion respectively. We can identify $(x^1, x^2, x^3, x^4)\in R^4$ with $x^1\cdot 1+x^2\cdot i+x^3\cdot j+x^4\cdot k\in\boldsymbol{Q}$ and $(y^1, y^2, y^3)\in R^3$ with $y^1\cdot i+y^2\cdot j+y^3\cdot k\in\boldsymbol{Q}^*$. Then we write $S^3(1)$ and $S^2(1/2)$ of forms $\{a\in\boldsymbol{Q} \mid \|a\|=1\}$

and $\{b \in \boldsymbol{Q}^* \mid \|b\| = 1/2\}$ respectively, where $\|\cdot\|$ is the canonical norm on $\boldsymbol{Q}$. We define a map $\phi: S^3(1) \to S^2(1/2)$ by $\phi(a) = (1/2)\bar{a} \cdot i \cdot a$ ($\bar{a}$ is the conjugate to $a \in \boldsymbol{Q}$). This is well-defined. Then $\phi: S^3(1) \to S^2(1/2)$ is the hopf fibering and a riemannian submersion with connected totally geodesic fibres (cf. [2]). Thus $S^3(1)$ is a foliated riemannian manifold of codimension two with leaves as fibres and the metric is a bunlde-like metric. The leaf through $a \in S^3(1)$ is given by the intersection of the plane in R^4 spanned by vectors a and $i \cdot a$ with $S^3(1)$. The tangent space at $a \in S^3(1)$ is spanned by $i \cdot a, j \cdot a, k \cdot a$. An orthonormal adapted frame $\{X_1, X_2, X_3\}$ is given by $X_1(a) = i \cdot a$, $X_2(a) = j \cdot a$ and $X_3(a) = k \cdot a$. Take a great circle (geodesic) $\sigma(t)$ parametrized by arc-length and suppose that, at $t = t_0$, $\langle \dot{\sigma}(t), X_2 \rangle |_{t=t_0} = \langle \dot{\sigma}(t), X_3 \rangle |_{t=t_0} = l(t_0) \neq 0$. Then $\sigma(t)$ is the intersection of the plane in R^4 spanned by vectors $\sigma(t_0)$ and $\dot{\sigma}(t_0)$ with $S^3(1)$. Therefore, we have $\langle \dot{\sigma}(t), X_2 \rangle = \langle \dot{\sigma}(t), X_3 \rangle = l(t) = \text{constant} = l(t_0)$ for every t.

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DEPARTMENT OF MATHEMATICS
COLLEGE OF LIBERAL ARTS
KANAZAWA UNIVERSITY