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ON KAEHLERIAN TORSE-FORMING VECTOR FIELDS

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§1. Introduction. K. Yano has studied in [7] the concurrency of a direction defined along a curve $x^h(s)$ in M, when it satisfies the differential equations

$$\frac{dx^h}{ds}+\frac{\delta \alpha v^h}{ds}=0,$$

where α is a suitable function of s. Moreover, generalizing these concepts of parallelism and concurrency, K. Yano [8] has introduced the notion of torse-forming directions in M as follows: Consider a vector field $v(s)$ defined along a curve $x^h(s)$. If, after the development, the directions defined by $v(s)$ form a developable surface or torse, the directions defined by $v(s)$ are called torse-forming along the curve in M.

In order that the directions $v(s)$ defined along a curve $x^h(s)$ be torse-forming, it is necessary and sufficient that

$$\frac{dx^h}{ds}+\frac{\delta \alpha v^h}{ds}=\beta v^h,$$

β being another suitable function of the parameter s. A vector field which is always torse-forming along any curve traced in M is called a torse-forming vector field. As for such a vector field, we have known the following theorems [8];

THEOREM A. *In order that a Riemannian manifold M admits a torse-forming vector field, it is necessary and sufficient that M contains a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are geodesics.*

THEOREM B. *In order that a Riemannian manifold M admits a torse-forming vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form*

$$ds^2=f(x^h)g_{ab}(x^c)dx^a dx^b+dx^n dx^n$$

$$(a, b, c=1, 2, \cdots, n-1).$$

The complex analogue of a torse-forming vector field is, as far as we know, not yet studied. So it might be interesting to develope complex versions of the theory of torse-forming vector fields. In §2, let us recall first of all definitions and formulas concerning Kaehlerian manifolds and hypersurfaces in a Kaehlerian

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manifold for later use. We shall introduce in § 3 the notion of a Kaehlerian torse-forming vector field along a curve, and investigate in § 4 a Kaehlerian torse-forming vector field along any curve, which will be called for simplicity a K-torse-forming vector field. § 5 is devoted to establish some formulas for later use. In § 6, a kind of hypersufaces called f-hypersurfaces will be defined and prove Theorems 4 and 5. Some examples of Kaehlerian manifolds admitting a K-torse-forming vector field will be given in § 7.

§ 2. **Preliminaries.** Let M be a real $2n$-dimensional Kaehlerian manifold from now on. Denote by g_{ji} and $J_j{}^h(h, i, j, \cdots =1, 2, \cdots, 2n)$ the componentes of the Hermitian metric tensor g and those of the complex structure tensor J of M respectively. Then we have by definition

$$(2.1) \qquad J_j{}^r J_r{}^i = -\delta_j{}^i, \quad g_{ji} = J_j{}^r J_i{}^s g_{rs}{}^{1)}, \quad \nabla_h J_j{}^i = 0,$$

∇ being the operator of covariant derivation with respect to the Riemannian connection defined by g.

The Kaehlerian manifold M is called a space of constant holomorphic sectional curvature if the curvature tensor of M has components of the form

$$(2.2) \qquad R_{kji}{}^h = \frac{K}{4}(\delta_k{}^h g_{ji} - \delta_j{}^h g_{ki} + J_k{}^h J_{ji} - J_j{}^h J_{ki} - 2J_{kj} J_i{}^h).$$

Next we shall recall definitions and terminologies in the theory of hypersurfaces in a Kaehlerian manifold. Let us consider a $(2n-1)$-dimensional orientable submanifold M' differentiably immersed in M. We fix orientation of M and M' and take an open covering $\{U_\beta\}(\beta\in\Lambda)$ of M by coordinate neighborhoods and an open covering $\{V_\alpha\}(\alpha\in\Lambda)$ of M' by coordinate neighborhoods so that they are coherent with the orientations, namely, in each coordinate neighborhoods U_β of M and V_α of M' natural frames determine positive positive orientations of those manifolds. Now, each non-empty set $U_\beta\cap V_\alpha$ can be expressed parametrically as $x^h = x^h(u^a)(a, b, c, \cdots =1, 2, \cdots, 2n-1)$, where $\{x^h\}$ are local coordinates in U_β and $\{u^a\}$ are those in V_α. We now put

$$(2.3) \qquad B_a{}^h = \frac{\partial x^h}{\partial u^a}.$$

Then B are linearly independent local vector fields tangent to M'. The induced Riemannian metric g' of M' is given by

$$g'_{ab} = B_a{}^h B_b{}^i g_{hi}.$$

The manifolds M and M' being both orientable, we can choose a unit normal vector field C^h along M' in such a way that (C, B) determine a frame having the positive sense of M on each non-empty $U_\beta\cap V_\alpha$. Then we get

1) We adapt the identification between vector fields and 1-forms by virtue of Riemannian metric.

$$g_{ji}B_a{}^jC^i=0\,,\qquad g_{ji}C^jC^i=1\,. \tag{2.4}$$

The transform JB of B by J and JC of C by J are expressed as linear combinations of B and C as follows:

$$J_i{}^hB_a{}^i=\varphi_c{}^bB_b{}^h+\eta_bC^h\,,\qquad J_i{}^hC^i=-\eta^aB_a{}^h\,, \tag{2.5}$$

because JC is tangent to M'. It follows from (2.1) and (2.5) that

$$\varphi_c{}^b\varphi_b{}^a=-\delta_c^a+\eta_c\eta^a\,,\qquad \varphi_a{}^b\eta^a=0\,,\qquad \eta_a\eta^a=1\,. \tag{2.6}$$

This means that M' admits an almost contact metric structure (φ, η, g').

Denoting by ∇' the symbol of the covariant derivation along M', we have the equations of Gauss and Weingarten:

$$\nabla'_aB_b{}^h\equiv\partial_aB_b{}^h+B_a{}^jB_b{}^i\left\{{h \atop j\ i}\right\}-B_c{}^h\left\{{c \atop a\ b}\right\}'=h_{ab}C^h\,,$$

$$\nabla'_aC^h\equiv\partial_aC^h+B_a{}^iC^j\left\{{h \atop j\ i}\right\}=-h_a{}^bB_b{}^h\,,$$

where $\left\{{h \atop j\ i}\right\}$ $\left(\text{resp. } \left\{{a \atop b\ c}\right\}'\right)$ are the Christoffel symbols with respect to g (resp. g') and h_{ab} are components of the second fundamental form of M'.

When the second fundamental form h of M' has the form

$$h_{ab}=\alpha g_{ab}+\beta\eta_a\eta_b\,, \tag{2.7}$$

α and β being certain functions along M', then we say that the almost contact contact metric hypersurface M' is contact umbilic. As for such M', it is well known that a necessary and sufficient condition for an almost contact hypersurface M' to be normal and contact metric is that it is contact umbilic [4, 10].

§3. **Kaehlerian torse-forming vector field along a curve.** In what follows M is assumed to be a $2n$-dimensional Kaehlerian manifold. Let $\xi(s)$ be a vector field along a curve $x^h(s)$ in M. Such a vector field $\xi(s)$ will be said to be Kaehlerian torse-forming, if the differential equation

$$\frac{dx^h}{ds}+\frac{\delta(\alpha\xi^h+\beta\tilde{\xi}^h)}{ds}=\alpha'\xi^h+\beta'\tilde{\xi}^h \tag{3.1}$$

holds along the curve for any functions α and β of the parameter s, α' and β' being certain functions of s, where we have put $\tilde{\xi}^h=J_r{}^h\xi^r$. If $\alpha=\beta=0$, then ξ is contained in the section spaned by dx^h/ds and $J_r{}^hdx^r/ds$. If we have $\alpha^2+\beta^2\neq0$, then we have from (3.1)

$$\frac{\delta\xi^h}{ds}=a\frac{dx^h}{ds}+bJ_r{}^h\frac{dx^r}{ds}+\lambda\xi^h+\mu\tilde{\xi}^h \tag{3.2}$$

for certain functions a, b, λ and μ along the cueve. We now note that $\alpha^2+\beta^2\neq 0$ if and only if $a^2+b^2\neq 0$.

Coversely, if a vector field $\xi(s)$ defined along a curve $x^h(s)$ satisfies the differential equations of the form (3.2) with a and b satisfied $a^2+b^2\neq 0$, then it is easily verified that $\xi(s)$ satisfies a differential equation of the form (3.1). Thus we have

THEOREM 1. *Let $\xi(s)$ be a vector field defined along a curve $x^h(s)$ and not contained in the section spaned by dx^h/ds and $J_r{}^h dx^r/ds$. Then in order that $\xi(s)$ be a Kaehlerian torse-forming vector field along the curve $x^h(s)$, it is necessary and sufficient that the covariant derivative of $\xi(s)$ along the curve be a linear combination of ξ, $\tilde{\xi}$, dx^h/ds and $J_r{}^h dx^r/ds$.*

When $\xi(s)$ satisfies a differential equation (3.2) with $a=b=0$, the two-dimensional distribution spaned by ξ and $\tilde{\xi}$ is parallel.

§ 4. K-torse-forming vector field.

In this paragraph, let us introduce first of all the notion of a Kaehlerian torse-forming vector field in M.

If a vector field ξ satisfies a differential equation of the form (3.2) along any curve traced in M, then we call such a vector field ξ a Kaehlerian torse-forming vector field, simply a K-torse-forming vector field. Since the equation (3.2) can be rewritten as follows:

$$\frac{dx^r}{ds}\nabla_r\xi^h=a\frac{dx^h}{ds}+bJ_r{}^h\frac{dx^r}{ds}+\lambda\xi^h+\mu\tilde{\xi}^h\,, \tag{4.1}$$

it is easy seen that for a K-torse-forming vector field

$$\nabla_j\xi^h=a\delta_j{}^h+bJ_j{}^h+\alpha_j\xi^h+\beta_j\tilde{\xi}^h\,, \tag{4.2}$$

or equivalently

$$\nabla_j\tilde{\xi}^h=aJ_j{}^h-b\delta_j{}^h+\alpha_j\tilde{\xi}^h-\beta_j\xi^h \tag{4.2$'$}$$

for suitable functions a and b and 1-forms α and β. The functions a and b (resp. 1-forms α and β) appearing in (4.2) will be called the associated functions (resp. forms) of ξ. Moreover if the associated functions a and b satisfy $a^2+b^2\neq 0$ in M, then we call such a vector field a proper K-torse-forming vector field.

We are now going to obtain some identities containing a K-torse-forming vector field for later use. Operating ∇_k to (4.2) and making use of (4.2) and (4.2)$'$, we can easily obtain

$$\begin{aligned}\nabla_k\nabla_j\xi^h=&a_k\delta_j{}^h+b_kJ_j{}^h+\nabla_k\alpha_j\xi^h+\nabla_k\beta_j\tilde{\xi}^h\\&+\alpha_j(a\delta_k{}^h+bJ_k{}^h+\alpha_k\xi^h+\beta_k\tilde{\xi}^h)+\beta_j(aJ_k{}^h-b\delta_k{}^h+\alpha_k\tilde{\xi}^h-\beta_k\xi^h)\,,\end{aligned} \tag{4.3}$$

from which

$$R_{kjr}{}^h\xi^r = X_k\delta_j^h - X_j\delta_k^h + Y_kJ_j{}^h - Y_jJ_k{}^h \tag{4.4}$$
$$+(\nabla_k\alpha_j - \nabla_j\alpha_k)\xi^h + (\nabla_k\beta_j - \nabla_j\beta_k)\tilde{\xi}^h\,,$$

where we have put

$$X_k = a_k - a\alpha_k + b\beta_k\,,\quad Y_k = b_k - b\alpha_k - a\beta_k\,,\quad a_k = \nabla_k a\,,\quad b_k = \nabla_k b\,. \tag{4.5}$$

Concerning K-torse-forming vector fields in a space of constant holomorphic sectional curvature, we have

PROPOSITION 2. *In a space M of dimensions $2n$ (>4) with constant holomorphic sectional curvature K, for any non-vanishing K-torse-forming vector field ξ its associated form α is locally a gradient of function and $d\beta=(K/4)\Phi$, where Φ is the fundamental two form of Kaehlerian structure of M.*

Proof. Substituting (2.2) into (4.4), we have

$$X'_k\delta_j^h - X'_j\delta_k^h + Y'_kJ_j{}^h - Y'_jJ_k{}^h + \alpha_{kj}\xi^h + \beta_{kj}\tilde{\xi}^h = 0\,, \tag{4.6}$$

where we have put

$$\begin{cases} X'_k = X_k + (K/4)\xi_k\,, & Y'_k = Y_k - (K/4)\tilde{\xi}_k\,, \\ \alpha_{kj} = \nabla_k\alpha_j - \nabla_j\alpha_k\,, & \beta_{kj} = \nabla_k\beta_j - \nabla_j\beta_k + (K/2)J_{kj}\,. \end{cases} \tag{4.7}$$

Hence, since $\dim M>4$, we can take unit vectors y and $\tilde{y}$ in such a way that $y, \tilde{y}, \xi$ and $\tilde{\xi}$ are mutually perpendicular. So, contracting (4.6) with $y_h, \tilde{y}_h, \xi_h$ and $\tilde{\xi}_h$, we get by a straightforward computation respectively

$$X'_ky_j - X'_jy_k - Y'_k\tilde{y}_j + Y'_j\tilde{y}_k = 0\,, \tag{4.8}$$

$$X'_k\tilde{y}_j - X'_j\tilde{y}_k + Y'_ky_j - Y'_jy_k = 0\,, \tag{4.9}$$

$$X'_k\xi_j - X'_j\xi_k - Y'_k\tilde{\xi}_j + Y'_j\tilde{\xi}_k + \alpha_{kj}|\xi|^2 = 0\,, \tag{4.10}$$

$$X'_k\tilde{\xi}_j - X'_j\tilde{\xi}_k + Y'_k\xi_j - Y'_j\xi_k + \beta_{kj}|\xi|^2 = 0\,. \tag{4.11}$$

From (4.8) and (4.9) it is evident that

$$\begin{cases} X'_k - X'(y)y_k + Y'(y)\tilde{y}_k = 0\,, & Y'_k - X'(y)\tilde{y}_k - Y'(y)y_k = 0\,, \\ X'_k - X'(\tilde{y})\tilde{y}_k - Y'(\tilde{y})y_k = 0\,, & Y'_k - Y'(\tilde{y})\tilde{y}_k + X'(\tilde{y})y_k = 0\,, \end{cases} \tag{4.12}$$

where we have put $X'(y) = X'_ky^k$ etc.. Transvecting (4.12) with $\tilde{y}^k$, ξ^k and $\tilde{\xi}^k$, we find respectively

$$Y'(\xi) = Y'(\tilde{\xi}) = X'(\xi) = X'(\tilde{\xi}) = 0\,, \tag{4.13}$$

$$X'(\tilde{y}) + Y'(y) = 0\,,\qquad X'(y) - Y'(\tilde{y}) = 0\,. \tag{4.14}$$

On the other hand, by contraction over h and i in (4.6), we can easily verify

$$(2n-1)X'_k+Y'_k+\alpha_{kr}\xi^r+\beta_{kr}\tilde{\xi}^r=0\,. \tag{4.15}$$

Further we transvect (4.10) (resp. (4.11)) with ξ^j (resp. $\tilde{\xi}^j$) and take account of (4.13) so that we obtain

$$X'_k+\alpha_{kr}\xi^r=0\,,\qquad X'_k+\beta_{kr}\tilde{\xi}^r=0\,, \tag{4.16}$$

which and (4.15) imply

$$(2n-3)X'_k+Y'_k=0\,.$$

Since $n>2$, this together with (4.14) gives

$$X'(y)=X'(\tilde{y})=Y'(y)=Y'(\tilde{y})=0\,,$$

which and (4.12) imply $X'_k=Y'_k=0$. Thus (4.6) implies $\alpha_{kj}=\beta_{kj}=0$. Accordingly, Proposition 2 is proved.

For the compact case, we have

THEOREM 3. *Let M be a $2n$ (>4) dimensional compact space of constant holomorphic sectional curvature $K\neq 0$. Then a K-torse-forming vector field in M vanishes identically.*

Proof. We assume that ξ is a non-vanishing K-torse-forming vector field in M. Then by Proposition 2 we obtain

$$\nabla_k\beta_j-\nabla_j\beta_k+(K/2)J_{kj}=0\,.$$

Contracting this with J^{kj}, we get $\nabla_r\tilde{\beta}^r=nK/2$, from which we have by Green's Theorem

$$\int_M K\,dV=0\,,$$

where dV denotes the volume element of M. Thus we have $K=0$. This completes the proof.

§5. Analytic K-torse-forming vector field.

From now on suppose that a K-torse-forming vector field ξ in M is contravariant analytic. Then the vector field ξ must satisfy (4.2) and

$$\nabla_j\xi_k=J_j{}^rJ_k{}^s\nabla_r\xi_s\,. \tag{5.1}$$

We can easily see that in order that for a K-torse-forming vector field ξ to be analytic it is necessary and sufficient that $\beta_j=\tilde{\alpha}_j(=-J_j{}^r\alpha_r)$ holds. Since ξ is analytic, (4.2), (4.3), (4.4) and (4.5) reduce respectively to

$$\nabla_j\xi^h=a\delta_j{}^h+bJ_j{}^h+\alpha_j\xi^h+\tilde{\alpha}_j\tilde{\xi}^h\,, \tag{5.2}$$

$$(5.2)' \qquad \nabla_j\tilde{\xi}^h = aJ_j{}^h - b\delta_j{}^h + \alpha_j\tilde{\xi}^h - \tilde{\alpha}_j\xi^h ,$$

$$(5.3) \qquad \nabla_k\nabla_j\xi^h = a_k\delta_j{}^h + b_kJ_j{}^h + \nabla_k\alpha_j\xi^h + \nabla_k\tilde{\alpha}_j\tilde{\xi}^h$$
$$+\alpha_j(a\delta_k{}^h + bJ_k{}^h + \alpha_k\xi^h + \tilde{\alpha}_k\tilde{\xi}^h) + \tilde{\alpha}_j(aJ_k{}^h - b\delta_k{}^h + \alpha_k\tilde{\xi}^h - \tilde{\alpha}_k\xi^h) ,$$

$$(5.4) \qquad R_{kjr}{}^h\xi^r = X_k\delta_j{}^h - X_j\delta_k{}^h + Y_kJ_j{}^h - Y_jJ_k{}^h + \alpha_{kj}\xi^h + \tilde{\alpha}_{kj}\tilde{\xi}^h ,$$

$$(5.5) \qquad X_k = a_k - a\alpha_k + b\tilde{\alpha}_k , \qquad Y_k = b_k - b\alpha_k - a\tilde{\alpha}_k ,$$

where we have put

$$(5.6) \qquad \alpha_{kj} = \nabla_k\alpha_j - \nabla_j\alpha_k , \qquad \tilde{\alpha}_{kj} = \nabla_k\tilde{\alpha}_j - \nabla_j\tilde{\alpha}_k .$$

Hence using (5.4) and the Bianchi identity $R_{kji}{}^h + R_{jik}{}^h + R_{ikj}{}^h = 0$, we have

$$(5.7) \qquad 2(Y_kJ_{jh} + Y_jJ_{hk} + Y_hJ_{kj})$$
$$+\alpha_{kj}\xi_h + \alpha_{jh}\xi_k + \alpha_{hk}\xi_j + \tilde{\alpha}_{kj}\tilde{\xi}_h + \tilde{\alpha}_{jh}\tilde{\xi}_k + \tilde{\alpha}_{hk}\tilde{\xi}_j = 0 .$$

By the way, taking account of

$$g_{ri}\underset{\xi}{L}\left\{ {r \atop j\ k} \right\} = \nabla_k\nabla_j\xi_i + R_{rkji}\xi^r ,$$

we have

$$g_{ri}\underset{\xi}{L}\left\{ {r \atop j\ k} \right\} = a_kg_{ji} + a_jg_{ki} - X_ig_{jk} - Y_iJ_{jk} - b_kJ_{ij}$$

$$(5.8) \qquad +(b_j - 2b\alpha_j - 2a\tilde{\alpha}_j)J_{ik} + \alpha_{ji}\xi_k + \tilde{\alpha}_{ji}\tilde{\xi}_k$$
$$+(\nabla_k\alpha_j + \alpha_j\alpha_k - \tilde{\alpha}_j\tilde{\alpha}_k)\xi_i + (\nabla_k\tilde{\alpha}_j + \alpha_j\tilde{\alpha}_k + \tilde{\alpha}_j\alpha_k)\tilde{\xi}_i ,$$

because of (5.3) and (5.4), where $\underset{\xi}{L}$ denotes the Lie derivation with respect to ξ. Since our manifold M is Kaehlerian and ξ is analytic, it is well known that

$$J_r{}^h\underset{\xi}{L}\left\{ {r \atop j\ k} \right\} = J_k{}^r\underset{\xi}{L}\left\{ {h \atop j\ r} \right\} ,$$

from which, using (5.9), we get

$$(5.9) \qquad Z_kg_{jh} - Z_hg_{jh} + \tilde{Z}_hJ_{kj} - \tilde{Z}_kJ_{hj}$$
$$+J_k{}^ru_{rj}\tilde{\xi}_h - J_h{}^ru_{rj}\tilde{\xi}_k + u_{kj}\xi_h - u_{hj}\xi_k = 0 ,$$

where we have put

$$(5.10) \qquad Z_k = a_k + \tilde{b}_k , \qquad u_{kj} = \nabla_k\alpha_j + J_k{}^rJ_j{}^s\nabla_r\alpha_s .$$

Again, changing k, j, i cyclically in (5.9) and adding those two obtained to (5.9), we get

$$
(5.11)\quad \begin{aligned} &2(\tilde{Z}_h J_{kj}+\tilde{Z}_k J_{jh}+\tilde{Z}_j J_{hk}) \\ &+J_k{}^r u_{rj}\tilde{\xi}_h+J_j{}^r u_{rh}\tilde{\xi}_k+J_h{}^r u_{rk}\tilde{\xi}_j-J_h{}^r u_{rj}\tilde{\xi}_k-J_k{}^r u_{rh}\tilde{\xi}_j-J_j{}^r u_{rk}\tilde{\xi}_h \\ &+u_{kj}\xi_h+u_{jh}\xi_k+u_{hk}\xi_j-u_{hj}\xi_k-u_{kh}\xi_j-u_{jk}\xi_h=0\,. \end{aligned}
$$

In the next place, we assume that the associated form α is gradient, that is, it satisfies $\alpha_i=\nabla_i\alpha$. (This condition is established in any K-torse-forming vector field in a space of constant holomorphic sectional curvature. (See Proposition 2)). So the equations (5.7) and (5.11) can be rewritten as follows:

$$
(5.12)\quad \begin{aligned} &2(Y_k J_{jh}+Y_j J_{hk}+Y_h J_{kj}) \\ &\qquad +J_k{}^r u_{rj}\tilde{\xi}_h+J_j{}^r u_{rh}\tilde{\xi}_k+J_h{}^r u_{rk}\tilde{\xi}_j=0\,, \end{aligned}
$$

$$
(5.13)\quad \tilde{Z}_h J_{kj}+\tilde{Z}_k J_{jh}+\tilde{Z}_j J_{hk}+J_k{}^r u_{rj}\tilde{\xi}_h+J_j{}^r u_{rh}\tilde{\xi}_k+J_h{}^r u_{rk}\tilde{\xi}_j=0\,,
$$

because of $J_k{}^r u_{rj}+J_j{}^r u_{rk}=0$ and $u_{jk}=u_{kj}$, and consequently

$$
(2Y_k-\tilde{Z}_k)J_{jh}+(2Y_j-\tilde{Z}_j)J_{hk}+(2Y_h-\tilde{Z}_h)J_{kj}=0\,.
$$

This together with (5.6) and (5.10) gives

$$
(5.14)\quad b_k+\tilde{a}_k=2(a\tilde{\alpha}_k+b\alpha_k)\,,
$$

or equivalently

$$
(5.14)'\quad a_k-\tilde{b}_k=2(a\alpha_k-b\tilde{\alpha}_k)\,.
$$

Thus we obtain

$$
(5.15)\quad X_k=-\tilde{Y}_k\,.
$$

Also, it follows from (5.14) and (5.14)′ that

$$
(5.16)\quad aa_k+bb_k+b\tilde{a}_k-a\tilde{b}_k=2(a^2+b^2)\alpha_k\,.
$$

In the third place, suppose that ξ is an analytic proper K-torse-forming vector field. Then (5.16) gives

$$
(5.16)'\quad \nabla_k\Big(\frac{1}{4}\log(a^2+b^2)-\alpha\Big)=\frac{-b\tilde{a}_k+a\tilde{b}_k}{a^2+b^2}\,,
$$

because of $\alpha_i=\nabla_i\alpha$. On the other hand, contracting (5.9) with g^{hj}, ξ^h and $\xi^h J^{kj}$ and taking account of $J_k{}^r u_{rj}+J_j{}^r u_{rk}=0$ and $u_{jk}=u_{kj}$, we obtain respectively

$$
(5.17)\quad (n-1)(a_k+\tilde{b}_k)+u_{kr}\xi^r-\nabla_r\alpha^r\xi_k=0\,,
$$

$$
(5.18)\quad \begin{aligned} &|\xi|^2 u_{kj}-u_{rj}\xi^r\xi_k-\tilde{\xi}^r u_{rj}\tilde{\xi}_k+(a_k+\tilde{b}_k)\xi_j \\ &\qquad -(a_r+\tilde{b}_r)\xi^r g_{jk}+(b_r-\tilde{a}_r)\xi^r J_{kj}-(b_k-\tilde{a}_k)\tilde{\xi}_j=0\,, \end{aligned}
$$

(5.19) $$(b_r-\tilde{a}_r)\xi^r=0\,.$$

Further, transvecting (5.8) with ξ^j and using (5.19), we have

$$|\xi|^2u_{kj}\xi^j=[u_{rs}\xi^r\xi^s+(a_r+\tilde{b}_r)\xi^r]\xi_k-(a_k+\tilde{b}_k)|\xi|^2\,,$$

from which, comparing this with (5.17)

(5.20) $$a_k+\tilde{b}_k=2\rho\xi_k\,,$$

or equivalently

(5.20)′ $$b_k-\tilde{a}_k=-2\rho\tilde{\xi}_k$$

for a certain function ρ. By virtue of (5.16), (5.20) and (5.20)′, it is clear that

(5.21) $$\frac{1}{2}\nabla_k\log(a^2+b^2)-\alpha_k=\frac{\rho}{a^2+b^2}(a\xi_k-b\tilde{\xi}_k)\,.$$

Here we put

(5.22) $$f=\frac{1}{2}\log(a^2+b^2)-\alpha\,.$$

Then, applying ∇_j to (5.22) and using (5.21), we find

(5.23) $$f_j=\frac{\rho}{a^2+b^2}(a\xi_j-b\tilde{\xi}_j),\quad (f_j=\nabla_jf)\,,$$

or equivalently

(5.23)′ $$\tilde{f}_j=\frac{\rho}{a^2+b^2}(a\tilde{\xi}_j+b\xi_j)\,.$$

So we have just shown that

(5.24) $$af_j+b\tilde{f}_j=\rho\xi_j\,,$$

or equivalently

(5.24)′ $$a\tilde{f}_j-bf_j=\rho\tilde{\xi}_j\,.$$

§6. **f-hypersurfaces.** Let ξ be an analytic proper K-torse-forming vector field whose associated form α is locally gradient. A point P of M is called an ordinary point of ξ, if both of ξ and f_h given by (5.23) do not vanish at P. Let M_1 be the set of all ordinary points of M. Then M_1 is a non-empty open subset of M. We also see from (5.24) that ρ has not zero points over M_1.

In the sequel we perform our discussions in M_1. Differentiating (5.23) covariantly and making use of (5.16), (5.21)∼(5.24) and (5.23)′∼(5.24)′, we find

(6.1) $$\nabla_kf_j=\rho g_{kj}+(\nabla_k\log\rho-f_k)f_j+\tilde{f}_k\tilde{f}_j\,,$$

from which

(6.2) $$\nabla_k \log \rho = \lambda f_k$$

for a certain function λ, since f_j is gradient. Thus (6.1) can be rewritten as follows:

(6.3) $$\nabla_k f_j = \rho g_{kj} + (\lambda - 1) f_k f_j + \tilde{f}_k \tilde{f}_j .$$

In a sufficiently small neighborhood of an ordinary point we consider the integral curve of the vector field f^h. By means of (6.3), we can easily find that such an integral curve is a geodesic arc.

Let Q be an ordinary point in M and U a coordinate neighborhood of Q which contain only ordinary points. So we can define in U a family of hypersurfaces by the equations $f(x)=$constant which will be called f-hypersurface. Given a point in M_1, there exists in the family one and only one f-hypersurface $V(P)$ passing P. It is clear that the f-curves form the normal congruence to the family of the f-hypersurfaces in U.

Put

$$C^h = \frac{1}{\sigma} f^h , \qquad \sigma = \sqrt{f_r f^r}$$

in M_1, then C^h is differentiable in M_1. As this equation and (6.3) yield that

$$\nabla_k \sigma C_j + \sigma \nabla_k C_j = \rho g_{kj} + (\lambda - 1) \sigma^2 C_k C_j + \sigma^2 \tilde{C}_k \tilde{C}_j ,$$

we get by transvection of this with C^j

(6.4) $$\nabla_k \sigma = (\rho + (\lambda - 1) \sigma^2) C_k ,$$

which implies

(6.5) $$\nabla_k C_j = \frac{\rho}{\sigma} (g_{kj} - C_k C_j) + \sigma \tilde{C}_k \tilde{C}_j .$$

Let P be a point in U and $V(P)$ the f-hypersurfaces in U passing through the point P. Then the vector field C^h is the normal unit vector to $V(P)$ at any point of $V(P)$. We choose a system of local coordinates $\{u^a\}$ in $V(P)$ and suppose that $V(P)$ is expressed by parametric equations $x^h = x^h(u^a)$ in U. We notice that the second fundamental form h of the f-hypersurface $V(P)$ is given by

$$h_{ab} = B_a{}^i B_b{}^j \nabla_j C_i .$$

By virtue of (2.5) and (6.5), it is evident that

(6.6) $$h_{ab} = \frac{\rho}{\sigma} g'_{ab} + \sigma \eta_a \eta_b .$$

So we can see that $V(P)$ is nothing but contact umbilic. By virtue of (6.2) and (6.4), we find respectively $\partial_a \rho = 0$ and $\partial_a \sigma = 0$ and consequently, we see that the functions ρ and σ are constant over $V(P)$.

Now we can choose a system of coordinates $\{x^h\}$ in U such that f-hypersurfaces defined by x^{2n}=constant are the f-hypersurfaces in U and the curves defined by the equations x^a=constant are the f-curves in U. Then it is easy to see that

$$g_{a\,2n}=g_{2n\,a}=0\,.$$

Since the f-curves are geodesics, we have

$$\left\{ {h \atop 2n\ \ 2n} \right\}=\gamma\delta_{2n}{}^h\,,$$

where γ is in U a function depending only on x^{2n}. Especially, if we put $h=a$, then it follows that

$$\left\{ {a \atop 2n\ \ 2n} \right\}=0\,.$$

Recalling $g_{a\,2n}=0$ and $g^{a\,2n}=0$, we have

$$\partial_a g_{2n\,2n}=0\,,$$

which means that $g_{2n\,2n}$ depends only on x^{2n}. Hence, taking a suitable transformation of the $2n$-th coordinate, we have $g_{2n\,2n}=1$ in U. Then we find explicitly

$$\left\{ {h \atop 2n\ \ 2n} \right\}=0\,.$$

And the variable x^{2n} is the arc-length of f-curves in U. So the line element of the Kaehlerian manifold M is written in the form

$$ds^2=g_{ab}(x^h)dx^a dx^b+(dx^{2n})^2\,. \tag{6.7}$$

Thus we get

THEOREM 4. *If a Kaehlerian manifold M admits an analytic proper K-torse-forming vector field ξ such that the associated form is locally gradient, then for any ordinary point P of the vector field ξ, there exists a coordinate neighborhood U of the point P in such a way that there is in U a system of coordinates $\{x^h\}$ having the following properties The function f depends only on the* 2n-*th variable x^{2n} in U. The line element of M is given by* (6.7) *in U. The hypersurfaces defined by the equation x^{2n}=constant are the f-hypersurfaces and the curves defined by the equation x^a=constant are the f-curves and x^{2n} indicates the arc length along the f-curves. Moreover, f-hypersurfaces are contact umbilic.*

Conversely, we assume that in a Kaehlerian manifold M there exists a coordinate neighborhood U in M such that there exists a family of contact umbilical hypersurfaces

$$f(x^h)=\text{constant} \tag{6.8}$$

whose orthogonal trajectories are geodesics. Then operating ∇'_a to (6.8), we can easily find that

$$f_h B_a{}^h = 0 .$$

Furthermore, differentiation of the above equation gives

$$\nabla_k f_j B_a{}^k B_b{}^j + f_k C^k h_{ab} = 0 ,$$

which means that

$$[\nabla_k f_j + f_r C^r (\alpha g_{kj} + \beta J_{kh} C^h J_{jm} C^m)] B_b{}^k B_a{}^j = 0 ,$$

because $h_{ab} = \alpha g'_{ab} + \beta \eta_a \eta_b$, where C^h denotes the unit normal vector of the hypersurface. Consequently we see that $\nabla_k f_j$ must take the form

$$\nabla_k f_j = \rho g_{kj} + a f_k f_j + b \tilde{f}_k \tilde{f}_j , \tag{6.9}$$

or equivalently

$$\nabla_k \tilde{f}_j = \rho J_{kj} + a f_k \tilde{f}_j - b \tilde{f}_k f_j \tag{6.9$'$}$$

for certain functions ρ, a and b. If we put

$$\xi^h = c f^h + e f^h$$

for any functions c and e such that $c^2 + e^2 \neq 0$, then we have

$$\nabla_j \xi^h = c\rho \delta_j{}^h + e\rho J_j{}^h + \alpha_j \xi^h + \beta_j \tilde{\xi}^h ,$$

α_j and β_j being certain 1-forms. The above equation means that ξ is a K-torse-forming vector field. Therefore we have

THEOREM 5. *If there exists a coordinate neighborhood U in a Kaehlerian manifold M such that there exists a family of contact umbilical hypersurfaces whose orthogonal trajectories are geodesics, then there exists a K-torese-forming vector field in U.*

§ 7. **Examples.** In [5] we have proved that in order that a Kaehlerian manifold M is holomorphically subprojective, it is necessary and sufficient that there exists a local coordinate system $\{x^h\}$ such that the Christoffel symbols $\left\{ {h \atop j\ i} \right\}$ of M take the form

$$\left\{ {h \atop j\ i} \right\} = \rho_j \delta_i{}^h + \rho_i \delta_j{}^h + \tilde{\rho}_j J_i{}^h + \tilde{\rho}_i J_j{}^h + f_{ji} x^h - f_{jr} J_i{}^r \tilde{x}^h \tag{7.1}$$

$$f_{[jk]} = 0 , \qquad f_{r[j} J_{i]}{}^h = 0 , \tag{7.2}$$

where ρ_i and f_{jk} are 1-form and a covariant tensor field respectively. Now, consider a vector field V such that V are given by $V^h = x^h$ with respect to a sysyem of coordinate $\{x^h\}$ having the properties above mentioned. Differentiate it covariantly with respect to the connection (7.1), we have by virtue of (7.2)

$$\nabla_j V^h = (1+\rho_r V^r)\delta_j{}^h + \tilde{\rho}_r V^r J_j{}^h + \alpha_j V^h + \tilde{\alpha}_j \tilde{V}^h ,$$

where we have put $\alpha_j = \rho_j + f_{jr} V^r$. Moreover we have proved in [5] that the associated form is gradient. These facts tell us that the vector field V is nothing but an analytic K-torse-forming vector field whose associated form is gradient.

In [5, III] we have also shown that the Christoffel symbols of the holomorphically subprojective Kaehlerian manifold of the first kind take the form

$$\left\{ {h \atop j\ i} \right\} = \rho_j \delta_i{}^h + \rho_i \delta_j{}^h + \tilde{\rho}_i J_j{}^h + \tilde{\rho}_j J_i{}^h + f_{ji} \xi^h - f_{jr} J_i{}^r \tilde{\xi}^h ,$$

for suitable coordinate system $\{x^h\}$, where $f_{[jk]}=0$, $f_{r[j} J_{i]}{}^h = 0$ and ξ^h is an analytic K-torse-forming vector field whose associated form is gradient.

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