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ON THE PICK-NEVANLINNA PROBLEM

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Introduction

Let there be given a finite number of points z_j in the unit disc Δ and assigned data w_j, $|w_j|<1$ at z_j, $j=1, \cdots, N$. The classical Pick-Nevanlinna (interpolation) problem asks whether there exist functions analytic, bounded by unity in Δ and satisfying $f(z_j)=w_j$, $j=1, \cdots, N$ (Pick [16], Nevanlinna [14], [15]). When this class of functions is found to be non-void, the set $\{f(z_0)\}$, called the "Wertevorrat" should be investigated [15] and the problem can be transformed into a linear extremal problem for the functional $\mathrm{Re}(e^{i\theta}f(z_0))$ under the given data. The problem was generalized for multiply-connected domains and the linear extremal problem was solved by Garabedian [8]. He formulated a dual extremal problem for the Schwarz lemma there, which has been a useful tool for extremal problems.

Duality in a problem with side conditions as in the Pick-Nevanlinna problem was not known for a long time until Havinson [9] found a dual extremal problem for the general Carleman-Milloux problem. Recently a formulation of dual extremal problem for the general Pick-Nevanlinna problem was given by Gamelin [6], [7]. More recently Hejhal [12] has shown how the method of dual extremal problems can be applied to both problems.

In the present paper we are concerned with the Pick-Nevanlinna problem. We treat the problem under the formulation of Carathéodory-Fejér [3] i. e. minimize the norm of f among the functions with side conditions (e. g. $f(z_j)=w_j$, $j=1, \cdots, N$). This formulation will allow us a symmetric treatment of the problem. By using a well-known duality in Banach spaces, the problem is reduced to a linear extremal problem for a single functional, which was investigated by Hejhal [11] a great deal. It should be noted that another duality relation was used by Lax [13] many years ago and that it was a fundamental technique for the case of regular regions in Hejhal [11]. Our duality, a counterpart of theirs, provides us with a conjugate differential conveniently. We also note that linear extremal problems in Gamelin [6], [7] and Hejhal [12] can be reduced to our formulation.

In § 1 we shall show how the Pick-Nevanlinna problem under Carathéodory-Fejér's formulation is reduced to a linear extremal problem for a single functional. The relationship with Gamelin's formulations will be discussed there. We also show the uniqueness of extremal functions in the space of bounded functions.

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In § 2, conjugate differentials will be obtained from the duality relation on a compact bordered Riemann surface. Here Royden's result [18], an extension of F and M Riesz's theorem, is useful.

§ 3 will be devoted to discussion of uniqueness of the extremal for the Pick-Nevanlinna problem in a more general situation. We shall need a sort of Cauchy kernel for differentials in order to apply Hejhal's method to a subdomain of a compact Riemann surface which does not belong to O_{AB} [21].

In § 4 we shall treat the classical Pick-Nevanlinna problem for meromorphic or multiplicative functions. To this case, while the problem is transformed into the single-valued case, Hejhal's result [11] cannot be applied directly.

§ 1. General principles, interpolation for bounded functions.

1. *Problems.* Let Ω be a subdomain of a compact Riemann surface, which does not belong to O_{AB} and let $X(\Omega)$ be a Banach space of functions f analytic in Ω with norms $\|f\|$. A general Pick-Nevanlinna interpolation problem for a finite number of data will be formulated in the following way: let there be given a finite number of linear functionals L_j continuous with respect to the supnorm on compact subsets K_j of Ω, each of which does not separate the boundary $\partial\Omega$ of Ω and the same number of data a_j, $j=1, \cdots, N$. Do there exist functions of $X(\Omega)$ which satisfy $\|f\|\leqq 1$ and $L_j(f)=a_j$, $j=1, \cdots, N$? In the classical Pick-Nevanlinna problem we just consider the Banach space $AB(\Omega)$ of bounded analytic functions f, with supnorm $\|f\|=\sup|f(z)|$, $z\in\Omega$ and take the values of f or more generally the values of successive derivatives at a finite number of points z_j, $j=1, \cdots, N$ as the data of the linear functionals. Here, for simplicity, we used z_j as a fixed value of a local parameter at a given point. The latter condition is equivalent to giving Taylor sections

$$D_j=\sum_{\nu=0}^{N_j} a_\nu(z-z_j)^\nu \text{ at } z_j,\ j=1, \cdots, N. \tag{1}$$

Quite recently Heins [10] proved uniqueness of the extremal function f_0 which maximizes $\mathrm{Re}\,(e^{i\theta} f(z_0))$ among the class of analytic functions f bounded by unity and with given Taylor sections (1) at z_j, $z_j\neq z_0$, $j=1, \cdots, N$ on a compact bordered Riemann surface Ω. He also proved the extremal f_0 maps Ω onto a finite sheeted covering of the unit disc and gave a bound of the number of sheets called the *Garabedian bound.*

In No. 4 of this section we show uniqueness for extremal functions for the general Pick-Nevanlinna problems for the class of bounded functions on a subdomain Ω of a compact Riemann surface. Other properties such as the Garabedian bound will be discussed in § 2.

2. *Fundamental lemma.* We state a well-known lemma, a duality result in a Banach space X.

LEMMA 1. *Let X be a Banach space with norm $\|\ \|$. Let S be a closed subspace of X. Let $S^\perp$ denote the annihilator of S, that is, the set of all con-*

tinuous linear functionals ψ such that $\psi(x)=0$ for $x\in S$. Then for each fixed $x\in X$,

$$\max_{\psi\in S^{\perp},\|\psi\|=1} |\psi(x)| = \inf_{y\in S} \|x+y\| .$$

Here "max" indicates that the supremum is attained.

For a proof the reader is referred to Duren [4] p. 111.

3. *Relation to other formulations.* First of all we see that our Pick-Nevanlinna problem with finite data can be reduced to that with a single datum. We state it in the most general form.

PROPOSITION 1. *Let X be a Banach space of functions analytic in a open Riemann surface Ω normed by $\| \quad \|$. Let $\{L_\nu\}_{\nu=1}^N$ be continuous linear functionals on X. Suppose that there exists an extremal function f_0 which minimizes the norm $\|f\|$ in the family $\mathfrak{F}$ of functions $f\in X$ satisfying $L_\nu(f)=a_\nu$, $\nu=1, \cdots N$. Then there exists a linear combination*

$$\psi_0=\sum_{\nu=1}^{N} c_\nu L_\nu$$

for which the f_0 is an extremal function of the Pick-Nevanlinna problem with a single datum $L_0(f)=\sum_{\nu=1}^N c_\nu a_\nu$.

Proof. Let S be a closed subspace of X defined by $S=\{f\,|\,f\in X, L_\nu(f)=0, \nu=1, \cdots, N\}$.

By Lemma 1, we have

$$\max_{\|\psi\|=1,\psi\in S^{\perp}} |\psi(f_0)| = |\psi_0(f_0)| = \|f_0\|,\ \psi_0\in S^{\perp}.$$

It is easy to show that $\psi_0\in S^{\perp}$ implies that ψ_0 is a linear combination of $\{L_\nu\}_{\nu=1}^N$ i. e.

$$\psi_0=\sum_{\nu=1}^{N} c_\nu L_\nu .$$

Since $\|\psi_0\|=1$, for every $f\in\mathfrak{F}$ we have $\|f\|\geqq|\psi_0(f)|=|\sum_{\nu=1}^N c_\nu L_\nu(f)|=|\sum_{\nu=1}^N c_\nu a_\nu|=\|f_0\|$.

We may suppose that $\psi_0(f_0)=|\psi_0(f_0)|$ multiplying by a constant. This means that the f_0 is extremal for the Pick-Nevanlinna problem with a single datum $\psi_0(f)=\sum_{\nu=1}^N c_\nu a_\nu$.

We consider the Garabedian-Hejhal-Gamelin formulation. Let B be the unit ball in X. Consider a linear extremal problem: maximize $|L_0(f)|$, $f\in B$ under the side condition $L_\nu(f)=a_\nu$, $\nu=1, \cdots, N$, where L_0 is a given continuous linear functional which cannot be expressed by a linear combinations of $\{L_\nu\}_{\nu=1}^N$. Suppose that there exists an extremal function f_0 for the linear extremal problem. Then the relation to the Pick-Nevanlinna problem is given by

PROPOSITION 2. *The extremal f_0 is an extremal function for the Pick-Nevanlinna problem with $N+1$ data $L_\nu(f)=a_\nu$, $\nu=1, \cdots, N$ and $L_0(f)=L_0(f_0)=a_0$.*

Proof. If f_0 were not extremal for the Pick-Nevanlinna problem, there would be a function f_1 with the same data and satisfying $\|f_1\|<\|f_0\|$. Since L_0 is independent of $\{L_\nu\}_{\nu=1}^N$, there exists a function $h\in X$ such that $L_\nu(h)=0$, $\nu=1, \cdots, N$ and $L_0(h)\neq 0$. Then $f_1+\varepsilon h\in B$ for sufficiently small ε and we get a contradiction that $|L_0(f_1+\varepsilon h)|>|L_0(f_0)|$ for a suitable ε.

4. *A uniqueness theorem.* In this section we consider the Banach space $AB(\Omega)$ of bounded functions f analytic in an arbitrary plane domain $\Omega\notin O_{AB}$ normed by the supnorm $\|f\|=\sup|f(z)|$, $z\in\Omega$. Let $\{L_\nu\}_{\nu=1}^N$ be linear functionals which are continuous with respect to the supnorm on a compact subset K of Ω.

$$\|f\|_K=\sup_{z\in K}|f(z)| .$$

Let $\mathfrak{F}$ be the family of functions f satisfying

$$L_\nu(f)=a_\nu,\ \nu=1, \cdots, N .$$

To avoid a trivial case, assume that $\inf\|f\|$, $f\in\mathfrak{F}$, is positive. There exists a minimal sequence $\{f_n\}_{n=1}^\infty$ such that

$$\|f_n\| \longrightarrow \inf_{f\in F}\|f\|=M_0\ (n\longrightarrow\infty) .$$

Since $\{f_n\}$ forms a normal family, we may suppose that $f_n\to f_0$ on every compact subset of Ω. $\underline{\lim}\,\|f_n\|\geqq\|f_0\|$. By continuity with respect to $\|f\|_K$, $f_0\in\mathfrak{F}$. Hence f_0 is an extremal function.

By using Fisher's method [5], we show uniqueness of the extremal function f_0.

THEOREM 1. *The extremal function f_0 is unique.*

Proof. Suppose that there were another extremal function f_0^*. Set

$$g=(f_0+f_0^*)/2,\ h=(f_0-f_0^*)/2 .$$

Then we have

$$|g|^2+|h|^2=\frac{1}{2}(|f_0|^2+|f_0^*|^2)\leqq M_0^2 .$$

and

$$|g|\leqq M_0-\frac{1}{2}\frac{|h^2|}{M_0}$$

Since g is an extremal function for the Pick-Nevanlinna problem with data $L_\nu(f)=a_\nu$, $\nu=1, \cdots, N$, by Proposition 1, g is an extremal function for the Pick-Nevanlinna problem with a datum $\phi_0(f)=\sum_{\nu=1}^N c_\nu a_\nu$. Since the problem is conformally invariant, we may assume $\infty\notin\Omega$.

Let $\{z_\nu\}_{\nu=1}^m$ be the totality of zero points of h^2 on K counting multiplicity. Then

$$H(z)=\frac{\eta}{2}\frac{h^2}{M_0}\prod_{\nu=1}^m\frac{1}{(z-z_\nu)} \quad \text{with} \quad \left|\eta\prod_{\nu=1}^m\frac{1}{\zeta-z_\nu}\right|\leqq 1,\ \zeta\in\partial\Omega.$$

satisfies

$$|g(z)|+|H(z)|\leqq M_0,\ z\in\Omega$$

and

$$H(z)\neq 0, \qquad z\in K.$$

By making use of a variation $g+\varepsilon H$, $|\varepsilon|=1$, we have $\psi_0(H)=0$. Since $|Hf/\|f\||\leqq|H|$, we get $\psi_0(Hf)=0$, for $f\in AB(\Omega)$.

We want to use Bishop's approximation theorem [2] p. 48. Every domain $\Omega\notin O_{AB}$ has a minimal prolongation for which every $f\in AB(\Omega)$ can be analytically extended onto the prolongation (Rudin [20]). We denote it by the same Ω. By adding relatively compact components of $\Omega-K$ to K, we may assume that the ideal boundary of every connected component of $\Omega-K$ does not belong to the class N_B (for definitions see Sario-Oikawa [21]). Then from Bishop's approximation theorem cited above, every analytic function on the compact set K is approximated by functions of $AB(\Omega)$. For an arbitrary function $f\in AB(\Omega)$, f/H is analytic on K. Hence there exists a function f_ε such that

$$\|f/H-f_\varepsilon\|_K<\varepsilon.$$

Since $\psi_0(Hf_\varepsilon)=0$, we have $\psi_0(f)=0$ which is a contradiction.

§ 2. Conjugate differentials.

5. *Duality relation.* In this section we consider a compact bordered Riemann surface Ω. Let $\chi(z)$ be a real valued function harmonic in Ω and continuous on $\bar{\Omega}$. We define a norm with respect to χ for analytic functions f by

$$(2) \qquad \chi\text{-}\|f\|=\sup_{z\in\Omega}|f(z)e^{-\chi(z)}|.$$

We denote by X the Banach space of analytic functions f on Ω satisfying χ-$\|f\|<\infty$. Let $\{L_\nu\}_{\nu=1}^N$ be linear functionals on X continuous with respect to the supnorm $\|\ \ \|_K$ on a compact set K on Ω. We consider the Pick-Nevanlinna problem under the conditions $L_\nu(f)=a_\nu$, $\nu=1,\cdots,N$.

Since L_ν is defined on a linear subspace X of the space $C(K)$ of continuous functions on K, by the Hahn-Banach extension theorem it can be regarded as a continuous linear functional on $C(K)$. By the Riesz representation theorem there exists a finite Borel measure $d\mu_\nu$ supported by K and such that

$$L_\nu(f)=\int_K f\,d\mu_\nu,\ f\in C(K).$$

Let $g(z, \zeta)$ be the Green's function of Ω. For simplicity, we use the local parameters z, ζ as the points of Ω in this section. We define a transform of L_ν by

(3) $$l_\nu(z)dz=\frac{i}{\pi}\left(\frac{\partial}{\partial z}\int_K g(z, \zeta)d\mu_\nu(\zeta)\right)dz, \ z\notin K,$$

$\frac{\partial}{\partial z}$ being $\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$, $z=x+iy$. Then we have

(4) $$\int_{\partial\Omega} f(z)l_\nu(z)dz=L_\nu(f) \qquad \text{for every } f\in X.$$

Here $f(z)$ denotes the Fatou boundary value of f on $\partial\Omega$. Note that f is bounded on Ω.

Let $\mathfrak{F}$ be the family of functions $f\in X$ satisfying $L_\nu(f)=a_\nu$, $\nu=1, \cdots, N$. We state

THEOREM 2. *Suppose that K does not separate the boundary of Ω. If*

$$M_0=\inf_{f\in\mathfrak{F}} \chi-\|f\|>0,$$

then there exists a unique extremal function $f_0\in\mathfrak{F}$ satisfying χ-$\|f_0\|=M_0$. Moreover, in a simply-connected neighborhood U of any boundary point of Ω, the function $f_0(z)e^{-(\chi(z)+i\chi^*(z))}$ can be analytically continued onto $\partial\Omega\cap U$ and $|f_0e^{-\chi}|$ is identically equal to M_0 on $\partial\Omega$. Here χ^* is a conjugate harmonic function of $\chi(z)$ in U.

6. *Proof of Theorem* 2. As in No. 4, a routine use of the normal family arguments proves existence of an extremal function f_0 for which χ-$\|f_0\|=M_0$. Set

$$S=\{f\,|\,L_\nu(f)=0,\ \nu=1, \cdots, N, f\in X\}.$$

From Lemma 1, we can deduce that there exists a ψ_0 for which $\psi_0(f_0)=\chi$-$\|f_0\|$ and

$$\max_{\psi\in S^\perp, \|\psi\|=1} |\psi(f_0)|=\psi_0(f_0),\ \|\psi_0\|=1,\ \psi_0\in S^\perp.$$

Let $\bar{A}(\Omega)$ denote the class of functions analytic in Ω and continuous on $\bar{\Omega}$. Clearly $\bar{A}(\Omega)$ is a linear subspace of X. When we consider the restriction of ψ_0 to $\bar{A}(\Omega)$, its norm $\|\psi_0\|_{\bar{A}(\Omega)}$ never exceeds one. Transforming $\bar{A}(\Omega)$ to $\chi-\bar{A}(\Omega)=\{\chi-f\,|\,\chi-f=e^{-\chi}f,\ f\in\bar{A}(\Omega)\}$ and ψ_0 to $\chi-\psi_0: e^{-\chi}f\to\psi_0(f)$, we see from the Hahn-Banach extension theorem that ψ_0 is extended to the space $C(\partial\Omega)$ of continuous functions on $\partial\Omega$ with norm $\|\psi_0\|_{\bar{A}(\Omega)}$. By the Riesz representation theorem, there exists a finite Borel measure $d\mu$ supported by $\partial\Omega$ such that

$$\psi_0(f)=\int_{\partial\Omega} fd\mu,\ f\in C(\partial\Omega)$$

and that the total variation $\|e^{\chi}d\mu\|$ of $e^{\chi}d\mu$ is equal to $\|\phi_0\|_{\bar{A}(\Omega)}$.

By Proposition 1, $\phi_0=\sum_{\nu=1}^{N} c_\nu L_\nu$ and we have from (4)

$$\int_{\partial\Omega} f(d\mu-\sum_{\nu=1}^{N} c_\nu l_\nu dz)=0\,, \qquad \text{for} \quad f\in\bar{A}(\Omega)\,.$$

From Royden's extension of F. and M. Riesz's theorem [18], we deduce that $d\mu$ is of the form

$$d\mu=(\sum_{\nu=1}^{N} c_\nu l_\nu(z)+\phi(z))dz\,, \qquad z\in\partial\Omega\,,$$

where $\varphi_0 dz$ belongs to the class Q[18], that is, for a nonvanishing analytic differential ω on $\bar{\Omega}$, $\phi dz/\omega$ is an analytic function of the Hardy class of index one and $\varphi_0(z)$ is considered as its Fatou boundary value. It is well known that $\phi(z)$ is locally integrable under a boundary uniformizer and that for every bounded analytic function f on Ω

$$\int_{\partial\Omega} f\varphi_0 dz=0\,.$$

Hence we have

$$\phi_0(f_0)=\int_{\partial\Omega} f_0(\sum_{\nu=1}^{N} c_\nu l_\nu+\varphi_0)dz\,.$$

Since

$$\|e^{\chi}d\mu\|=\int_{\partial\Omega} e^{\chi}|\sum_{\nu=1}^{N} c_\nu l_\nu+\phi|\,|dz|=\|\phi_0\|_{\bar{A}(\Omega)}\leqq 1$$

and

$$M_0\leqq\int_{\partial\Omega}|f_0e^{-\chi}|e^{\chi}|\sum_{\nu=1}^{N} c_\nu l_\nu+\varphi_0|\,|dz|\,,\ |f_0e^{-\chi}|\leqq M_0\,,$$

we have $\|e^{\chi}d\mu\|=1$. By Riesz's uniqueness theorem $\sum_{\nu=1}^{N}(c_\nu l_\nu+\varphi_0)dz$ never vanishes except for a set of linear measure zero. We infer that

$$|f_0(z)e^{-\chi(z)}|=M_0 \quad \text{a.e. on} \quad \partial\Omega \tag{5}$$

and

$$f_0(z)(\sum_{\nu=1}^{N} c_\nu l_\nu(z)+\varphi_0(z))dz\geqq 0 \quad \text{a.e. on} \quad \partial\Omega\,. \tag{6}$$

To show the boundary property of f_0, take a simply-connected neighborhood U of a boundary point ζ. U is supposed to be represented as the demidisc V, $|z|<1$, $\text{Im}\,z\geqq 0$ so that $\partial\Omega\cap U$ corresponds to the interval $(-1, 1)$ and ζ to the origin. Take a conjugate harmonic function χ^* of χ in V and set $E(z)=e^{-\chi(z)-i\chi^*(z)}$ in V. Then we have from (5), (6)

(7) $$|f_0(x)E(x)|=M_0 \quad \text{a.e.} \quad x\in(-1, 1)$$

and

(8) $$f_0(x)E(x)E^{-1}(x)\Big(\sum_{\nu=1}^{N} c_\nu l_\nu(x)+\varphi_0(x)\Big)\geqq 0$$
$$\text{a.e.} \quad x\in(-1, 1).$$

Since the left hand side of (8) belongs to the Hardy class of index one, it can be continued analytically to every $x\in(-1, 1)$. By Rudin's lemma [19], this together with (7) implies f_0E is continued analytically to $(-1, 1)$ and clearly $|f_0e^{-\chi}|=M_0$ on $\partial\Omega$.

We will show uniqueness. Let f_0^* be any extremal function. By the same argument, we have $|f_0|=|f_0^*|$ on $\partial\Omega$. From (8) after being analytically continued we conclude that $\arg f_0=\arg f_0^*$. We have $f_0=f_0^*$ on $\partial\Omega$ whence $f_0\equiv f_0^*$. This completes the proof.

Remark. If we drop the assumption that K does not separate the boundary, there might be components of $\Omega-K$ for which $(\sum_{\nu=1}^{N} c_\nu l_\nu+\varphi_0)dz\equiv 0$. Then we have no information about the value of f_0 on the part of $\partial\Omega$ on those components except that $|f_0(z)e^{-\chi(z)}|\leqq M_0$. However, since there is a component of $\Omega-K$ for which the differential does not vanish identically the uniqueness of the extremal function still holds. Such an example does exist (Hejhal [11]).

The relation given by (5) and (6) is called a *duality relation* and the differential

(9) $$d\Phi_0=\Big(\sum_{\nu=1}^{N} c_\nu l_\nu+\phi\Big)dz \quad \text{with} \quad \int_{\partial\Omega} e^{\chi}|d\Phi_0|=1$$

is called a *conjugate differential* of f_0 in the Pick-Nevanlinna problem.

Following Garabedian [8], we state

COROLLARY 1. *The conjugate differential $d\Phi_0$ minimizes the integral*

$$\int_{\partial\Omega} e^{\chi}|d\Phi_0+\phi dz|, \quad \phi dz\in Q.$$

Any duality relation for $d\Phi_0$ in (9) *characterizes the extremal function.*

Proof. The extremality of $d\Phi_0$ follows from

$$1=\int_{\partial\Omega}\frac{f_0}{M_0}(d\Phi_0+\phi dz)\leqq\int_{\partial\Omega} e^{\chi}|d\Phi_0+\phi dz|.$$

Next for $f\in\mathfrak{F}$, we have

$$M_0=\int_{\partial\Omega} f\, d\Phi_0\leqq\chi-\|f\|\int_{\partial\Omega}|e^{\chi}d\Phi_0|=\chi-\|f\|.$$

7. *The classical Pick-Nevanlinna problem.* Let Ω be a bordered Riemann surface. To discuss this problem, we give a datum at a point p_ν in the following form. For a fixed local parameter $z(p)$ around p_ν with $z(p_\nu)=0$, take a finite Taylor section

$$D_\nu(z)=\sum_{j=0}^{n_\nu} a_j^{(\nu)} z^j .$$

For given N points $\{p_\nu\}_{\nu=1}^N$ and associated Taylor sections $\{D_\nu\}_{\nu=1}^N$, consider the family $\mathfrak{F}$ of analytic functions in Ω satisfying that $f(z)-D_\nu(z)$ has a zero of order at least $n_\nu+1$ at p_ν, $\nu=1, \cdots, N$. Then we show

THEOREM 3. *If*

$$M_0=\inf_{f\in\mathfrak{F}} \|f\|>0 ,$$

there exists a unique extremal function $f_0\in\mathfrak{F}$ such that $|f_0|=M_0$ on $\partial\Omega$. Here $\|\ \|$ denotes the supnorm of f. If Ω is of genus g and has h contours, then f_0 maps Ω onto an at most $\sum_{\nu=1}^N(n_\nu+1)+2g+h-2$ sheeted disc $|z|<M_0$.

Proof. Let K be the union of N mutually disjoint closed parameter discs Δ_ν of $\{p_\nu\}_{\nu=1}^N$. We may suppose that Δ_ν corresponds to $|z|\leqq 1$ with $z(p_\nu)=0$, $\nu=1, \cdots, N$. Let $\{L_j\}_{j=1}^{N_0}$, $N_0=\sum_{\nu=1}^N(n_\nu+1)$, be linear functionals defined by $L_1(f)=f(p_1)$, $L_2(f)=f'(p_1), \cdots, L_{N_0}(f)=f^{(n_N)}(p_N)$, where derivatives are those with respect to the respective local parameters. Then the present problem is equivalent to the Pick-Nevanlinna problem for those data. We remark that in this case L_j is explicitly expressed by a measure $d\mu_j$ on $\bar{\Delta}_j$: $d\mu_1=(2\pi i z)^{-1}dz$ on $|z|=1$; $=0$ in the interior of $\Delta_1, \cdots, d\mu_{N_0}=(2\pi i z^{n_N+1})^{-1}n_N!\ dz$ on $|z|=1$; $=0$ in the interior of Δ_N. From Theorem 2, we obtain a unique extremal function f_0 and its conjugate differential $d\Phi_0$ satisfying

$$f_0 d\Phi_0\geqq 0 \quad \text{along } \partial\Omega .$$

We call the divisor $\delta=p_1^{n_1}p_2^{n_2}\cdots p_N^{n_N}$ the *interpolation divisor* of the (classical) Pick-Nevanlinna problem and N_0 the *degree* of δ. It is easy to construct a function $F(z)$ which is analytic on $\bar{\Omega}$ and has the same divisor as δ (Heins [10]). For every $f\in\bar{A}(\Omega)$, $fF\in S$ and

$$\int_{\partial\Omega} f\,F\,d\Phi_0=0 .$$

Again by Royden's result [18] cited in No. 5, $Fd\Phi_0$ is an analytic differential on $\bar{\Omega}$. Thus $d\Phi_0$ is an meromorphic differential which is a multiple of δ^{-1}.

Let $\hat{\Omega}$ be the double of Ω. Since $f_0d\Phi_0\geqq 0$ along $\partial\Omega$, $f_0d\Phi_0$ can be analytically continued to $\hat{\Omega}$. The genus of $\hat{\Omega}$ is equal to $2g+h-1$. Hence the degree of $(f_0d\Phi_0)$ is equal to $2(2g+h-2)$. Since $f_0d\Phi_0$ has possible poles at p_j, the sum of whose orders is equal to at most N_0 in $\bar{\Omega}$, by symmetry the amount of the orders of the poles $f_0 d\Phi_0$ on $\hat{\Omega}$ is not greater than $2N_0$. Hence the number of zeros on Ω is at most $2g+h-2+N_0$.

§ 3. Generalizations.

8. *The Pick-Nevanlinna problem for arbitrary domains.* Let Ω be a subdomain of a compact Riemann surface R of genus g. Suppose that Ω does not belong to O_{AB}. We may suppose that the genus of Ω is equal to g. Let $\chi(z)$ be a real valued harmonic function in Ω. As in No. 5, we define the χ-norm by (2) and set

$$X=\{f \mid f \text{ is analytic in } \Omega \text{ and } \chi\text{-}\|f\|<\infty\}.$$

To avoid a trivial case $\chi(z)$ is supposed to be taken so that $X\neq\{0\}$. We take a canonical exhaustion $\{\Omega_n\}_{n=1}^{\infty}$ of Ω. Let $\chi\text{-}\|f\|_n$ denote the χ-norm of f in Ω_n. Let there be given N linear functionals $\{L_\nu\}_{\nu=1}^{N}$, each of which is continuous with respect to the supnorm $\|f\|_K$ of f on a compact subset K of Ω. We may suppose $K\subset\Omega_1$. Moreover we suppose that K does not separate the boundary of Ω. Then each Ω_n enjoys the same property with respect to $\partial\Omega_n$ and K. Let $\mathfrak{F}_n$ be the family of functions f analytic in Ω_n and satisfying $L_\nu(f)=a_\nu$, $\nu=1, \cdots, N$. We denote by $\mathfrak{F}$ the corresponding family in Ω.

Suppose

$$0<M_0=\inf_{f\in\mathfrak{F}}\chi-\|f\|<\infty .$$

Then a normal family argument shows that there exists an extremal function f_0 such that $M_0=\chi-\|f_0\|$, $L_\nu(f_0)=a_\nu$, $\nu=1, \cdots, N$. By Proposition 1, we get a linear combination

$$\Psi_0=\sum_{\nu=1}^{N} c_\nu L_\nu$$

such that

$$\Psi_0(f_0)=\max_{\|\Psi\|=1,\Psi\in S^\perp}|\Psi(f_0)|=\chi-\|f_0\| ,$$

and that f_0 is an extremal function for the Pick-Nevanlinna problem with a single datum $\Psi_0(f)=\sum_{\nu=1}^{N} c_\nu a_\nu$.

Now we consider the Pick-Nevanlinna problems for the norm $\chi\text{-}\|f\|_n$ with a single datum $\Psi_0(f)=\sum_{\nu=1}^{N} c_\nu a_\nu$ in each Ω_n. Then there exists a unique extremal function F_n and its conjugate differential $d\Phi_n$ satisfying $|e^{-\chi(z)}F_n(z)|=M_n$ on $\partial\Omega_n$. On the other hand there exists a unique extremal function f_n satisfying

$$M_n'=\chi\text{-}\|f_n\|=\inf_{f\in\mathfrak{F}_n}\chi\text{-}\|f\|.$$

Those are direct results from Theorem 2. Clearly $M_n\leqq M_n'$. We state

THEOREM 4. *The sequence $\{f_n\}$ converges to a unique extremal function f_0 uniformly on every compact subset of Ω.*

9. *A Cauchy kernel.* Before proving Theorem 4 we prepare

LEMMA 2. *Let q be a non-Weierstrass point of R in Ω. Then for a fixed parameter disc U of q there exists a meromorphic function $C(p, r)$, $r\in U$, called a Cauchy kernel, satisfying the condition that $C(p, r)$ is analytic apart from $p=r$, at which it has an expansion*

$$C(p, r)=\frac{1}{2\pi i(z(p)-\zeta)}+\text{regular terms},\ p\in U,\ \zeta=z(r).$$

and that $C(p, r)$ is uniformly bounded in a neighborhood of the boundary of Ω if r lies in any compact subset of U.

Proof. We take a parameter disc U whose closure contains no Weierstass points. Then for $r\in U$ there exists a meromorphic function $Q(p, r)$ on R with a single pole of order $g+1$ at $p=r$, where it has an expansion

$$Q(p, r)=\frac{1}{2\pi i(z(p)-\zeta))^{g+1}}+\frac{b_g}{(z(p)-\zeta)^g}+\cdots+\frac{b_1}{z(p)-\zeta}$$
$$+\text{regular terms},\ p\in U,\ \zeta=z(r)$$

Such a function is uniquely determined as a definite integral of a linear combination of normalized differentials of the first and the second kind. From the symmetry law [22], we can deduce that those differentials and their periods are continuous with respect to the parameter r.

Let $f_0(p, r)$ be the Ahlfors function of Ω which satisfies $f_0(r, r)=0$ and

$$\frac{d}{dz}f_0(p(z), r)|_{z=\zeta}=\max|df(p(z))/dz|_{z=\zeta}|,$$
$$|f|\leqq 1,\ f(r)=0.$$

It is easy to see the Ahlfors function is unique for such a Riemann surface Ω. In fact, in order to use Fisher's method [5] we need a meromorphic function with a simple pole at r, analytic in $\Omega-r$ and bounded near the boundary. Since $\Omega\notin O_{AB}$, there exists a bounded function $h(p)$ which has a zero of some order, say n, at r. On the compact Riemann surface R containing Ω, there exist meromorphic functions $g(p)$ with a single pole at r. The order of the pole can be taken as an arbitrary positive integer m greater than an integer m_0. Set $m=nk+1>m_0$. Then $h^k g$ is a desired function. Write $f_0(p, r)=f_0(z, \zeta)$. We show that $f_0'(\zeta, \zeta)$ is continuous with respect to ζ, $r\in U$. Since

$$\frac{f_0(z, \zeta')-f_0(\zeta, \zeta')}{1-\overline{f_0(\zeta, \zeta')}f_0(z, \zeta')},\ r'\in U,\ \zeta'=z(r')$$

denotes a competing function in the problem to determine the maximum at ζ, we have

$$f_0'(\zeta, \zeta) \geqq \frac{|f_0'(\zeta, \zeta')|}{1-|f_0(\zeta, \zeta')|^2},$$

whence

$$f_0'(\zeta', \zeta')-f_0'(\zeta, \zeta) \leqq f_0'(\zeta', \zeta')-\frac{|f_0'(\zeta, \zeta')|}{1-|f_0(\zeta, \zeta')|^2}.$$

Similarly we have

$$f_0'(\zeta, \zeta)-f_0'(\zeta', \zeta') \leqq f_0'(\zeta, \zeta)-\frac{|f_0'(\zeta', \zeta)|}{1-|f_0(\zeta', \zeta)|^2}.$$

Since by uniqueness

$$\lim_{\zeta' \to \zeta} f_0(z, \zeta')=f_0(z, \zeta)$$

uniformly on $\bar{U}$, we get the continuity of $f_0'(\zeta, \zeta)$ for $r \in U$.

Now set

$$C(p, r)=Q(p, r) f_0(p, r)^g / f_0'(\zeta, \zeta)^g, \quad r=r(\zeta) \in U$$

Then $C(p, r)$ has the desired singularity and since $f_0'(\zeta, \zeta)$ is continuous and positive [21], $C(p, r)$ is uniformly bounded in a neighborhood of $\partial\Omega$ if r lies in any compact subset of U.

10. *Proof of Theorem* 4. Under the notations in No. 8, we have $M_n \leqq M_n' \leqq M_0$ and $\lim M_n=M_0$. Clearly $\{F_n\}$ forms a normal family. If we establish the uniqueness of the extremal function, the proof of Theorem 4 is completed. Hence we may suppose that $\{F_n\}$ tends to a function F_0 uniformly on every compact subset of Ω. It is easily verified that F_0 is an extremal function of the original Pick-Nevanlinna Problem in Ω.

Define a transform l_ν of L_ν by (3). For every f analytic in Ω_n and with $\chi-\|f\|_n<\infty$, there holds

$$L_\nu(f)=\int_{\partial\Omega} f(z) l_\nu(z) dz, \quad \nu=1, \cdots, N.$$

Hence for a conjugate differential $d\Phi_n$ of F_n, we have an expression

$$d\Phi_n=(k_n \sum_{\nu=1}^{N} c_\nu l_\nu(z)+\psi_n) dz \quad \text{along} \quad \partial\Omega_n.$$

Here k_n is defined by

$$k_n=(\sum_{\nu=1}^{N} c_\nu a_\nu)^{-1} M_n=\frac{M_n}{M_0},$$

and $\psi_n\, dz$ belongs to the class Q on Ω_n.

We shall show that $\{F_0 \psi_n dz\}$ forms a normal family in the following sence: for every compact $B \subset \Omega$ and a nonvanishing analytic differential ω_B on B, $\{F_0 \psi_n dz/\omega_B\}$ is normal on B.

Take a non-Weierestrass point p_0 in $\Omega-\bar{\Omega}_1$ and its parameter disc U in Lemma 2 and contained in $\Omega-\bar{\Omega}_1$. We have

$$F_0(z)(\psi_n(z)+k_n\sum_{\nu=1}^{N}c_\nu l_\nu(z))$$

$$=\int_{\partial\Omega_n-\partial\Omega_1}F_0(t)(\psi_n(t)+k_n\sum_{\nu=1}^{N}c_\nu l_\nu(t))C(t,\ z)dt\,,$$

$$C(q(t),\ p(z))=C(t,\ z),\ p(z)\in U\,.$$

By Lemma 2, $|C(t,\ z)|\leqq L$ on $\partial\Omega_n\times U$ for sufficiently large n. We have

$$\left|\int_{\partial\Omega_n}F_0(\psi_n+k_n\sum_{\nu=1}^{N}c_\nu l_\nu)C(t,\ z)dt\right|\leqq LM_0\,,$$

since $|e^{-\chi}F_0|\leqq M_0$ on Ω and $\int_{\partial\Omega_n}|\psi_n+k_n\sum_{\nu=1}^{N}c_\nu l_\nu|e^{\chi}|dt|=1$, and

$$\int_{\partial\Omega_1}F_0\psi_nC(t,\ z)dt=0\,,$$

since $C(q,\ p)$ is analytic in Ω_1. Since $k_n\rightarrow 1$ $(n\rightarrow\infty)$, $\{F_0\psi_n\}$ is uniformly bounded on U. Since the set of all Weierstrass points of R is finite, we may suppose that $\partial\Omega_n$, $n=1,\ 2,\ \cdots$ contains no Weierstrass points. Then $\partial\Omega_n$, $n\geqq 2$ is covered by a finite number of parameter discs for which $\{F_0\psi_n\}$ is uniformly bounded. Then for a non-vanishing analytic differential ω_{Ω_m} on $\bar{\Omega}_m$ by the maximum principle $\{F_0\psi_ndz/\omega_{\Omega_m}\}_{n=m}^{\infty}$ is uniformly bounded on $\bar{\Omega}_m$. Hence $\{F_0\psi_ndz/\omega_{\Omega_m}\}$ forms a normal family. Changing notations we may suppose that $\{F_0\psi_ndz/\omega_{\Omega_m}\}$ converges to $F_0\psi_0dz/\omega_{\Omega_m}$ uniformly on $\bar{\Omega}_m$. We say that $\{F_0\psi_ndz\}$ *converges* to an analytic differential $f_0\psi_0dz$. Since convergence of $\{F_0\psi_ndz/\omega_{\Omega_m}\}$ implies that of $\{\psi_ndz/\omega_{\Omega_m}\}$, $\{\psi_ndz\}$ converges to an analytic differential ψ_0dz uniformly on every compact subset of Ω. Note that $F_0\not\equiv 0$. Thus $d\Phi_n=(k_n\sum_{\nu=1}^{N}c_\nu l_\nu+\psi_n)dz$ tends to an analytic differential $d\Phi_0=(\sum_{\nu=1}^{N}c_\nu l_\nu+\psi_0)dz$ in $\Omega-K$.

Now Hejhal's method [11] provides the uniqueness of the extremal functions. Let $F_0{}^*$ be any extremal function of the Pick-Nevanlinna problem in Ω. Let $C(\Omega)$ be the Banach space of complex valued functions h continuous in Ω and normed by the supnorm $\|h\|$. We define three sequeces of linear functionals

$$T_n^{(1)}(h)=\int_{\partial\Omega_n}hF_nd\Phi_n\,,$$

$$T_n^{(2)}(h)=\int_{\partial\Omega_n}h\frac{M_n}{M_0}F_0d\Phi_n\,,$$

$$T_n^{(3)}(h)=\int_{\partial\Omega_n}h\frac{M_n}{M_0}F_0{}^*d\Phi_n,\ n=1,\ 2,\ \cdots$$

We have $\|T_n^{(j)}\| \leqq M_n \leqq M_0$, $j=1, 2, 3$. In fact

$$\int_{\partial\Omega_n} |F_n d\Phi_n| = \int_{\partial\Omega_n} F_n d\Phi_n = M_n ,$$

and $|F_0 e^{-\chi}|$ and $|F_0^* e^{-\chi}|$ are bounded by M_0 in Ω. By Hejhal's lemma ([11] p. 102) for those three sequences we can find a subnet $\{n\alpha\}$ of $\{n\}$ for which they have a common weak star limit T_0.

We take a non-Weierstrass point p_0 and its parameter disc U so near to the boundary $\partial\Omega$ that there exists an Ahlfors function $f_0(p, r)$ such that

$$\inf_U |f_0(p, r)| > \max_{\partial\Omega_1} |f_0(p, r)|$$

Since $\Omega \notin O_{AB}$ and since $\sup_\Omega |f_0(p, r)|=1$, such a point p_0 exists. We have

$$\frac{M_n}{M_0}(F_0(p)-F_0^*(p))\Phi_n'(p(z))$$

$$=\int_{\partial\Omega_n-\partial\Omega_1} \frac{M_n}{M_0}(F_0-F_0^*)\left(\frac{f_0(q, r)}{f_0(p, r)}\right)^n C(q, p)d\Phi_n(q) ,$$

$$\Phi_n'(p(z))=k_n\left(\sum_{\nu=1}^{N} c_\nu l_\nu(z)+\psi_n(z)\right),\ k_n=M_n/M_0 .$$

Taking the subnet $\{n\alpha\}$ and letting it tend to ∞, the integral along $\partial\Omega_1$ tends to zero. Since $C(q, p)$ is regarded as the restriction of a bounded continuous function to a boundary neighborhood, the integral along $\partial\Omega_{n\alpha}$ also tends to zero. We get $(F_0(p)-F_0^*(p))d\Phi_0(p)\equiv 0$ in U. Since $\int_{\partial\Omega_1} F_0 d\Phi_0 = M_0 > 0$, $d\Phi_0 \not\equiv 0$. We get $F_0 \equiv F_0^*$ in Ω which completes the proof.

§4. Special problems

11. *Meromorphic functions.* In this section we consider the classical Pick-Nevanlinna problem for wider classes of functions. For simplicity we restrict ourselves to a plane domain Ω. For a moment, suppose that Ω is bounded by k analytic Jordan curves ($k \geqq 1$). Let t_j, $j=1, \cdots, l$, be a finite number of mutually distinct points. We take $\chi(z)$ as the superharmonic function

$$\chi(z)=\sum_{j=1}^{l} m_j g(z, t_j) ,$$

m_j being positive integers and $g(z, t)$ being the Green's function of Ω. Let $\{\zeta_\nu\}_{\nu=1}^{N}$ be mutually distinct points in Ω. We give data at ζ_ν by Taylor or Laurent sections:

$$D_\nu(z)=\sum_{m=0}^{N_\nu} a_m^{(\nu)}(z-\zeta_\nu)^m,\ \zeta_\nu \neq t_j , \tag{10}$$

$$D_\nu(z)=\sum_{m=-m_j}^{N_\nu} a_m^{(\nu)}(z-\zeta_\nu)^m,\ \zeta_\nu = t_j .$$

Let $\mathfrak{F}$ be the family of functions meromorphic in Ω which have the given data at ζ_ν. Our problem is to minimize

$$\chi\text{-}\|f\| = \sup |f(z)e^{-\chi(z)}| .$$

Write M_0 for the minimum and suppose $M_0 > 0$.

The set X of meromorphic functions f defined by

$$X = \{f \mid \chi\text{-}\|f\| < \infty\}$$

forms a Banach space. Every function $f \in X$ has possible poles at t_j of order m_j.

Let h be a function analytic on $\bar{\Omega}$ which vanishes at t_j precisely of order m_j and has no zeros other than t_j. We can set $h = \prod_{j=1}^{l} (z-t_j)^{m_j}/(z-\xi)^{m_j}$, $\xi \notin \bar{\Omega}$.

Then hf is a bounded analytic function if $f \in X$. Consider a Banach space X_h of bounded analytic functions f normed by $\chi_h\text{-}\|f\| = \sup |f(z)h(z)^{-1}e^{-\chi(z)}|$, $z \in \Omega$. By this transformation we have the new data

$$D_\nu^*(z) = [h(z)D_\nu(z)]_{N_\nu + m_\nu}, \tag{11}$$

where $[\ \ ]_{N_\nu+m_\nu}$ denotes a Taylor section up to order $N_\nu + m_\nu$ with $m_\nu = 0$ if $\zeta_\nu \neq t_j$. Since $-\log|h| - \chi(z)$ is harmonic on $\bar{\Omega}$, Theorem 3 is applicable. We get

THEOREM 5. *There exists a unique extremal function f_0 and its conjugate differential $d\Phi_0$ for the Pick-Nevanlinna problem with data* (10). *f_0 is meromorphic on $\bar{\Omega}$ and $|f_0|$ is equal to M_0 on $\partial\Omega$. $d\Phi_0$ is a meromorphic differential on $\bar{\Omega}$ which is a multiple of the divisor $\delta = \prod_{\nu=1}^{N} \zeta_\nu^{-N_\nu - m_\nu - 1} \prod t_j^{m_j}$, when j runs over non-interpolation points t_j. $d\Phi_0$ satisfies*

$$\int_{\partial\Omega} |d\Phi_0| = 1$$

and the duality relation

$$f_0 d\Phi_0 \geqq 0 \quad \text{along} \quad \partial\Omega ,$$

which characterizes the extremal function f_0.

Proof. It is easy to see that the extremal function F_0 for the transformed Pick-Nevanlinna problem with the data (11) in χ_h yields the solution $f_0 = F_0 h^{-1}$. $\sup |F_0(z)h^{-1}(z)e^{-\chi(z)}| = M_0$ on $\partial\Omega$ and $e^{-\chi(z)} = 1$ on $\partial\Omega$. Thus $|f_0| = M_0$ on $\partial\Omega$. As in the proof of Theorem 2, there exists a conjugate differential $d\Phi_1$ for F_0 satisfying

$$\int_{\partial\Omega} |he^{\chi} d\Phi_1| = 1$$

and $F_0\, d\Phi_1 \geqq 0$ along $\partial\Omega$. Put $d\Phi_0 = h\, d\Phi_1$. We have

$$\int_{\partial\Omega} |d\Phi_0| = 1$$

and $f_0\, d\Phi_0 \geqq 0$ along $\partial\Omega$. As in the proof of Theorem 3 $d\Phi_0$ is a multiple of $\prod_{\nu=1}^{N} \zeta_\nu^{-N_\nu-1}$. Hence $d\Phi_0 = h\, d\Phi_1$ has the desired property concerning the divisor of $d\Phi_0$. The uniqueness of f_0 and its characterization follow from Theorem 2 and Corollary 1.

Many years ago R. M. Robinson [17] proved a generalization of the maximum principle for meromorphic function with one pole in an annulus Ω, $R^{-1}<|z|<R$ $(R>1)$. It follows from a solution of our extremal problem. We state it as the following

COROLLARY 2. *Let $f(z)$ be meromorphic in an annulus Ω, $R^{-1}<|z|<R$ $(R>1)$, which has possibly one simple pole at $z=-t$, $R^{-1}<t<R$. If $\overline{\lim}_{z\to\partial\Omega}|f(z)|\leqq 1$, then $|f(x)|\leqq 1$ for $R^{-1}<x<R$.*

Proof. The Green's function $g(z, x)$ of Ω yields a positive differential

$$d\Phi_0 = \frac{i}{\pi}\frac{\partial}{\partial z} g(z, x)dz > 0 \qquad \text{along} \quad \partial\Omega .$$

Since $g(z, x)$ is symmetric with respect to the real axis, it has a critical point t_0 on $(-R, -R^{-1})$. By the argument principle there exists only one critical point. Set $d\omega = idz/z$ which is positive along $|z|=R^{-1}$ and negative along $|z|=R$. We construct positive differentials

$$d\Phi_\lambda = d\Phi_0 + \lambda d\omega ,$$

where the real parameter λ runs on a interval $(-\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2>0$ so that $d\Phi_\lambda$ remains positive along $\partial\Omega$. The zero of $d\Phi_\lambda$ moves from t_0 to $-R$ and from t_0 to $-R^{-1}$ as λ moves from 0 to λ_2 and from 0 to $-\lambda_1$ respectively. The there exists a λ_t such that $d\Phi_{\lambda_t}=0$ at $z=-t$.

Consider the Pick-Nevanlinna problem with single datum $f(x)=1$ for the class of meromorphic functions with possible pole at $-t$. Here the norm of f is given by $\chi-\|f\| = \sup|f(z)e^{-g(z,t)}|$, $z\in\Omega$, $\chi=g(z, t)$. Clearly $f_0(z)\equiv 1$ is an extremal function. Indeed

$$\int_{\partial\Omega} d\Phi_{\lambda_t} = \int_{\partial\Omega} |d\Phi_{\lambda_t}| = 1$$

and

$$f_0 d\Phi_{\lambda_t} > 0 \qquad \text{along} \quad \partial\Omega$$

is a duality relation. If $|f(x)|>1$ we get a contradiction $\chi\text{-}\|f/f(x)\|<1$.

12. *General domains.* Let Ω be a plane domain which does not belong to O_G. For the Green's function $g(z, t)$ of Ω we set as before

$$\chi(z) = \sum_{j=1}^{l} m_j g(z, t_j),\ t_j\in\Omega .$$

We consider the Pick-Nevanlinna problem with data (10) for the same as in

No. 11. We take a canonical exhaustion $\{\Omega_n\}_{n=1}^{\infty}$ such that all t_j and $\zeta_\nu \in \Omega_1$. Then from Theorem 5 we obtain a sequence of extremal functions $\{f_n\}$ and conjugate differentials $\{d\Phi_n\}$ with $\chi_n = \sum_{j=1}^{l} m_j g_n(z, t_j)$, $g_n(z, t_j)$ being the Green's function of Ω_n. We show

THEOREM 6. *If*

$$0 < M_0 = \inf_{f \in \mathfrak{F}} \chi\text{-}\|f\| < \infty ,$$

the sequence of extremal functions $\{f_n\}$ *converges to a unique extremal function* f_0 *uniformly on every compact subset of* Ω. *Here* $\mathfrak{F}$ *is as in No.* 11.

Proof. Consider the Pick-Nevanlinna problem with the same data (10) for the norm $\chi\text{-}\|f\|_n = \sup |f(z)e^{-\chi(z)}|$, $z \in \Omega_n$, in Ω_n. Let $\mathfrak{F}_n$ be the corresponding family for the problem.

Then $M_n = \inf_{f \in \tilde{\mathfrak{F}}_n} \chi\text{-}\|f\|_n > 0$ and is increasing with respect to n. In fact, if $M_n = 0$, all the data vanish and $M_0 = 0$. The monotonicity of M_n follows from the fact that $\tilde{\mathfrak{F}}_{n+1} \subset \tilde{\mathfrak{F}}_n$. Similarly as in the proof of Theorem 5, there exists a unique extremal function $\tilde{f}_n$ and its conjugate differential $d\tilde{\Phi}_n$ which satisfy

$$|\tilde{f}_n e^{-\chi(z)}| = M_n , \qquad z \in \partial\Omega_n , \tag{12}$$

$$\tilde{f}_n d\tilde{\Phi}_n \geqq 0 \qquad \text{along } \partial\Omega_n , \tag{13}$$

and

$$M_n = \int_{\partial\Omega_n} \tilde{f}_n d\tilde{\Phi}_n, \text{ with } \int_{\partial\Omega_n} e^{\chi} |d\tilde{\Phi}_n| = 1 . \tag{14}$$

$d\tilde{\Phi}_n$ is a multiple of $\prod_{\nu=1}^{N} \zeta_\nu^{-N\nu-1-m\nu} \prod t_j^{m_j}$, where j runs over non-interpolation points t_j. Since $|\tilde{f}_n e^{-\chi(z)}|$ satisfies the maximum principle, $|\tilde{f}_n e^{-\chi(z)}|$ is uniformly bounded. Then we may suppose that $\{\tilde{f}_n\}$ tends to a meromorphic function f_0 uniformly on every compact subset of Ω. Here spherical distances are taken in the convergence. It is easy to see that f_0 is one of the extremal functions in $\mathfrak{F}$. On the other hand, by the same reason, $|f_n e^{-\chi_n}|$ is also bounded and every infinite subsequence of $\{f_n\}$ contains a subsequence converging to an extremal function. If we show that the extremal function f_0 is unique, the proof is complete.

Let S be a subspace of X consisting of meromorphic functions with vanishing data, that is, $D_\nu(z) \equiv 0$, $\nu = 1, 2, \cdots, N$.

If $S = \{0\}$, then the extremal f_0 is unique. Next suppose that $h \not\equiv 0$ belongs to S. For the sequence of conjugate differentials $\{d\tilde{\Phi}_n\}$, we have

$$\int_{\partial\Omega_n} |h\, d\tilde{\Phi}_n| = \int_{\partial\Omega_n} |h e^{-\chi} \cdot e^{\chi} d\tilde{\Phi}_n| \leqq \chi\text{-}\|h\| .$$

Since

$$h(z)\tilde{\Phi}_n'(z) = \frac{1}{2\pi i} \int_{\partial\Omega_n} \frac{h(\zeta) d\tilde{\Phi}_n(\zeta)}{\zeta - z} ,$$

$\{h(z)\tilde{\Phi}_n'(z)\}$ is locally uniformly bounded. Hence we may suppose that $\{\tilde{\Phi}_n'\}$ tends to a meromorphic function $\Phi_0'(z)$ uniformly on every compact subset of Ω. Since

$$M_n=\int_{\partial\Omega_n}\tilde{f}_n d\tilde{\Phi}_n$$

from (14), $\Phi_0(z)\not\equiv 0$.

Let f_0^* be any other extremal function. We define three sequences of measures,

$$d\mu_n=\begin{cases}\dfrac{\tilde{f}_n d\tilde{\Phi}_n}{M_n} & \text{on } \partial\Omega_n\\ 0 & \text{otherwise}\end{cases}$$

$$d\tilde{\mu}_n=\begin{cases}\dfrac{f_0 d\tilde{\Phi}_n}{M_0} & \text{on } \partial\Omega_n\\ 0 & \text{otherwise}\end{cases}$$

and

$$d\mu_n^*=\begin{cases}\dfrac{f_0^* d\tilde{\Phi}_n}{M_0} & \text{on } \partial\Omega_n\\ 0 & \text{otherwise}.\end{cases}$$

We have $\|d\mu_n\|=1$ from (13) and (14), $\|d\tilde{\mu}_n\|\leqq 1$ and $\|d\mu_n^*\|\leqq 1$, since $|f_0e^{-\chi_n}|\leqq 1$, $|f_0^*e^{-\chi_n}|\leqq 1$ on $\partial\Omega_n$.

By taking subsequences, we may suppose that

$$\{d\mu_n\}\to d\mu_0,\ \{d\tilde{\mu}_n\}\to d\tilde{\mu}_0,\ \{d\mu_n^*\}\to d\mu_0^*$$

in weak star convergence as $n\to\infty$. Since f_n, f_0 and f_0^* have the same data, we find by the calculus of residues

$$\|d\mu_n\|=\int d\mu_n=1$$

and

$$\int d\tilde{\mu}_n=\int d\mu_n^*=\frac{M_n}{M_0}$$

Clearly $|d\tilde{\mu}_n|\leqq d\mu_n$ and $|d\mu_n^*|\leqq d\mu_n$. Since $M_n/M_0\to 1$ as $n\to\infty$, we have $d\mu_0=d\tilde{\mu}_0=d\mu_0^*$. For $h=f_0-f_0^*\in S$

$$(f_0(z)-f_0^*(z))\Phi_n'(z)=\frac{1}{2\pi i}\int\frac{1}{\zeta-z}(d\tilde{\mu}_n-d\mu_n^*)\to 0$$

as $n\to\infty$, since $1/(\zeta-z)$ is continuous in a neighborhood of $\partial\Omega$.

$$\Phi_0'(z)=\lim\Phi_n'(z)\not\equiv 0.$$

We have $f_0(z)\equiv f_0{}^*(z)$.

Remark. The case where $S=\{0\}$ does exist. For example, suppose that Ω belongs to $O_{AB}-O_G$. Set $\chi(z)=-g(z, t)$, $t\in\Omega$. We give a datum $D_t(z)=(z-t)^{-1}$ so that $f(z)-D_t(z)=0$ at $z-t\neq\infty$. Then the extremal function is $(z-t)^{-1}$ and $S=\{0\}$.

13. *Function with characters.* Extremal problems for a class of mutiple-valued functions were recently dealt with by Widom [23]. Let f be a multiple-valued function whose modulus is single-valued. Then continuation of a function element of f along a closed curve c results in multiplication by a constant $\Gamma_f(c)$. $\Gamma_f(c)$ depends on the homotopy class containing c. Since $\Gamma_f(c)$ is given by

$$\Gamma_f(c)=e^{\int_c(f'/f)dz},$$

$\Gamma_f(c)$ is a character on the homotopy group $\pi(\Omega)$ of Ω or the homology group $H_1(\Omega)$. We consider the classical Pick-Nevanlinna problem with fixed $\Gamma(c)$.

Let Ω be a plane domain bounded by k analytic curves. In this case we fix a point $\zeta_0\in\Omega$. We give N data at $\{\zeta_\nu\}_{\nu=1}^N$, $\zeta_\nu\in\Omega$

$$D_\nu(z)=\sum_{j=0}^{n_\nu} a_j^{(\nu)}(z-\zeta_\nu)^j \tag{15}$$

and fix a system of curves $\{\gamma_\nu\}_{\nu=1}^N$ such that γ_ν connects ζ_0 with ζ_ν. Let $\mathfrak{F}_\Gamma$ be a family of functions f multiple-valued, analytic, with character Γ, and such that the analytic continuation of a fixed element of f along γ_ν has the same Taylor section (15).

Then we show

THEOREM 7. *Under the nontriviality condition*

$$M_0=\inf_{f\in\mathfrak{F}_\Gamma}\|f\|>0,$$

there exists a unique extremal function f_0 which minimizes $\|f\|$. $|f_0|=M_0$ on $\partial\Omega$ and f_0 has at most

$$\sum_{\nu=1}^N(n_\nu+1)+k-2$$

zeros.

Proof. It is easy to transform the problem into that for single-valued functions. Let $\{C_j\}_{j=1}^{k-1}$ be the boundary contours except C_k, which form a homology basis. Let $\omega_j(z)$ be the harmonic measure of C_j. We construct a linear combination $\chi(z)=\sum_{j=1}^{k-1}x_j\omega_j(z)$ so that

$$\int_{C_i}\sum_{j=1}^{k-1}x_j d\omega_j{}^*=-\Gamma(C_i),\ i=1,\cdots,k-1.$$

This has clearly a solution [1]. Set $E(z)=e^{\chi(z)+i\chi^*(z)}$ where $\chi^*(z)$ is a conjugate harmonic function of $\chi(z)$. Then for $f\in\mathfrak{F}_\Gamma$, $fE(z)$ is single-valued. By taking Taylor sections of $D_\nu(z)E(z)$ up to n_ν, we get the new data,

$$D_\nu^*(z)=\sum_{j=0}^{n_\nu} b_j^{(\nu)}(z-\zeta_\nu)^j \,. \tag{16}$$

We consider the Pick-Nevanlinna problem with data (16) for the Banach space of analytic functions f normed by $\chi-\|f\|=\sup|f(z)e^{-\chi(z)}|$, $z\in\Omega$. Then Theorem 2 is applicable and we obtain a unique extremal function F_0 with $|F_0(z)e^{-\chi(z)}|=M_0$ on $\partial\Omega$. Now it is easy to check that $f_0=F_0/E$ is the desired extremal function.

To examine the number of zero points of f_0, take a conjugate differential $d\Phi_0$ of F_0, which has poles of order at most $n_\nu+1$ at ζ_ν. Since $F_0d\Phi_0\geqq 0$ along $\partial\Omega$, it is continued analytically onto the double $\hat{\Omega}$ of Ω. Similarly as in the proof of Theorem 3, we get the bounds of the number of zeros of F_0, $\sum_{\nu=1}^{N}(n_\nu+1)+k-2$. Since $E\neq 0$, the proof is complete.

Let Ω be an arbitrary domain $\notin O_G$. Let a character Γ on $H_1(\Omega)$ be given. We consider the classical Pick-Nevanlinna problem with data (15). We take an exhaustion $\{\Omega_n\}$ on Ω such that Ω_1 contains $\{\zeta_\nu\}_{\nu=1}^{N}$, ζ_0 and $\{\gamma_\nu\}_{\nu=1}^{N}$. The restriction of Γ to Ω_n, denoted by Γ_n, is a character on $H_1(\Omega_n)$ and the Pick-Nevanlinna problem with the same data has a unique solution $f_n(z)$. We state

THEOREM 8. *If*

$$0<M_0=\inf_{f\in\mathfrak{F}_\Gamma}\|f\|<\infty\,,$$

the sequence of extremal functions $\{f_n\}$ *converges to a unique extremal function* f_0 *uniformly on every compact subset of* Ω.

The proof is verbatim of that of Theorem 6 and will be omitted.

REFERENCES

[1] AHLFORS, L. V., Complex analysis. McGraw Hill (1958).
[2] BISHOP, E., Subalgebras of functions on a Riemann surface. Pacific J. Math. **8** (1958), 29-50.
[3] CARATHÉODORY, C., AND L. FEJÉR, Über den Zusammenhang der Extremen von harmonischen Funktionen mit ihren Koeffizienten und über den Picard-Landauschen Satz. Rend. Circolo Math. Palermo, **32** (1911), 218-239.
[4] DUREN, P., Theory of H^P spaces. Academic Press (1970).
[5] FISHER, S., On Schwarz's lemma and inner functions. Trans. Amer. Math. Soc. **132** (1969), 229-240.
[6] GAMELIN, T. W., Extremal problems in arbitrary domains. Michigan Math. J. **20** (1972), 3-11.
[7] GAMELIN, T. W., Extremal problems in arbitrary domains II. Ibid. **21** (1974), 297-307.
[8] GARABEDIAN, P. R., Schwarz's lemma and the Szegö kernel function. Trans.

Amer. Math. Soc. **67** (1949), 1-35.

[9] HAVINSON, S. Ja, Extremal problems for bounded analytic functions with interior side conditions. Russian Math. Surveys **18** (1963), 23-96.

[10] HEINS, M., Nonpersistence of the Grenzkreis phenomenon for Pick-Nevanlinna interpolation on annuli. Ann. Acad. Sci. Fenn. Ser. A. **596** (1975), 1-21.

[11] HEJHAL, D. A. Linear extremal problems for analytic functions. Acta Math. **128** (1972), 91-122.

[12] HEJHAL, D. A., Linear extremal problems for analytic function with interior side conditions. Ann. Acad. Sci. Fenn. Ser. A. **586** (1974), 1-36.

[13] LAX, P. D., Reciprocal extremal problems in function theory. Comm. Pure Appl. Math. **8** (1955), 437-453.

[14] NEVANLINNA, R., Ueber beschränkte analytische Funktionen, die in gegebenen Punkten vorgeschriebene Werte annehmen. Ann. Acad. Sci. Fenn. BXV (1919) 71pp.

[15] NEVANLINNA, R., Ueber beschränkte analytische Funktionen. Ibid. A XXXII (1929), 75pp.

[16] PICK, G., Ueber die Beschränkungen analytischen Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden. Math. Ann. **77** (1916), 7-23.

[17] ROBINSON, R. M., Analytic functions in circular rings. Duke Math. J. **10** (1943), 341-354.

[18] ROYDEN, H. L., The boundary values of analytic and harmonic function. Math. Z. **78** (1962), 1-24.

[19] RUDIN, W., Analytic functions of class H_p. Trans. Amer. Math. Soc. **78** (1955), 46-66.

[20] RUDIN, W., Some theorems on bounded analytic functions. Ibid. 333-342.

[21] SARIO, L., AND K. OIKAWA., Capacity functions. springer (1969).

[22] WEYL, H., Die Idee der Riemannschen Fläche. **3**. Aufl. Teubner (1955).

[23] WIDOM, H., The maximum principle for multiple-valued analytic functions. Acta Math. **126** (1971) 63-82.

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