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AN ENTIRE FUNCTION WITH LINEARLY DISTRIBUTED VALUES

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1. Introduction. The purpose of this paper is to prove the following theorem.

THEOREM. *Let $G(z)$ be a transcendental entire function of finite lower order. Assume that all the zero-points $\{a_n\}$ and all the one-points $\{b_n\}$ of $G(z)$ lie on the lines* $\mathrm{Re}\, z=0$ *and* $\mathrm{Re}\, z=1$, *respectively. Then*

$$G(z)=P(\exp Cz)\,,$$

where $P(z)$ is a polynomial and C is a non-zero real constant.

Here of course, $P(z)$ should satisfy suitable conditions, which will be explained in the final section 7. In this theorem it would be interesting to remove the restriction that $G(z)$ be of finite lower order, but we are unable to do this.

The present paper is a continuation of our previous papers [3] and [4], in which we have investigated entire functions with three linearly distributed values. Since we need constantly to refer to these papers [3] and [4], they will be referred to henceforth as *CE* and *EL*, respectively. The notation and terminology generally follow that of *CE* and *EL*.

2. Preliminary results. Let $G(z)$ be an entire function satisfying the assumptions of our theorem. Then from Theorem 4 of *CE*, $G(z)$ has at most order one and mean type, that is,

$$\limsup_{r\to\infty}\frac{T(r,\,G)}{r}<+\infty\,. \tag{2.1}$$

In particular, the genera of $G(z)$ and $G(z)-1$ are at most one. Hence we easily have

$$\mathrm{Re}\frac{G'(z)}{G(z)}=A+\sum_n\frac{\mathrm{Re}\, z}{|z-a_n|^2}\,, \tag{2.2}$$

$$\mathrm{Re}\frac{G'(z)}{G(z)-1}=B+\sum_n\frac{\mathrm{Re}\, z-1}{|z-b_n|^2}\,, \tag{2.3}$$

and

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$$\overline{G(\bar{z})}=G(-z)\exp(2Az+iA'), \tag{2.4}$$

$$\overline{G(\bar{z}+1)}-1=(G(-z+1)-1)\exp(2Bz+iB') \tag{2.5}$$

with suitable real constants A, B, A' and B'. From these (2.4) and (2.5),

$$\begin{aligned}G(z)=&\exp(2Az-iA')\\&+(G(z+2)-1)\exp(2Az-2Bz-2B-iA'+iB').\end{aligned} \tag{2.6}$$

In CE and EL, we have shown the next fact on the real constants A and B.

LEMMA 1. *The quantities A and B which appear in the functional equations* (2.4) *and* (2.5) *must be $AB\neq 0$. Further if AB is negative, then A is negative and B is positive.*

Hence the following six cases may occur.

case 1) $A>B>0$.

case 2) $B>A>0$.

case 3) $A<B<0$.

case 4) $B<A<0$.

case 5) $A<0$ and $B>0$.

case 6) $A=B\neq 0$.

For the case 6), we have proved the following theorem by making use of the functional equation (2.6).

THEOREM 1 (CE; Lemma 11). *If $A=B\neq 0$, then*

$$G(z)=a_1+a_2\exp(Az)+a_3\exp(2Az),$$

where a_1, a_2 and a_3 are constants with $a_1a_3\neq 0$, $a_1\neq 1$.

By this Theorem 1, we therefore obtain a part of the desired result in the case 6).

In the cases 3) and 4), let us consider the function defined by

$$G^*(z)=1-G(1-z).$$

Then we can easily see that the cases 3) and 4) reduce to the cases 2) and 1), respectively. Hence from this point on we may discuss only the cases 1), 2) and 5). The next lemma plays an important role for the cases 1) and 2).

LEMMA 2 (EL; Lemma E). *If $A>B>0$, then*

$$\lim_{r\to+\infty} G(-re^{it})=0$$

uniformly for $|t|\leqq t^*$, *where* t^* *is an arbitrarily fixed number in* $(0, \pi/2)$. *Further if* $B>A>0$, *then*

$$\lim_{r\to+\infty} G(-re^{it})=1$$

uniformly for $|t|\leqq t^*$.

Assume that $A>B>0$. Then by the above Lemma 2 and (2.5), for an arbitrarily fixed number t^* with $0<t^*<\pi/2$, it is possible to find a positive number r^* such that

$$\frac{1}{2}\leqq|G(re^{it})|\exp(2B-2Br\cos t)\leqq\frac{3}{2} \tag{2.7}$$

for $|t|\leqq t^*$ and $r\geqq r^*$. Hence

$$\lim_{r\to+\infty}\frac{\log^+|G(re^{it})|}{r}=2B\cos t \tag{2.8}$$

for values of t with $|t|<\pi/2$. It also follows from Lemma 2 that

$$\lim_{r\to+\infty}\frac{\log^+|G(re^{it})|}{r}=0 \tag{2.9}$$

for values of t with $|t-\pi|<\pi/2$. Here let us note that

$$\log^+|G(re^{it})|\leqq\log^+ M(r, G)\leqq 3T(2r, G),$$

where

$$M(r, G)=\max_{|z|=r}|G(z)|.$$

Since the characteristic function $T(r, G)$ satisfies (2.1), from (2.8) and (2.9), we thus have

$$\lim_{r\to+\infty}\frac{T(r, G)}{r}=\frac{1}{2\pi}\int_0^{2\pi}\lim_{r\to+\infty}\frac{\log^+|G(re^{it})|}{r}\,dt=\frac{2B}{\pi}. \tag{2.10}$$

On the other hand by means of (2.4) and (2.7), for an arbitrarily fixed number t^* with $0<t^*<\pi/2$, it is possible to choose a positive number r^* such that

$$\begin{aligned}-\log 2+2B+2(A-B)r\cos t&\leqq\log^+\frac{1}{|G(-re^{it})|}\\&\leqq\log 2+2B+2(A-B)r\cos t\end{aligned} \tag{2.11}$$

and

$$|G(re^{it})|\geqq 1 \tag{2.12}$$

for $|t|\leqq t^*$ and $r\geqq r^*$. Therefore from (2.11),

$$m(r,\ 0,\ G)\geqq\frac{1}{2\pi}\int_{-t^*}^{t^*}\log^+\frac{1}{|G(-re^{it})|}dt$$

$$\geqq O(1)+\frac{2(A-B)}{\pi}r\sin t^*,$$

so that

$$\liminf_{r\to+\infty}\frac{m(r,\ 0,\ G)}{r}\geqq\frac{2(A-B)}{\pi}\sin t^*. \tag{2.13}$$

Further it follows from (2.11) and (2.12) that

$$m(r,\ 0,\ G)=\frac{1}{2\pi}\int_{-t^*}^{t^*}\log^+\frac{1}{|G(-re^{it})|}dt$$

$$+\frac{1}{2\pi}\int_{I(t^*)}\log^+\frac{1}{|G(re^{it})|}dt$$

$$\leqq O(1)+\frac{2(A-B)}{\pi}r\sin t^*+\left(1-\frac{2}{\pi}t^*\right)\log^+M(r,\ 0,\ G)\,,$$

where $I(t^*)$ indicates the union of the open intervals $(t^*,\ \pi-t^*)$ and $(t^*-\pi,\ -t^*)$, and

$$M(r,\ 0,\ G)=\sup_{|z|=r}\frac{1}{|G(z)|}.$$

Hence by making use of Petrenko's result [7] and (2.10), we have

$$\liminf_{r\to+\infty}\frac{m(r,\ 0,\ G)}{r}\leqq\frac{2(A-B)}{\pi}\sin t^*$$

$$+\left(1-\frac{2}{\pi}t^*\right)\frac{2B}{\pi}\liminf_{r\to+\infty}\frac{\log^+M(r,\ 0,\ G)}{T(r,\ G)}. \tag{2.14}$$

Since t^* is an arbitrary number with $0<t^*<\pi/2$, it therefore follows from (2.13) and (2.14) that

$$\liminf_{r\to+\infty}\frac{m(r,\ 0,\ G)}{r}=\frac{2(A-B)}{\pi},$$

so that

$$\delta(0,\ G)=\liminf_{r\to+\infty}\frac{m(r,\ 0,\ G)}{T(r,\ G)}=\frac{A-B}{B}.$$

In particular, for the case 1), the value 0 is a deficient value of $G(z)$ and $0<B<A\leqq 2B$.

The case 2) can be treated by the same way as the above case 1). Consequently, we obtain the following lemma.

LEMMA 3. *If* $A>B>0$, *then* $2B\geqq A$ *and*

$$\lim_{r\to\infty}\frac{T(r,G)}{r}=\frac{2B}{\pi},\qquad \delta(0,G)=\frac{A-B}{B}.$$

If $B>A>0$, *then* $2A\geqq B$ *and*

$$\lim_{r\to\infty}\frac{T(r,G)}{r}=\frac{2A}{\pi},\qquad \delta(1,G)=\frac{B-A}{A}.$$

3. Case 5). In this section we shall treat the case 5) and our goal is to show the impossibility of this case. Our consideration is divided into several steps, since it needs a little bit complicated process.

The first step. In this case 5), by virtue of (2.2) and (2.3),

$$\mathrm{Re}\frac{G'(z)}{G(z)}\leqq A<0 \tag{3.1}$$

for $\mathrm{Re}\,z\leqq 0$ unless $z=a_n$, and

$$\mathrm{Re}\frac{G'(z)}{G(z)-1}\geqq B>0 \tag{3.2}$$

for $\mathrm{Re}\,z\geqq 1$ unless $z=b_n$. Hence $G'(z)$ has no zeros for $\mathrm{Re}\,z\leqq 0$ and $\mathrm{Re}\,z\geqq 1$ except for the zero-points and the one-points of $G(z)$. Further $G(z)$ approaches infinity when z tends to infinity along the real axis.

The second step. In this step we shall prove the next fact.

LEMMA 4. *Assume that* A *is negative and* B *is positive. Then all the roots of the equation* $G(z)=x$ *must be contained in the open strip* $0<\mathrm{Re}\,z<1$, *where* x *is an arbitrary real number with* $0<x<1$.

Proof. Assume that there exist a real number x^* and a complex number z^* such that $0<x^*<1$, $\mathrm{Re}\,z^*\leqq 0$ and $G(z^*)=x^*$. Then by what mentioned above, $G'(z^*)\neq 0$. By $E(w,x^*)$, we denote the regular element of the inverse function of $G(z)$ with center x^* and satisfying $E(x^*,x^*)=z^*$. Now let us continue analytically this element $E(w,x^*)$ along the segment $I=\{x^*\leqq t\leqq 1\}$ toward the point $t=1$. Then we have an analytic continuation $G^{-1}(I_s)$ with algebraic character along the segment I up to some point $t=s$ $(x^*<s\leqq 1)$, with the possible exception of this end point. Hence from this continuation $G^{-1}(I_s)$, we can define the simple path $C^*=\{z(t):\,x^*\leqq t<s\}$ such that $z(x^*)=z^*$ and

$$G(z(t))=t \tag{3.3}$$

for $x^*\leqq t<s$. For a moment, assume that $\mathrm{Re}\,z(t^*)=0$ for some t^* with $x^*\leqq t^*<s$. Then $G'(z(t^*))\neq 0$, since $0<G(z(t^*))<1$. Thus the path C^* is differentiable at $t=t^*$ and it follows from (3.1) and (3.3) that

$$\mathrm{Re}\frac{G'(z(t^*))}{G(z(t^*))}=\mathrm{Re}\frac{1}{t^*z'(t^*)}\leqq A<0,$$

so that $\operatorname{Re} z'(t^*)<0$. Hereby if $\operatorname{Re} z(t^*)=0$, the real part of $z(t)$ decreases as t increases in a neighborhood of t^*. By this fact, we can easily see that this path C^* must be contained entirely in the open half plane $\operatorname{Re} z<0$ save for the initial point $z(x^*)$. Therefore the continuation $G^{-1}(I_s)$ does not continue along the segment I to the point $t=1$, since all the one-points of $G(z)$ are distributed on the line $\operatorname{Re} z=1$ only. Consequently, we may assume that the continuation $G^{-1}(I_s)$ defines a transcendental singularity at the point $t=s$. It thus follows from Iversen's theorem [6] that the path C^* must be an asymptotic path of $G(z)$ and as z tends to infinity along this path C^*, $G(z)$ converges to the value s. Further since $G(z)$ omits the values 0 and 1 in the open half plane $\operatorname{Re} z<0$, Lindelöf-Iversen-Gross' theorem [6] yields that $G(z)$ approaches the value s when z tends to infinity along the negative real axis. This clearly contradicts what mentioned in the first step. Accordingly, for each x with $0<x<1$, $G(z)\neq x$ in the closed half plane $\operatorname{Re} z\leqq 0$.

Next let us assume that there exist a real number x_* and a complex number z_* such that $0<x_*<1$, $\operatorname{Re} z_*\geqq 1$ and $G(z_*)=x_*$. Then $G'(z_*)\neq 0$. As before, by $E(w, x_*)$, let us denote the regular element of the inverse function of $G(z)$ with center x_* and satisfying $E(x_*, x_*)=z_*$. In this case we continue analytically this element $E(w, x_*)$ along the segment $J=\{0\leqq t\leqq x_*\}$ toward the point $t=0$, and we have an analytic continuation $G^{-1}(J_r)$ with algebraic character along this segment J up to some point $t=r$ $(0\leqq r<x_*)$, with the possible exception of this end point. Hence from this $G^{-1}(J_r)$, we can define the simple path $C_*=\{z(t): 0\leqq t<x_*-r\}$ such that $z(0)=z_*$ and

$$G(z(t))=x_*-t \tag{3.4}$$

for $0\leqq t<x_*-r$. By the same way as above, it thus follows from (3.2) and (3.4) that the path C_* must be contained entirely in the open half plane $\operatorname{Re} z>1$ with the possible exception of the point $z(0)$. By this fact, we can see that the path C_* is an asymptotic path of $G(z)$ and as z tends to infinity along C_*, $G(z)$ converges to some finite real number. Therefore we arrive at a contradiction. Consequently, for each x with $0<x<1$, $G(z)\neq x$ in the closed half plane $\operatorname{Re} z\geqq 1$. Lemma 4 is thus proved.

The third step. By Lemma 4, we can show the following lemma.

LEMMA 5. *Let the assumptions of Lemma* 4 *be satisfied. Then for each real number* x *with* $0<x<1$, *either*

$$\operatorname{Re}\frac{G'(z)}{G(z)-x}>0$$

for $\operatorname{Re} z\geqq 1$, *or*

$$\operatorname{Re}\frac{G'(z)}{G(z)-x}<0$$

for $\operatorname{Re} z\leqq 0$.

Proof. Let x be a real number with $0<x<1$. From (2.1), the genus of $G(z)-x$ is one or zero. Hence we can write $G(z)-x$ in the form

$$G(z)-x=\exp(az+b)\prod_{n\geqq 1}\left(1-\frac{z}{c_n}\right),$$

or else

$$G(z)-x=\exp(az+b)\prod_{n\geqq 1}\left(1-\frac{z}{c_n}\right)\exp\left(\frac{z}{c_n}\right),$$

where a and b are constants and $\{c_n\}$ denotes the x-points of $G(z)$. It thus follows from these expressions that

$$\frac{G'(z)}{G(z)-x}=a+\sum_{n\geqq 1}\frac{1}{z-c_n}, \tag{3.5}$$

or else

$$\frac{G'(z)}{G(z)-x}=a+\sum_{n\geqq 1}\left(\frac{1}{z-c_n}+\frac{1}{c_n}\right). \tag{3.6}$$

Now assume that the assertion of this lemma is false. Then there exist two points z_1 and z_2 such that $\operatorname{Re} z_1\geqq 1$, $\operatorname{Re} z_2\leqq 0$,

$$\operatorname{Re}\frac{G'(z_1)}{G(z_1)-x}\leqq 0$$

and

$$\operatorname{Re}\frac{G'(z_2)}{G(z_2)-x}\geqq 0.$$

Hence we have

$$\operatorname{Re}\frac{G'(z_1)}{G(z_1)-x}-\operatorname{Re}\frac{G'(z_2)}{G(z_2)-x}\leqq 0. \tag{3.7}$$

On the other hand from (3.5) or (3.6),

$$\frac{G'(z_1)}{G(z_1)-x}-\frac{G'(z_2)}{G(z_2)-x}=\sum_{n\geqq 1}\left(\frac{1}{z_1-c_n}-\frac{1}{z_2-c_n}\right),$$

so that

$$\operatorname{Re}\frac{G'(z_1)}{G(z_1)-x}-\operatorname{Re}\frac{G'(z_2)}{G(z_2)-x}$$

$$\geqq\sum_{n\geqq 1}\left(\frac{1-\operatorname{Re} c_n}{|z_1-c_n|^2}+\frac{\operatorname{Re} c_n}{|z_2-c_n|^2}\right).$$

By the above Lemma 4, $0<\operatorname{Re} c_n<1$. Therefore

$$\operatorname{Re}\frac{G'(z_1)}{G(z_1)-x}-\operatorname{Re}\frac{G'(z_2)}{G(z_2)-x}>0,$$

which clearly contradicts (3.7). This completes the proof of Lemma 5.

The fourth step. By Lemma 5, we have the two possibility.

(i) $$\mathrm{Re}\frac{G'(z)}{2G(z)-1}>0$$

in the closed half plane $\mathrm{Re}\, z\geqq 1$.

(ii) $$\mathrm{Re}\frac{G'(z)}{2G(z)-1}<0$$

in the closed half plane $\mathrm{Re}\, z\leqq 0$.

In what follows, we may assume that (i) holds, since the case where (ii) holds is treated by the same fashion. Now we are in a position to prove the next lemma.

LEMMA 6. *Let the assumptions of* Lemma 4 *be satisfied and let the inequality* (i) *hold. If* $G(1+iy)$ *is real and non-negative for some real number* y, *then* $G(1+iy)=1$.

Proof. First of all, it follows from the inequality (i) that $G'(z)\neq 0$ in the closed half plane $\mathrm{Re}\, z\geqq 1$. Further from Lemma 4, for an arbitrary real number x with $0\leqq x<1$, $G(z)\neq x$ there.

Assume now that the assertion is false. Then it is possible to take some real number y^* such that $G(1+iy^*)=s^*>1$. Hereafter we shall lead a contradiction by using the same method developed in the proof of Lemma 4. By $E(w, s^*)$, we denote the regular element of the inverse function of $G(z)$ with center s^* and satisfying $E(s^*, s^*)=1+iy^*$. Let us continue analytically this $E(w, s^*)$ along the segment $L=\{1\leqq x\leqq s^*\}$ toward the point $x=1$. Then we get an analytic continuation $G^{-1}(L_u)$ with algebraic character along L up to some point $x=u$ $(1\leqq u<s^*)$, with the possible exception of this end point. From this continuation $G^{-1}(L_u)$, we can thus define the simple path $C_u=\{z(t): 0\leqq t<s^*-u\}$ such that $z(0)=1+iy^*$ and

(3.8) $$G(z(t))=s^*-t$$

for $0\leqq t<s^*-u$. It therefore follows from (i), (3.1) and (3.8) that the path C_u must be contained entirely in the open strip $0<\mathrm{Re}\, z<1$ save for the point $z(0)$, so that the continuation $G^{-1}(L_u)$ does not continue along L to the point $x=1$. Hereby we may assume that $G^{-1}(L_u)$ defines a transcendental singularity at the point $x=u$. Consequently, the path C_u is an asymptotic path of $G(z)$ and as z tends to infinity along this C_u, $G(z)$ converges to the value $u(\geqq 1)$. Here let us denote by D the simply connected domain which is contained in the open strip $0<\mathrm{Re}\, z<1$ and is surrounded by C_u and by a part of the line $\mathrm{Re}\, z=1$. Evidently there exists a one-point z' of $G(z)$ which lies on the boundary of this domain D. Of course, $\mathrm{Re}\, z'=1$. Therefore, as above, using the regular element $E(w, 1)$ of the inverse function of $G(z)$ with center at 1 and satisfying $E(1, 1)=z'$, we can obtain an asymptotic path C_v of $G(z)$ such that C_v is contained in the domain D and as z tends to infinity along this C_v, $G(z)$ converges to some real value v with $0 \leqq v < 1$. However since $G(z)$ fails to take the values 0 and 1 in this domain D, it must be $u=v$, which is clearly absurd. Hence we have a desired contradiction. Lemma 6 is thus proved.

The final step. From the above Lemma 6 and the inequalities (i) and (3.2), by making use of the exactly same argument which we used to prove Lemma 10 of *CE*, it can be shown that the one-points of $G(z)$, which we write as $\{1+ib_n^*\}$, are simple and satisfy

$$(m-n-1)\pi \leqq B(b_m^* - b_n^*) \leqq (m-n+1)\pi \tag{3.9}$$

for arbitrary integers m and n with $m \geqq n$. Further with the help of the observation which we used to prove Lemma E of *EL*, it follows from (3.9) that $G(x+ib_0^*)$ is bounded for negative real values of x. This clearly contradicts the fact that A is negative. Hence we have proved the impossibility of the case 5) under the assumption that (i) holds.

For the case where (ii) holds, let us consider the function defined by

$$G^*(z) = 1 - G(1-z).$$

Then it is clear that $G^*(z)$ satisfies the assumptions of Lemma 6. Hence we also arrive at a contradiction. Consequently, the case 5) never occurs.

4. Further consequences of Lemma 2. By virtue of the foregoing results, we may consider only the cases 1) and 2). On these cases, from Lemma 2, we have obtained Lemma 3 in the section 2. In addition to this information, we require further lemmas to obtain our desired result.

Let $G(z)$ be an entire function satisfying the assumptions of our theorem, and let A, B, A' and B' be the real constants in the identities (2.4) and (2.5).

Firstly assume that $A > B > 0$. Then by means of Lemma 2 and the identity (2.5), we can conclude that

$$\lim_{r\to+\infty} G(re^{it}) \exp(-2Bre^{it}) = -\exp(-2B-iB') \tag{4.1}$$

uniformly for $|t| \leqq t^*$, where t^* is an arbitrarily fixed number with $0 < t^* < \pi/2$. Further by making use of the functional equation (2.6), we can prove the following Lemma 7. As a matter of convenience, we shall prepare some notations. For each non-negative integer k, let us set

$$C_k = \exp(2k(k-1)A - 2k^2 B). \tag{4.2}$$

With these positive real constants C_k, let us further set

$$H_n(z) = \frac{b^n}{C_n} G(z) \exp(2nAz - 2nBz - 2Bz) \tag{4.3}$$

$$+\sum_{k=0}^{n-1}\frac{C_{n-k}}{C_n}ab^k\exp(2(k+1)Az-2(k+1)Bz)$$

for each natural number n, where $a=\exp(-\imath A')$ and $b=\exp(\imath B'-iA')$.

LEMMA 7. *Assume that* $0<nB<nA<(n+1)B$ *for some natural number* n. *Then for an arbitrarily fixed number* t^* *in* $(0, \pi/2)$,

$$\lim_{r\to+\infty}H_n(re^{it})=-\frac{a}{b}\exp(4nA-4nB-2B)$$

uniformly for $|t|\leqq t^*$.

Proof. This assertion can be shown by induction. Let us recall the functional equation (2.6), which can be written as

$$\begin{aligned}G(z)=a\exp(2Az)-b\exp(2Az-2Bz-2B)\\ +bG(z+2)\exp(2Az-2Bz-2B)\,.\end{aligned}\tag{4.4}$$

Then by (4.2) and (4.3),

$$\begin{aligned}G(z)\exp(-2Bz)&=a\exp(2Az-2Bz)\\ &\quad+bG(z+2)\exp(2Az-4Bz-2B)\\ &\quad-b\exp(2Az-4Bz-2B)\\ &=H_1(z+2)\exp(4B-4A)-b\exp(2Az-4Bz-2B)\,.\end{aligned}\tag{4.5}$$

Hence if $0<B<A<2B$, it follows from (4.1) and (4.5) that

$$\lim_{r\to+\infty}H_1(2+re^{it})\exp(4B-4A)=-\frac{a}{b}\exp(-2B)\,,$$

so that

$$\lim_{r\to+\infty}H_1(re^{it})=-\frac{a}{b}\exp(4A-6B)$$

uniformly for $|t|\leqq t^*$ with $0<t^*<\pi/2$. Therefore our assertion is true when $n=1$. We now assume that it also holds for $n\geqq 1$. Combining (4.3) and (4.4), we can easily have

$$\begin{aligned}H_n(z)&=H_{n+1}(z+2)\exp(4B-4A)\\ &\quad-\frac{b^{n+1}}{C_{n+1}}\exp(2(n+1)Az-2(n+2)Bz+4nA-4(n+1)B)\,.\end{aligned}\tag{4.6}$$

Here let us note that $0<(n+1)B<(n+1)A<(n+2)B$ implies $0<nB<nA<(n+1)B$. Hence if A and B satisfy $0<(n+1)B<(n+1)A<(n+2)B$, it follows from (4.6) that

$$\lim_{r\to+\infty} H_n(re^{it}) = \lim_{r\to+\infty} H_{n+1}(2+re^{it}) \exp(4B-4A)$$

$$= -\frac{a}{b}\exp(4nA-4nB-2B),$$

so that

$$\lim_{r\to+\infty} H_{n+1}(re^{it}) = -\frac{a}{b}\exp(4(n+1)A-4(n+1)B-2B)$$

uniformly for $|t| \leqq t^*$ with $0 < t^* < \pi/2$. Hereby the assertion is also true for $n+1$. Lemma 7 is thus proved.

Secondly, let us consider the case 2), that is, $B > A > 0$. The functional equation (2.6) can be rewritten in the form

$$\begin{aligned} G(z)-1 &= a\exp(2Az)-1 \\ &\quad + b(G(z+2)-1)\exp(2Az-2Bz-2B) \end{aligned} \tag{4.7}$$

with $a=\exp(-iA')$, $b=\exp(iB'-iA')$. Hence it follows from Lemma 2 that

$$\begin{aligned} &\lim_{r\to+\infty}(G(-re^{it})-1)\exp(2(B-A)re^{it}) \\ &\qquad = \frac{1}{b}\exp(4A-2B) \end{aligned} \tag{4.8}$$

uniformly for $|t| \leqq t^*$, where t^* is an arbitrarily fixed number in $(0, \pi/2)$. Now let us set

$$C_k^* = \exp(2k(k+1)A-2k^2B),$$

$$\begin{aligned} H_n^*(z) &= \frac{b^n}{C_n^*}(G(z)-1)\exp(2(n+1)Az-2(n+1)Bz) \\ &\quad - \sum_{k=0}^{n-1}\frac{C_{n-k}^*}{C_n^*} b^k \exp(2(k+1)Az-2(k+1)Bz) \end{aligned} \tag{4.9}$$

for each natural number n. Combining (4.7) and (4.9), we at once obtain

$$\begin{aligned} H_n^*(z) &= \frac{ab^n}{C_n^*}\exp(2(n+2)Az-2(n+1)Bz) \\ &\quad + H_{n+1}^*(z+2)\exp(4B-4A). \end{aligned} \tag{4.10}$$

Therefore by making use of (4.8) and (4.10), we can achieve the following lemma.

LEMMA 8. *Assume that* $0 < nA < nB < (n+1)A$ *for some natural number* n. *Then*

$$\lim_{r\to+\infty} H_n^*(-re^{it}) = \frac{1}{b}\exp(4(n+1)A-4(n+1)B+2B)$$

uniformly for $|t| \leqq t^*$, *where* t^* *is an arbitrarily fixed number with* $0 < t^* < \pi/2$.

5. Case 1). The purpose of this section is to treat the case 1). Let the notations be as in the above section 4. For each natural number k, with the real constant C_k defined by (4.2), we now introduce the entire function

$$f_k(z) = -\frac{a}{b^k} C_k \exp(2kBz - 2(k-1)Az) - b^k C_k \exp(2kAz - 2kBz). \tag{5.1}$$

It is clear that

$$\overline{f_k(\bar{z})} = \bar{a} f_k(-z) \exp(2Az), \tag{5.2}$$

since $a\bar{a} = 1$ and $b\bar{b} = 1$. Further let us set

$$S_n(z) = \sum_{k=1}^{n} f_k(z) \tag{5.3}$$

for each natural number n. Of course, it follows from (5.2) that

$$\overline{S_n(\bar{z})} = \bar{a} S_n(-z) \exp(2Az). \tag{5.4}$$

Moreover, by the definitions (4.3), (5.1) and (5.3), we can easily see that

$$G(z) - S_n(z) = \frac{C_n}{b^n} H_n(z) \exp(2(n+1)Bz - 2nAz) + \sum_{k=1}^{n} C_k b^k \exp(2kAz - 2kBz). \tag{5.5}$$

Our task now is to prove the next lemma.

LEMMA 9. *Let* $G(z)$ *be an entire function satisfying the assumptions of the theorem, and let* A *and* B *be the real constants in the identities* (2.4) *and* (2.5). *Assume that* A *and* B *are both positive and that*

$$\frac{2n+2}{2n+1} < \frac{A}{B} < \frac{2n}{2n-1}$$

for some natural number n. *Then*

$$G(z) = S_n(z),$$

where $S_n(z)$ *is the entire function defined by* (5.3).

Proof. Let us set

$$F(z) = \frac{G(z)}{S_n(z)}.$$

Then it follows from (5.5) that

$$F(z)-1=\frac{C_nH_n(z)}{b^nS_n(z)}\exp(2(n+1)Bz-2nAz) \tag{5.6}$$
$$+\sum_{k=1}^{n}\frac{C_kb^k}{S_n(z)}\exp(2kAz-2kBz).$$

Combining (2.4) and (5.4), we at once have

$$\overline{F(\bar{z})}=F(-z). \tag{5.7}$$

Further by the assumption

$$\frac{2n+2}{2n+1}<\frac{A}{B}<\frac{2n}{2n-1}, \tag{5.8}$$

it is clear that

$$0<nB<nA<(n+1)B. \tag{5.9}$$

Now let t^* be an arbitrarily fixed number with $0<t^*<\pi/2$. Then by means of (5.1), (5.3), (5.9) and Lemma 7, it is possible to find a positive number r^* such that

$$|S_n(z)\exp(-2Bz)|\geqq\frac{C_1}{2}$$

and

$$|H_n(z)|\leqq 2\exp(4nA-4nB-2B)$$

for values of z with $|z|\geqq r^*$ and $|\arg z|\leqq t^*$. Hence it follows from (5.6) that

$$|F(z)-1|\leqq\frac{4C_n}{C_1}\exp(4nA-4nB-2B)|\exp(2n(B-A)z)| \tag{5.10}$$
$$+\sum_{k=1}^{n}\frac{2C_k}{C_1}|\exp(2kAz-2(k+1)Bz)|$$

there. Here let us set

$$I=\max(2nB-2nA,\ 2nA-2nB-2B). \tag{5.11}$$

Then from (5.10), with some positive constant M, we can get

$$|F(re^{it})-1|\leqq M\exp(Ir\cos t) \tag{5.12}$$

for $r\geqq r^*$ and $|t|\leqq t^*$.

Hereafter, assume that the meromorphic function $F(z)$ is not constant. Then by virtue of (5.7) and the inequality (5.12),

$$m(r,\ 1,\ F)=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\log^{+}\frac{1}{|F(re^{it})-1|}\,dt$$

$$\geqq\frac{1}{\pi}\int_{-t^*}^{t^*}\log^{+}\frac{1}{|F(re^{it})-1|}\,dt$$

$$\geqq-\frac{2t^*}{\pi}\log M-\frac{2}{\pi}Ir\sin t^*$$

for real values of r with $r\geqq r^*$, so that

$$\liminf_{r\to+\infty}\frac{m(r,\ 1,\ F)}{r}\geqq-\frac{2I}{\pi}\sin t^*.$$

Since t^* is an arbitrary number between 0 and $\pi/2$, it follows that

$$\liminf_{r\to+\infty}\frac{m(r,\ 1,\ F)}{r}\geqq-\frac{2}{\pi}I\,,$$

so that

$$\liminf_{r\to+\infty}\frac{T(r,\ F)}{r}\geqq-\frac{2}{\pi}I\,. \tag{5.13}$$

In particular, the lower order of $F(z)$ must be exactly one. In addition to this (5.13), we can see that

$$\liminf_{r\to+\infty}\frac{m(r,\ 0,\ F)}{T(r,\ F)}=0\,, \tag{5.14}$$

so that

$$\limsup_{r\to+\infty}\frac{N(r,\ 0,\ F)}{T(r,\ F)}=1\,. \tag{5.15}$$

In fact, by means of (5.12), we can choose a positive number r_* so that

$$|F(re^{it})|\geqq\frac{1}{2}$$

for $r\geqq r_*$ and $|t|\leqq t^*$. Taking into account of (5.7), we thus find

$$m(r,\ 0,\ F)=\frac{1}{\pi}\int_{-t^*}^{t^*}\log^{+}\frac{1}{|F(re^{it})|}\,dt+\frac{1}{\pi}\int_{J^*}\log^{+}\frac{1}{|F(re^{it})|}\,dt$$

$$\leqq\log 2+\left(1-\frac{2}{\pi}t^*\right)\log^{+}M(r,\ 0,\ F)\,,$$

so that

$$\liminf_{r\to+\infty}\frac{m(r,\ 0,\ F)}{T(r,\ F)}\leqq\left(1-\frac{2}{\pi}t^*\right)\liminf_{r\to+\infty}\frac{\log^{+}M(r,\ 0,\ F)}{T(r,\ F)}\,,$$

where J_* indicates the union of the open intervals $(t^*,\ \pi/2)$ and $(-\pi/2,\ -t^*)$, and

$$M(r,\ 0,\ F)=\sup_{|z|=r}\frac{1}{|F(z)|}.$$

Hence the result of Petrenko, which we used in the section 2, yields

$$\liminf_{r\to+\infty}\frac{m(r,\ 0,\ F)}{T(r,\ F)}\leqq\pi-2t^{*}.$$

Since t^{*} is arbitrary subject to $0<t^{*}<\pi/2$, the desired (5.14) follows immediately. On the other hand, by the definition of $F(z)$, it is clear that

$$N(r,\ 0,\ F)\leqq N(r,\ 0,\ G)$$

for positive real values of r. Hence

$$\frac{N(r,\ 0,\ F)}{T(r,\ F)}\frac{T(r,\ F)}{r}\leqq\frac{N(r,\ 0,\ G)}{T(r,\ G)}\frac{T(r,\ G)}{r}.$$

It therefore follows from (5.13) and (5.15) that

$$-\frac{2}{\pi}I\leqq\limsup_{r\to+\infty}\frac{N(r,\ 0,\ G)}{T(r,\ G)}\frac{T(r,\ G)}{r}.$$

Here recall Lemma 3. Then we obtain

$$-\frac{2}{\pi}I\leqq(1-\delta(0,\ G))\lim_{r\to+\infty}\frac{T(r,\ G)}{r}$$

$$=\frac{2}{\pi}(2B-A),$$

so that

(5.16) $$0\leqq 2B-A+I.$$

However this (5.16) contradicts (5.8) and (5.11). Indeed if

$$\frac{2n+2}{2n+1}<\frac{A}{B}\leqq\frac{2n+1}{2n},$$

then $I=2nB-2nA$, so that (5.16) gives

$$\frac{A}{B}\leqq\frac{2n+2}{2n+1},$$

which is absurd. Further if

$$\frac{2n+1}{2n}\leqq\frac{A}{B}<\frac{2n}{2n-1},$$

then $I=2nA-2nB-2B$, so that (5.16) implies

$$\frac{2n}{2n-1} \leqq \frac{A}{B}.$$

This is clearly untenable. Consequently, the meromorphic function $F(z)$ must be constant. Of course, by (5.12), $F(z)=1$. Therefore $G(z)=S_n(z)$, which is to be proved.

As an immediate consequence of Lemma 9, we can now prove our theorem under the condition (5.8).

LEMMA 10. *Let $G(z)$ be an entire function satisfying the hypotheses of* Lemma 9. *Then*

$$G(z)=-\sum_{k=1}^{2n} b^k \exp(kCz-k(2n+1-k)C),$$

where $C=B/n$.

Proof. By virtue of Lemma 9, $G(z)=S_n(z)$. Since $G(z)$ satisfies the functional equation (2.5), $S_n(z)$ also does

$$\overline{S_n(\bar{z}+1)}-1=(S_n(-z+1)-1)\exp(2Bz+\imath B').$$

It thus follows from the definition of $S_n(z)$ that

$$a\exp(4nAz-2(2n+1)Bz)=b^{2n+1}.$$

Hence we have $a=b^{2n+1}$ and $2nA=(2n+1)B$. Inserting these relations into $S_n(z)$, we at once obtain the desired result. Lemma 10 is thus proved.

There still remains the case where A and B are

$$\frac{A}{B}=\frac{2n}{2n-1} \tag{5.17}$$

with some natural number n. In this case, let us consider the entire function defined by

$$E(z)=G(z)\exp(-Az) \qquad (n=1),$$

$$E(z)=(G(z)-S_{n-1}(z))\exp(-Az) \qquad (n\geqq 2).$$

In what follows let t^* be an arbitrarily fixed number in $(0, \pi/2)$, as before. Firstly assume that $A=2B$, that is, $n=1$ in (5.17). Then it follows immediately from (4.1) that

$$\lim_{r\to+\infty} E(re^{\imath t})=-\exp(-A-\imath B')=-\frac{a}{b}\exp(-A) \tag{5.18}$$

uniformly for $|t|\leqq t^*$. Further the functional equation (2.4) implies

$$\overline{E(\bar{z})}=E(-z)\exp(\imath A').$$

Therefore from (5.18), we also obtain

$$\lim_{r\to+\infty} E(re^{it}) = -\exp(-A+iB'-\imath A') = -b\exp(-A) \tag{5.19}$$

uniformly for $|t-\pi|\leqq t^*$. By taking into account of these (5.18) and (5.19), Edrei's spread relation [1] yields that

$$E(z) = -b\exp(-A) = -\frac{a}{b}\exp(-A),$$

identically. Consequently,

$$G(z) = -b\exp(2Bz-2B).$$

Secondly, assume that A and B satisfy (5.17) with $n\geqq 2$. Then

$$\lim_{r\to+\infty} E(re^{it}) = -\frac{a}{b^n}C_n \tag{5.20}$$

and

$$\lim_{r\to+\infty} E(-re^{it}) = -b^n C_n \tag{5.21}$$

uniformly for $|t|\leqq t^*$. In fact, the assumption (5.17) implies

$$0<(n-1)B<(n-1)A<nB,$$

and by means of (5.5),

$$E(z) = \frac{C_{n-1}}{b^{n-1}}H_{n-1}(z)$$

$$+\sum_{k=1}^{n-1} C_k b^k \exp((2k-1)Az-2kBz).$$

Hence Lemma 7 and (4.2) yield

$$\lim_{r\to+\infty} E(re^{it}) = -\frac{a}{b^n}C_{n-1}\exp(4(n-1)A-(4n-2)B)$$

$$= -\frac{a}{b^n}C_n$$

uniformly for $|t|\leqq t^*$. This is the desired (5.20). The fact (5.21) follows at once by combining this (5.20) and the functional equation

$$\overline{E(\bar{z})} = E(-z)\exp(iA').$$

Therefore by making use of the spread relation, as above, we can conclude that the entire function $E(z)$ must be constant, so that

$$G(z) = S_{n-1}(z) - b^n C_n \exp(Az)$$

$$= -\sum_{k=1}^{2n-1} b^k \exp(2kCz-2k(2n-k)C)$$

with $C=B/(2n-1)$.

We have now treated the case 1) completely. Let us formulate the results of this section as a theorem.

THEOREM 2. *Let $G(z)$ be an entire function satisfying the hypotheses of our theorem. If $A>B>0$, then*

$$mA=(m+1)B$$

for some natural number m. Furthermore

$$G(z)=-\sum_{k=1}^{m}\exp(kCz-k(m+1-k)C+ikC'),$$

where C' is a real constant and

$$C=\frac{2}{m}B.$$

6. Case 2). In this section, we shall discuss the case 2) by appealing to Lemma 8. For each natural number n, with the real constants C_k^* defined by (4.9), let us set

$$S_n^*(z)=1+\sum_{k=1}^{n}\frac{C_k^*}{b^k}\exp(2kBz-2kAz)$$

(6.1)

$$+\sum_{k=1}^{n}ab^{k-1}C_{k-1}^*\exp(2kAz-2(k-1)Bz),$$

where $a=\exp(-iA')$ and $b=\exp(iB'-iA')$. It is easily verified by the definitions that

$$\overline{S_n^*(\bar{z}+1)}-1=\frac{b}{a}(S_n^*(-z+1)-1)\exp(2Bz) \tag{6.2}$$

and that

$$G(z)-S_n^*(z)=\frac{C_n^*}{b^n}H_n^*(z)\exp(2(n+1)Bz-2(n+1)Az)$$

(6.3)

$$-\sum_{k=1}^{n}ab^{k-1}C_{k-1}^*\exp(2kAz-2(k-1)Bz),$$

where $H_n^*(z)$ is the entire function defined by (4.9).

Assume now that the positive real constants A and B are

$$\frac{2n+2}{2n+1}<\frac{B}{A}<\frac{2n}{2n-1} \tag{6.4}$$

with some natural number n. Then it is clear that

$$0<nA<nB<(n+1)A. \tag{6.5}$$

Hereafter we compare the function $S_n^*(z)$ with $G(z)$, and conclude that $G(z)$ must coincide with $S_n^*(z)$. To this end, let us introduce the meromorphic function defined by

$$F^*(z)=\frac{G(z+1)-1}{S_n^*(z+1)-1}. \tag{6.6}$$

Then by (2.5), (6.2) and (6.3), this function $F^*(z)$ satisfies

$$\overline{F^*(\bar{z})}=F^*(-z) \tag{6.7}$$

and

$$\begin{aligned} F^*(z)-1= & \frac{C_n^*H_n^*(z+1)}{b^n(S_n^*(z+1)-1)}\exp(2(n+1)(B-A)(z+1)) \\ & -\sum_{k=1}^{n}\frac{ab^{k-1}C_{k-1}^*}{S_n^*(z+1)-1}\exp(2kA(z+1)-2(k-1)B(z+1)). \end{aligned} \tag{6.8}$$

Let t^* be an arbitrarily fixed number with $0<t^*<\pi/2$. Then by means of (6.1), (6.5) and Lemma 8, it is possible to find a positive real number R such that

$$|S_n^*(z+1)-1|\geqq\frac{C_1^*}{2}\exp(2B-2A)|\exp(2Bz-2Az)|$$

and

$$|H_n^*(z+1)|\leqq 2\exp(4(n+1)A-4(n+1)B+2B)$$

for values of z with $|z|\geqq R$ and $|\arg z-\pi|\leqq t^*$. Hence with suitable positive constants M_k, it follows from (6.8) that

$$\begin{aligned} |F^*(z)-1|\leqq & M_0|\exp(2nBz-2nAz)| \\ & +\sum_{k=1}^{n}M_k|\exp(2(k+1)Az-2kBz)| \end{aligned}$$

there. Setting

$$J=\max(2nA-2nB,\ 2nB-2nA-2A), \tag{6.9}$$

we thus obtain that

$$|F^*(re^{it})-1|\leqq M^*\exp(-Jr\cos t) \tag{6.10}$$

for values of r and t with $r\geqq R$ and $|t-\pi|\leqq t^*$, where

$$M^*=\sum_{k=0}^{n}M_k.$$

Therefore if $F^*(z)$ is not constant, it follows from (6.7) and (6.10) that

$$m(r,1,F^*)\geqq 0(1)-\frac{2}{\pi}Jr\sin t^*,$$

so that

$$\liminf_{r\to\infty}\frac{T(r, F^*)}{r}\geqq\liminf_{r\to\infty}\frac{m(r, 1, F^*)}{r}\geqq-\frac{2}{\pi}J. \tag{6.11}$$

Further it is also verified from (6.7) and (6.10) that

$$\liminf_{r\to\infty}\frac{m(r, 0, F^*)}{T(r, F^*)}=1-\limsup_{r\to\infty}\frac{N(r, 0, F^*)}{T(r, F^*)}=0. \tag{6.12}$$

Consequently, these (6.11) and (6.12) yield

$$\limsup_{r\to\infty}\frac{N(r, 0, F^*)}{r}\geqq-\frac{2}{\pi}J, \tag{6.13}$$

provided that $F^*(z)$ is not a constant function. On the other hand, by virtue of the definition (6.6), it is clear that

$$N(r, 0, F^*)\leqq N(r, 1, G(z+1)) \tag{6.14}$$

for positive real values of r. Moreover by means of Lemma 3,

$$\limsup_{r\to\infty}\frac{N(r, 1, G(z))}{r}=\frac{2}{\pi}(2A-B). \tag{6.15}$$

Here let us notice that

$$\limsup_{r\to\infty}\frac{N(r, 1, G(z))}{r}=\limsup_{r\to\infty}\frac{N(r, 1, G(z+1))}{r}$$

Then from (6.14), we can see that

$$\limsup_{r\to\infty}\frac{N(r, 0, F^*)}{r}\leqq\limsup_{r\to\infty}\frac{N(r, 1, G(z))}{r}.$$

so that (6.13) and (6.15) lead us to

$$0\leqq 2A-B+J, \tag{6.16}$$

provided that $F^*(z)$ is not constant. However this inequality (6.16) is evidently absurd by (6.4) and (6.9). Accordingly, the meromorphic function $F^*(z)$ must be constant, so that $F^*(z)=1$ from (6.10). Hence we can conclude that $G(z)=S_n^*(z)$, which is the required result. By this fact, the next lemma follows immediately.

LEMMA 11. *Let $G(z)$ be an entire function satisfying the hypotheses of our theorem, and let A and B be the real constants in the identities* (2.4) *and* (2.5). *Assume that A and B are both positive and that*

$$\frac{2n+2}{2n+1}<\frac{B}{A}<\frac{2n}{2n-1}$$

for some natural number n. Then

$$2nB=(2n+1)A\,,$$

and

$$G(z)=\sum_{k=0}^{2n}\frac{1}{b^{k}}\exp(kCz+k(2n-k)C)$$

with $C=A/n$.

Proof. By what mentioned just above, $G(z)$ coincides with $S_n^*(z)$. Hence the entire function $S_n^*(z)$ must satisfy

$$\overline{S_n^*(\bar{z})}=\frac{1}{a}S_n^*(-z)\exp(2Az)\,.$$

It therefore follows from this functional equation that

$$2nB=(2n+1)A\,,\qquad ab^{2n}=1\,.$$

Inserting these relations into $S_n^*(z)$, we at once arrive at the desired result.

In order to treat the case 2) completely, we must investigate the case where the real constants A and B are both positive and

$$(2n-1)B=2nA \tag{6.17}$$

with some natural number n. In this case let us consider the entire function defined by

$$E^*(z)=(G(z+1)-1)\exp(-Bz) \qquad (n=1)\,,$$

$$E^*(z)=(G(z+1)-S_{n-1}^*(z+1))\exp(-Bz) \qquad (n\geqq 2)\,.$$

Evidently, from (2.5) and (6.2),

$$\overline{E^*(\bar{z})}=\frac{b}{a}E^*(-z)\,. \tag{6.18}$$

If $n\geqq 2$, using (6.3) and (6.17), we can see that

$$\begin{aligned} E^*(z)=&\frac{C_{n-1}^*}{b^{n-1}}H_{n-1}^*(z+1)\exp(2nB-2nA)\\ &-\sum_{k=1}^{n-1}ab^{k-1}C_{k-1}^*\exp(2kAz-(2k-1)Bz+2kA-2(\;-\;B \end{aligned} \tag{6.19}$$

Since (6.17) implies

$$0<(n-1)A<(n-1)B<nA$$

and

$$2kA-(2k-1)B>0 \qquad (1\leqq k\leqq n-1)\,,$$

it thus follows from (4.9), (6.18), (6.19) and Lemma 8 that

$$\lim_{r\to+\infty} E^*(-re^{it})=\frac{C^*_{n-1}}{b^n}\exp(2nA-2nB+2B)$$

$$=\frac{1}{b^n}\exp(2n^2A-2n(n-1)B)\,,$$

$$\lim_{r\to+\infty} E^*(re^{it})=ab^{n-1}\exp(2n^2A-2n(n-1)B)$$

uniformly for $|t|\leqq t^*$, where t^* is an arbitrarily fixed number in the open interval $(0, \pi/2)$. This assertion is also true when $n=1$. Therefore we can conclude that

$$E^*(z)=\frac{1}{b^n}\exp(2n^2A-2n(n-1)B)$$

and $ab^{2n-1}=1$. By this fact, if $n=1$,

$$G(z+1)=1+\frac{1}{b}\exp(Bz+2A)\,,$$

so that

$$G(z)=1+\frac{1}{b}\exp(2Az)\,.$$

Further if $n\geqq 2$,

$$G(z+1)=S^*_{n-1}(z+1)+\frac{1}{b^n}\exp(Bz+2n^2A-2n(n-1)B)\,,$$

so that

$$G(z)=\sum_{k=0}^{2n-1}\frac{1}{b^k}\exp(2kCz+2k(2n-1-k)C)$$

with $C=A/(2n-1)$.

In conclusion we have now treated the case 2) completely, and we have established the following theorem.

THEOREM 3. *Let $G(z)$ be an entire function which satisfies the hypotheses of our theorem. If $B>A>0$, then*

$$mB=(m+1)A$$

for some natural number m. Furthermore

$$G(z)=\sum_{k=0}^{m}\exp(kCz+k(m-k)C+ikC')\,,$$

where C' is a real constant and

$$C=\frac{2}{m}A\,.$$

7. Conclusion. The foregoing facts, especially Theorems 1, 2 and 3, are sufficient to yield our theorem which we stated in the introduction. Actually we have established the following conclusion which clearly includes our theorem.

Let n be a natural number and let C be a non-zero real constant. In the sequel we shall say that the pair (n, C) is *admissible* if either $n=1$, or if $n\geqq 2$ and all the roots of the algebraic equations

$$\sum_{k=0}^{n} \exp(-k(n-k)C)z^k=0$$

and

$$\sum_{k=0}^{n-1} \exp(-(k+1)(n-k)C)z^k=0$$

are distributed on the unit circle. With this convention, we can state our conclusion in the following form.

THEOREM. *Let $G(z)$ be a transcendental entire function of finite lower order. Assume that all the zero-points and all the one-points of $G(z)$ are distributed only on the lines* $\operatorname{Re} z=0$ *and* $\operatorname{Re} z=1$, *respectively. Then $G(z)$ has exactly one finite deficient value a. Furthermore $G(z)$ can be written as follows.*

(I) *If the finite deficient value a is different from* 0 *and* 1, *then*

$$G(z)=P(\exp Cz),$$

where C is a non-zero real constant and $P(z)$ is a quadratic polynomial with $P(0)=a$.

(II) *If a is equal to* 0, *then*

$$G(z)=P_n(\exp Cz),$$

$$P_n(t)=-\sum_{k=1}^{n} \exp(-k(n+1-k)C+ikC')t^k$$

with an admissible pair (n, C) and with a real constant C'.

(III) *If a is equal to* 1, *then*

$$G(z)=Q_n(\exp Cz),$$

$$Q_n(t)=\sum_{k=0}^{n} \exp(k(n-k)C+ikC')t^k,$$

where C' is a real constant and $(n, -C)$ is an admissible pair.

Now let $G(z)$ be an entire function of the form (II), that is,

$$G(z)=-\sum_{k=1}^{n} \exp(kCz-k(n+1-k)C+ikC'),$$

where n is a natural number and C, C' are both real constants with $C\neq 0$. Then by taking into account of properties of the exponential function, a necessary and sufficient condition that all the roots of the equation $G(z)=1$ lie on the line $\operatorname{Re} z=1$ is that all the zeros of the polynomial

$$\sum_{k=0}^{n} \exp(-k(n-k)C)t^k$$

are contained in the unit circle. Further if $n \geqq 2$, then a necessary and sufficient condition for all the zeros of $G(z)$ to lie on the line $\operatorname{Re} z=0$ is that all the roots of the algebraic equation

$$\sum_{k=0}^{n-1} \exp(-(k+1)(n-k)C)t^k=0$$

are distributed on the unit circle. Therefore a necessary and sufficient condition for an entire function of the form (II) to satisfy the assumptions of our theorem is that the pair (n, C) is admissible. Similarly, a necessary and sufficient condition that an entire function of the form (III) satisfies the assumptions of our theorem is that the pair $(n, -C)$ is admissible.

It is now natural to ask whether there exist admissible pairs for each natural number, actually. With respect to this question, by using a straightforward calculation, we can easily see that $(2, C)$ is admissible if and only if $C \geqq -\log 2$. Furthermore, $(3, C)$ and $(4, C)$ are admissible if and only if $2C \geqq -\log 3$ and $2C \geqq -\log 2$, respectively. However it seems to be very difficult to find a necessary and sufficient condition for pairs to be admissible, in general.

Finally we shall show the following lemma which gives us a sufficient condition. The proof of this fact is a standard piece of work in the theory of algebraic equations, but we include it for completeness.

LEMMA 12. *Let n be a natural number and let C be an arbitrary positive real number. Then the pair (n, C) is admissible.*

Proof. Our goal is to show that all the roots of the algebraic equations

$$\sum_{k=0}^{n} \exp(-k(n-k)C)z^k=0$$

and

$$\sum_{k=0}^{n-1} \exp(-(k+1)(n-k)C)z^k=0$$

must lie on the unit circle. Consider the latter equation which can be rewritten as

$$\exp(-nC)\sum_{k=0}^{n-1} \exp(-k(n-1-k)C)z^k=0\,.$$

Then it is sufficient to prove that for each natural number n and for each positive number C, all the zeros of the self-inversive polynomial

$$\sum_{k=0}^{n} \exp(-k(n-k)C)z^k$$

are distributed on the unit circle. As a matter of convenience, let us set $a_k = \exp(-k(n-k)C)$ and set

$$P_n^*(z) = \sum_{k=0}^{n} a_k z^k . \tag{7.1}$$

Assume firstly that n is even. We may also assume that $n \geqq 4$, since the assertion can be verified directly when $n=2$. Then by setting $n=2m$,

$$P_n^*(z) = a_m z^m + \sum_{k=0}^{m-1} a_k (z^k + z^{n-k}) ,$$

since $a_k = a_{n-k}$. Therefore

$$z^{-m} P_n^*(z) = a_m + \sum_{k=0}^{m-1} a_k (z^{k-m} + z^{m-k}) . \tag{7.2}$$

Here let us introduce the trigonometric polynomials defined by

$$I_n(t) = a_m + \sum_{k=0}^{m-1} 2a_k \cos(m-k)t , \tag{7.3}$$

$$J_n(t) = \sum_{k=0}^{m} a_k^* \cos kt , \tag{7.4}$$

where $a_k^* = a_k$ for $0 \leqq k \leqq m-1$, and $a_m^* = a_m/2$. Then it is clear from (7.2) that

$$I_n(t) = e^{-imt} P_n^*(e^{it}) \tag{7.5}$$

for real values of t. Further by setting $c_j = j\pi/m\,(0 \leqq j \leqq m)$, we can see from (7.3) and (7.4) that

$$\begin{aligned} I_n(c_j) &= 2(-1)^j \sum_{k=0}^{m} a_k^* \cos kc_j \\ &= 2(-1)^j J_n(c_j) \end{aligned} \tag{7.6}$$

for each integer j with $0 \leqq j \leqq m$. On the other hand, by making use of Abel's transformation, it follows that

$$\begin{aligned} J_n(t) &= \sum_{k=0}^{m} a_k^* \cos kt \\ &= -\frac{a_0^*}{2} + \sum_{k=0}^{m} (a_k^* - a_{k+1}^*) D_k(t) \end{aligned}$$

with the convention $a_{m+1}^* = 0$, where $D_k(t)$ denotes Dirichlet's kernel, that is,

$$D_0(t)=\frac{1}{2}, \qquad D_k(t)=\frac{1}{2}+\sum_{j=1}^{k}\cos jt \qquad (k\geqq 1).$$

Further using Abel's transformation once more, we obtain

$$J_n(t)=\frac{a_0^*}{2}+(m+1)a_m^*K_m(t)$$

$$+\sum_{j=0}^{m-1}(j+1)(a_j^*-2a_{j+1}^*+a_{j+2}^*)K_j(t),$$

where $K_j(t)$ denotes Fejer's kernel, that is,

$$K_j(t)=\frac{1}{j+1}\sum_{i=0}^{j}D_i(t).$$

Hereby we can get

$$J_n(t)=\frac{1}{2}+\frac{a_m}{2}((m+1)K_m(t)-(m-1)K_{m-2}(t))$$

(7.7)

$$+m(a_{m-1}-a_m)K_{m-1}(t)+\sum_{j=0}^{m-2}(j+1)(a_j-2a_{j+1}+a_{j+2})K_j(t)$$

for real values of t. Let us recall the well known fact that Fejer's kernel $K_j(t)$ is always non-negative. Further since the real number C is positive, it follows that

$$a_{m-1}-a_m\geqq 0$$

and

$$a_ja_{j+2}=e^{2C}a_{j+1}^2>a_{j+1}^2,$$

so that

$$a_j-2a_{j+1}+a_{j+2}>0.$$

Combining these facts with (7.7), we at once reach

$$J_n(t)\geqq\frac{1}{2}+\frac{a_m}{2}(D_m(t)+D_{m-1}(t))$$

$$=\frac{1}{2}+\frac{a_m}{2}\cot\frac{t}{2}\sin mt$$

for real values of t. Therefore

(7.8) $$2J_n(c_j)\geqq 1$$

for each integer j with $0\leqq j\leqq m$. Hence by means of (7.6), this (7.8) yields that $(-1)^jI_n(c_j)$ must be positive, so that the real valued function $I_n(t)$ has at least one zero-point in each open interval (c_j, c_{j+1}). Consequently, by virtue of (7.5), the polynomial $P_n^*(z)$ has exactly $2m$ zeros in the unit circle, since the coefficients of

$P_n^*(z)$ are all real numbers. Thus we have proved our assertion when n is even.

Next we discuss the case where n is odd. Since the assertion can be verified directly when $n=3$, we may assume that the natural number n is odd and $n\geqq 5$. Let us set $n=2m+1$, and let us consider the trigonometric polynomial defined by

$$J_n^*(t)=\sum_{k=0}^{m} a_k \cos kt \,.$$

Then by using Abel's transformation twice, we can easily see that

$$J_n^*(t)=\frac{1}{2}+a_m D_m(t)+m(a_{m-1}-a_m)K_{m-1}(t)$$

$$+\sum_{j=0}^{m-2}(j+1)(a_j-2a_{j+1}+a_{j+2})K_j(t)\,.$$

As before, since the real number C is positive, it follows that $a_j-2a_{j+1}+a_{j+2}$ and $a_{m-1}-a_m$ must be positive. Hence we get

$$J_n^*(t)\geqq\frac{1}{2}+a_m D_m(t)$$

$$=\frac{1}{2}+\frac{a_m \sin{(m+1/2)t}}{2\sin{(1/2)t}}$$

for real values of t. Therefore by setting $x_j=2j\pi/(2m+1)$ $(0\leqq j\leqq m)$, this yields $2J_n^*(x_j)\geqq 1$ for each point x_j. Here we further introduce the trigonometric polynomial defined by

$$I_n^*(t)=\sum_{k=0}^{m} a_k \cos\left(m-k+\frac{1}{2}\right)t\,.$$

It is clear from the definitions that

$$I_n^*(x_j)=(-1)^j J_n^*(x_j)$$

for each integer j with $0\leqq j\leqq m$. Then this real valued function $I_n^*(t)$ must have at least m zeros in the open interval $(0, \pi)$. In addition to this fact, by taking into account of $a_{n-k}=a_k$, the polynomial $P_n^*(z)$ can be rewritten in the form

$$P_n^*(e^{it})=2\exp\left(i\left(m+\frac{1}{2}\right)t\right)\sum_{k=0}^{m} a_k \cos\left(m-k+\frac{1}{2}\right)t$$

$$=2\exp\left(i\left(m+\frac{1}{2}\right)t\right)I_n^*(t)$$

for real values of t. Accordingly, we can conclude that $P_n^*(e^{it})$ has at least m zeros in the open interval $(0, \pi)$, so that $P_n^*(z)$ must have at least $2m$ zeros in the unit circle except for the point -1. Clearly $P_n^*(-1)=0$. Therefore this polynomial $P_n^*(z)$ has exactly $2m+1$ zeros in the unit circle, which is to be proved. This completes the proof.

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