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HOLOMORPHIC CURVES WITH MAXIMAL DEFICIENCY SUM

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Introduction. Let f be a non-degenerate holomorphic mapping of the m-dimensional complex Euclidean space $\boldsymbol{C}^m$ into the n-dimensional complex projective space $\boldsymbol{P}^n$. Then, for any q hyperplanes $\{H_\nu\}_{\nu=1}^q \subset \boldsymbol{P}^n$ in general position, the inequality so called Nevanlinna's defect relation $\sum_{\nu=1}^q \delta(H_\nu, f) \leqq n+1$ is well known (See Stoll [6] or Vitter [9]). What can we say about a mapping f with maximal deficiency sum?

In the case $m=n=1$, the following result is known: If f is a meromorphic function of finite order which satisfies $\delta(\infty, f)=1$ and $\sum_{a\neq\infty} \delta(a, f)=1$, then f is of positive integral order and of regular growth (See Edrei-Fuchs [1]).

In the case when $m, n \geqq 1$, if there exist $n+1$ hyperplanes $\{H_\nu\}_{\nu=0}^n \subset \boldsymbol{P}^n$ in general position such that $\sum_{\nu=0}^n \delta(H_\nu, f)=n+1$, then f is of positive integral order or infinite order and is of regular growth (Mori [4] or Noguchi [5]).

In this note we treat the case of holomorphic curves with maximal deficiency sum.

§1. **Notations.** Let $f: \boldsymbol{C} \to \boldsymbol{P}^n$ be a non-degenerate holomorphic curve and let L be the standard line bandle over $\boldsymbol{P}^n$. For a homogeneous coordinate system $w=[w_0, \cdots, w_n]$ on $\boldsymbol{P}^n$,

$$h_\alpha(w) = \sum_{k=0}^n \left| \frac{w_k}{w_\alpha} \right|^2 \qquad \text{in} \quad U_\alpha = \{w : w_\alpha \neq 0\}$$

is a metric on $L \to \boldsymbol{P}^n$. Let $\psi = \{\psi_\alpha\} \in H^0(\boldsymbol{P}^n, O(L))$ be a holomorphic section. Since $|\psi_\alpha(w)|/h_\alpha(w) = |\psi_\beta(w)|/h_\beta(w)$ on $U_\alpha \cap U_\beta$, we put

$$|\psi|^2(w) \equiv \frac{|\psi_\alpha(w)|^2}{h_\alpha(w)}$$

and call it the norm of ψ. We put $\omega = \omega_L \equiv \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log h_\alpha$ which is the curvature form on L. The quantity

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$$T(r, f)\equiv\int_0^r \frac{dt}{t}\int_{|z|<t} f^*\omega$$

is called the characteristic function of f where $f^*\omega$ denotes the pull back of the form ω by f. For a divisor (ψ) of ψ, we denote by $n(t, (\psi))$ the number of points of $\{(f^*\psi)\cap(|z|<t)\}$ counting multiplicities, and put

$$N(r, (\psi))\equiv\int_0^r \frac{n(t, (\psi))}{t} dt .$$

For a hyperplane H in $\boldsymbol{P}^n$, we choose a global holomorphic section $\psi\in H^0(\boldsymbol{P}^n, O(L))$ such that $(\psi)=H$ and $|\psi|\leqq 1$, and put

$$m(r, H)\equiv\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{f^*|\psi|(z)}d\theta \qquad (z=re^{i\theta}) .$$

In particular, we use notations $T_1(r, g)$, $m_1(r, \infty)=m_1(r, g)$ and $N_1(r, \infty)=N_1(r, g)$ for a meromorphic function $g: \boldsymbol{C}\to\boldsymbol{P}^1=\boldsymbol{C}\cup\{\infty\}$.

By Nevanlinna's first main theorem, we have

$$T(r, f)=N(r, H)+m(r, H)+\log f^*|\psi|(0)$$

provided that $f^*H\not\ni 0$. For a hyperplane $H\subset\boldsymbol{P}^n$, the quantity

$$\delta(H, f)\equiv 1-\limsup_{r\to\infty}\frac{N(r, H)}{T(r, f)}\left(=\liminf_{r\to\infty}\frac{m(r, H)}{T(r, f)}\right)$$

is called the deficiency of H. We define the order λ and the lower order μ of f as follows:

$$\lambda=\limsup_{r\to\infty}\frac{\log T(r, f)}{\log r} \quad\text{and}\quad \mu=\liminf_{r\to\infty}\frac{\log T(r, f)}{\log r} .$$

§ 2. **Statement of Theroem.** Let $f: \boldsymbol{C}\to\boldsymbol{P}^n$ be a holomorphic curve and $w=[w_0, \cdots, w_n]$ a homogeneous coordinate system in $\boldsymbol{P}^n$. Then f can be represented as $f=[f_0, \cdots, f_n]$, where f_j are entire functions without common zeros. If $f=[g_0, \cdots, g_n]$ is another representation of f, then there is an entire function $\alpha(z)$ such that $g_j(z)=e^{\alpha(z)}f_j(z)$ $(j=0, \cdots, n)$. We note that the characteristic function $T(r, f)$ can be written as

$$T(r, f)=\frac{1}{4\pi}\int_0^{2\pi}\log\sum_{k=0}^n |f_k(re^{i\theta})|^2 d\theta-\log\Big(\sum_{k=0}^n |f_k(0)|^2\Big)^{1/2}$$

provided that $\sum_{k=0}^n |f_k(0)|\neq 0$.

THEOREM. *Let $f\cdot \boldsymbol{C}\to\boldsymbol{P}^n$ be a non-degenerate holomorphic curve of finite order λ. If there are q hyperplanes $\{H_\nu\}_{\nu=0}^{q-1}\subset\boldsymbol{P}^n$ $(n+1\leqq q\leqq+\infty)$ in general position such that $\lambda_{N(r,H_\nu)}<\lambda$ $(\nu=0, \cdots, n-1)$ and $\sum_{\nu=0}^{q-1}\delta(H_\nu, f)=n+1$, then f is of positive integral order, where $\lambda_{N(r,H_\nu)}$ denotes the order of $N(r, H_\nu)$.*

Since $\{H_\nu\}_{\nu=0}^{q-1}$ are in general position, we choose a homogeneous coordinate system $w=[w_0, \cdots, w_n]$ on $\boldsymbol{P}^n$ so that $H_\nu=\{w: w_\nu=0\}$ $(\nu=0, \cdots, n)$. Let $\psi_\alpha^\nu(w)=\sum_k a_k^\nu w_k/w_\alpha$ $(w_\alpha\neq 0)$ be the holomorphic section on $L\rightarrow \boldsymbol{P}^n$ such that $H_\nu=\{w: \psi_\alpha^\nu=0\ (w_\alpha\neq 0)\}$ $(\nu=0, \cdots, q-1)$. Then, for any p $(n+1\leqq p<\infty)$, we see

$$\prod_{\nu=0}^{p-1}\frac{\sum_{k=0}^{n}|w_k|^2}{|\sum_{k=0}^{n}a_k^\nu w_k|^2}\leqq \kappa\sum_{\binom{p}{n}}\prod_{j=1}^{n}\left\{\frac{\sum_{k=0}^{n}|w_k|^2}{|\sum_{k=0}^{n}a_k^{\nu_j}w_k|^2}\right\} \tag{1}$$

for some constant $\kappa>0$ depending only on $H_\nu(\nu=0, \cdots, p)$. We put

$$\phi_\nu(z)=\sum_{k=0}^{n}a_k^\nu f_k(z),\quad \text{so}\quad \phi_\nu=f_\nu\quad \text{for}\quad \nu=0, \cdots, n.$$

We can represent f as $f=[f_0, \cdots, f_n]$ so that f_ι are entire functions of order λ_ι and $\lambda_0=\lambda_{N(r,H_0)}$. In fact, f_0 can be represented as

$$f_0(z)=z^s e^{G(z)}\prod_{j=1}^{\infty}E\left(\frac{z}{a_j}, d\right)$$

where $E\left(\frac{z}{a_j}, d\right)$ is the Weierstrass' primary factor of genus d consisting of the zeros $\{a_j\}$ of f_0, $s\in\boldsymbol{Z}$, and $G(z)$ an entire function. We now divide $f_j(z)$ by $e^{G(z)}$ $(j=0, \cdots, n)$, and have $\lambda_0=\lambda_{N(r,H_0)}$. We also see $\lambda_j\leqq\lambda$, since $T_1(r, f_j/f_\iota)\leqq T(r, f)+O(1)$ $(\iota\neq j)$.

§ 3. **Two lemmas.** We need the following lemmas:

LEMMA 1. *Let $f: \boldsymbol{C}\rightarrow\boldsymbol{P}^n$ be a non-degenerate holomorphic curve of finite order and let $\{H_\nu\}_{\nu=0}^{p-1}$ be p hyperplanes $\subset\boldsymbol{P}^n$ in general position $(p<\infty)$. Then*

$$\sum_{\nu=0}^{p-1}m(r, H_\nu)\leqq nT(r, f)+\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|J(re^{i\theta})|}d\theta+\sum_{k=0}^{n-1}2T_1(r, f_k)+O(\log r)\ (r\longrightarrow\infty). \tag{2}$$

Here f_k are entire functions defined above and $J\equiv W(f_0, \cdots, f_n)$ denotes the Wronskian determinant of $\{f_j\}_{j=0}^n$.

Proof. We now estimate $\sum_{\nu=0}^{p-1}m(r, H_\nu)$ by using (1). We see

$$\sum_{\nu=0}^{p-1}m(r, H_\nu)=\sum_{\nu=0}^{p-1}\frac{1}{4\pi}\int_0^{2\pi}\log\frac{\sum_{k=0}^{n}|f_k(re^{i\theta})|^2}{|\phi_\nu(re^{i\theta})|^2}d\theta$$

$$=\frac{1}{4\pi}\int_0^{2\pi}\log\prod_{\nu=0}^{p-1}\frac{\sum_{k=0}^{n}|f_k(re^{i\theta})|^2}{|\phi_\nu(re^{i\theta})|^2}d\theta$$

$$\leqq \frac{1}{4\pi}\int_0^{2\pi}\log \kappa\left\{\sum_{\binom{p}{n}}\prod_{j=1}^{n}\frac{\sum_{k=0}^{n}|f_k|^2}{|\phi_{\nu_j}|^2}\right\}d\theta$$

$$=\frac{n}{4\pi}\int_0^{2\pi}\log\left(\sum_{k=0}^{n}|f_k|^2\right)d\theta$$

$$+\frac{1}{4\pi}\int_0^{2\pi}\log\left[\left\{1+\sum_{\substack{\binom{p}{n}'\\ \Pi\neq|\phi_0\cdots\phi_{n-1}|^2}}\frac{|\phi_0\cdots\phi_{n-1}|^2}{\prod_{j=1}^{n}|\phi_{\nu_j}|^2}\right\}\cdot\frac{1}{|\phi_0\cdots\phi_{n-1}|^2}\right]d\theta+\text{const.}$$

$$\leqq nT(r,f)+\frac{1}{4\pi}\int_0^{2\pi}\log^+\left\{\sum_{\substack{\binom{p}{n}'\\ \Pi\neq|\phi_0\cdots\phi_{n-1}|^2}}\frac{|\phi_0\cdots\phi_{n-1}|^2}{\prod_{j=1}^{n}|\phi_{\nu_j}|^2}\cdot\frac{|J|^2}{|\phi_0\cdots\phi_{n-1}|^2\left(\sum_{k=0}^{n-1}|\phi_k|^2\right)}\right\}d\theta$$

$$+\frac{1}{4\pi}\int_0^{2\pi}\log^+\frac{\sum_{k=0}^{n-1}|\phi_k|^2}{|J|^2}\,d\theta+\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|\phi_0\cdots\phi_{n-1}|}\,d\theta+\text{const}.$$

$$\leqq nT(r,f)+\frac{1}{4\pi}\int_0^{2\pi}\log^+\sum_{\substack{\binom{p}{n}'\\ \Pi\neq|\phi_0\cdots\phi_{n-1}|^2}}\frac{|J|^2}{\prod_{j=1}^{n}|\phi_{\nu_j}|^2\left(\sum_{k=0}^{n-1}|\phi_k|^2\right)}\,d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|J|}\,d\theta+\sum_{k=0}^{n-1}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\phi_k|\,d\theta$$

$$+\sum_{k=0}^{n-1}\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|\phi_k|}\,d\theta+\text{const}.$$

$$\leqq nT(r,f)+\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|J|}\,d\theta+2\sum_{k=0}^{n-1}T_1(r,f_k)+O(\log r)\qquad(r\longrightarrow\infty),$$

since $\{H_\nu\}_{\nu=0}^{p-1}$ are in general position, $\log^+|a\cdot b|\leqq\log^+|a|+\log^+|b|$, $\log^+|a+b|\leqq\log^+|a|+\log^+|b|+\log 2$ and $\frac{1}{2\pi}\int_0^{2\pi}\log^+|\phi'_{\nu_j}|/|\phi_{\nu_j}|d\theta=O(\log r)$ for a meromorphic function ϕ_{ν_j} of finite order. Here $\sum_{\binom{p}{n}'}$ means $\sum_{\binom{p}{n}}$ deleting the term of $|\phi_0\cdots\phi_{n-1}|^2$. This proves Lemma 1.

LEMMA 2 (Mori [3], p. 666-667). *Let f_j be $n+1$ linearly independent meromorphic functions of order λ_j (in the sense of order of $T_1(r, f_j)$) ($j=0, \cdots, n$). Put $J=W(f_0, \cdots, f_n)$ be the Wronskian determinant of $\{f_j\}$. If $\lambda_j<\lambda_n$ for $j=0, \cdots, n-1$, then J has order λ_n.*

§4. **Proof of Theorem.** If f has order λ which is not an integer, then f_j has order $<\lambda$ $(j=0, \cdots, n-1)$. For, if there exists an f_j $(0\leqq j\leqq n-1)$ with $\lambda_j=\lambda$, then $\lambda=\lambda_j$ is an integer, since $N_1(r, 1/f_j)=N(r, H_j)$ is of order $<\lambda=\lambda_j$. This is a contradiction. Hence f_n has order $\lambda_n=\lambda$.

We also see that J is of order λ. For, $f=[f_0, \cdots, f_n]$ repesented above is a non-degenerate holomorphic curve, so f_j are linearly independent entire functions of order $\lambda_j<\lambda$ $(j=0, \cdots, n-1)$ and $\lambda_n=\lambda$. Hence, by Lemma 2, J is of order $\lambda_n=\lambda$.

By the assumption $\sum_{\nu=0}^{q}\delta(H_\nu, f)=n+1$, for any $\eta>0$ there exists an integer $p=p(\eta)$ such that

$$\sum_{\nu=0}^{n-1}\delta(H_\nu, f)\geqq n+1-\frac{\eta}{2}.$$

By definition of $\delta(H_\nu, f)=\liminf_{r\to\infty}\{m(r, H_\nu)/T(r, f)\}$, for any $\varepsilon_\nu>0$ there is an $r_{\varepsilon_\nu}>0$ such that

$$m(r, H_\nu)>(\delta(H_\nu, f)-\varepsilon_\nu)\cdot T(r, f),$$

if $r\geqq r_{\varepsilon,\eta}$. We now choose ε_ν so small that $\varepsilon=\sum_{\nu=0}^{p-1}\varepsilon_\nu<\frac{\eta}{2}$. Then there is an $r_{\varepsilon,\eta}$ such that

$$\sum_{\nu=0}^{p-1}m(r, H_\nu)\geqq(n+1-\eta)\cdot T(r, f)$$

for all $r\geqq r_{\varepsilon,\eta}$. Thus, by (2), we have

$$(3)\qquad (1-o(1)-\eta)\cdot T(r, f)\leqq\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|J|}\,d\theta +2\sum_{k=0}^{n-1}T_1(r, f_k)+O(\log r), \qquad (r\longrightarrow\infty).$$

We note that

$$(4)\qquad T_1(r, J)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|J|\,d\theta\leqq\sum_{k=0}^{n}T_1(r, f_k)+O(\log r)$$
$$\leqq\sum_{k=0}^{n-1}T_1(r, f_k)+T_1(r, f_n/f_0)+T_1(r, f_0)+O(\log r)$$
$$\leqq T(r, f)+\sum_{k=0}^{n-1}T_1(r, f_k)+T_1(r, f_0)+O(\log r), \ (r\longrightarrow\infty).$$

We obtain from (3) and (4),

$$1-\eta=\liminf_{r\to\infty}\frac{(1-o(1)-\eta)T(r, f)}{(1+o(1))T(r, f)}\leqq\liminf_{r\to\infty}\frac{(1-o(1)-\eta)T(r, f)}{T_1(r, J)-\sum_{k=0}^{n-1}3T_1(r, f_k)}$$

$$\leqq \liminf_{r\to\infty}\left\{\frac{m_1(r, 1/J)+\sum_{k=0}^{n-1}T_1(r, f_k)}{T_1(r, J)-\sum_{k=0}^{n-1}3T_1(r, f_k)}\right\}.$$

Hence, for a sequence $\{r_n\}_{n=1}^{\infty}$ of $\lim_{r_n\to\infty}(\log T_1(r_n, J)/\log r_n)=\lambda$, we obtain

$$1-\eta\leqq \lim_{r_n\to\infty}\frac{m_1(r_n, 1/J)}{T_1(r_n, J)},$$

such a sequence exists since J has order λ. On the other hand, we see $T_1(r, J)=m_1(r, J)$, since J is an entire function. Thus we have

$$\lim_{r_n\to\infty}\frac{m_1(r_n, J)+m_1(r_n, 1/J)}{T_1(r_n, J)}\geqq 2-\eta .$$

Since we can choose Pólya peaks $\{r_n\}$ of $T(r, f)$ such that

$$\lim_{r_n\to\infty}(\log T_1(r_n, J)/\log r_n)=\lambda ,$$

we obtain

$$\lim_{r_n\to\infty}\frac{m_1(r_n, J)+m_1(r_n, 1/J)}{T_1(r_n, J)}\leqq \chi(\lambda),$$

where

$$\chi(\lambda)<2-\frac{([\lambda]+1-\lambda)(\lambda-[\lambda])}{2\lambda([\lambda]+1)(2+\log([\lambda]+1))} \qquad \text{if} \quad \lambda>1 ,$$

and

$$\chi(\lambda)<1+\lambda \qquad \text{if} \quad \lambda<1 ,$$

by a proof of Nevanlinna's theorem ([2], p. 101-103). Hence we obtain a contradiction if we take η sufficiently small. Therefore we have $\lambda\in \boldsymbol{Z}^+$. Here $\lambda>0$ is a consequence from [4] or [7], since there are $n+1$ hyperplanes $\{H_\nu\}$ in general position with $\delta(H_\nu, f)>0$. This completes the proof of Theorem.

Remark 1. We can construct a holomorphic curve f which satisfies the conditions of Theorem and has more than $n+1$ deficient hyperplanes. Here the deficient hyperplane means the hyperplane H with $\delta(H, f)>0$.

Let $w=[w_0, w_1, w_2]$ be a homogeneous coordinate system in $\boldsymbol{P}^2$. Consider $f: \boldsymbol{C}\to \boldsymbol{P}^2$ a non-degenerate holomorphic curve given by $f=[1, e^{2z}, e^{2z}-e^z]$, and hyperplanes $H_\nu=\{w : w_\nu=0\}$, $(\nu=0, 1, 2)$ and $H_3=\{w : w_0-w_1+w_2=0\}$. Then we see that $T(r, f)=\dfrac{2r}{\pi}+O(1)$, $N(r, H_\nu)=0$ $(\nu=0, 1)$ and $N(r, H_\nu)=\dfrac{r}{\pi}+O(\log r)$ $(\nu=2, 3)$. Hence we obtain that $\delta(H_0, f)=\delta(H_1, f)=1$, $\delta(H_2, f)=\delta(H_3, f)=\dfrac{1}{2}$ and f is of order one.

Remark 2. Prof. Toda told me that the conditions $\lambda_{N(H_j, f)}<\lambda$ and $\sum_{\nu=0}^{q}\delta(H_\nu, f)$

$=n+1$ implies $\delta(H_j, f)=1$ by using the theory of his modified deficiency. (See Toda [8]).

REFERENCES

[1] EDREI, A. AND W. H. J. FUCHS, On the growth of meromorphic functions with several deficient values, Trans. Amer. Math. Soc. **93** (1959), 293-328.

[2] HAYMAN, W. K., Meromorphic functions, Oxford Math. Mono., (1964).

[3] MORI, S., Sum of deficiencies and the order of a meromorphic function, Tôhoku Math. J., Vol. **22** (1970), 659-669.

[4] MORI, S., On the deficiencies of meromorphic mappings of C^n into $P^N C$, Nagoya Math. J., **67** (1977). 165-176.

[5] NOGUCHI, J., A relation between order and defect of meromorphic mapping of C^n into $P^N C$, Nagoya Math. J., Vol. **59** (1975), 97-106.

[6] STOLL, W., Die beiden Hauptsätze Wertverteilungstheorie bei Funktionen mehrerer komplexen Veränderlichen (I) (II), Acta Math., **90** (1953), 1-115 and **92** (1954), 55-169.

[7] TODA, N., Sur la croissance de fonctions algebroides á valeurs deficientes, Kōdai Math. Sem. Rep., **22** (1970), 324-337.

[8] TODA, N., On a modified deficiency of meromorphic functions, Tôhoku Math. J., Vol. **22** (1970), 635-658.

[9] VITTER, A., The lemma of the logarithmic derivative in several complex variables, Duke Math. J., Vol. **44** (1977), 89-104.

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