Y. HIRASAWA
KODAI MATH. J.
2 (1979), 43—53

ON A SINGULAR PERTURBATION PROBLEM FOR LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS, II

BY YOSHIKAZU HIRASAWA

1. Let us consider two linear systems of ordinary differential equations containing a small positive parameter ε:

$$\varepsilon\frac{d\boldsymbol{u}}{dt}=(A_0-\varepsilon A_1)\boldsymbol{u}+\boldsymbol{\delta}_0(\varepsilon)-\varepsilon\boldsymbol{\delta}_1(\varepsilon)\,, \tag{1}$$

$$\varepsilon\frac{d\boldsymbol{u}}{dt}=(A_0+\varepsilon A_1)\boldsymbol{u}+\boldsymbol{\delta}_0(\varepsilon)+\varepsilon\boldsymbol{\delta}_1(\varepsilon)\,, \tag{2}$$

where

$$\boldsymbol{u}=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},\quad A_0=\begin{pmatrix} 0 & 0 & 0 \\ \alpha & -\beta & \gamma \\ -\alpha & -\gamma & \beta \end{pmatrix},\quad A_1=\begin{pmatrix} \alpha & \alpha & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and α, β, γ are positive constants such that $\gamma<\beta$.

Further

$$\boldsymbol{\delta}_0(\varepsilon)=\begin{pmatrix} 0 \\ \delta_2(\varepsilon) \\ \delta_3(\varepsilon) \end{pmatrix},\quad \boldsymbol{\delta}_1(\varepsilon)=\begin{pmatrix} \delta_1(\varepsilon) \\ 0 \\ 0 \end{pmatrix}$$

are three-dimensional real vectors defined on $0\leqq\varepsilon\leqq\varepsilon_0$ and

$$\delta_j(\varepsilon)\longrightarrow 0\,(j=1,2,3)\qquad\text{for}\quad \varepsilon\longrightarrow +0\,.$$

For a given interval $t_1\leqq t\leqq t_2$ and for a given point t_0 such that $t_1\leqq t_0\leqq t_2$, we need a set of continuous functions $u_1(t;\varepsilon)$, $u_2(t;\varepsilon)$, $u_3(t;\varepsilon)$ of t with the following properties.

(I) The conditions

$$u_1(t_0;\varepsilon)=P(\varepsilon)\,,\ u_2(t_1;\varepsilon)=Q(\varepsilon)\,,\ u_3(t_2;\varepsilon)=R(\varepsilon)$$

are fulfilled, where $P(\varepsilon)$, $Q(\varepsilon)$ and $R(\varepsilon)$ are suitable positive quantities tending to

Received September 1, 1977.

zero with ε, such that $P_0(\varepsilon)\leqq P(\varepsilon)$, $Q_0(\varepsilon)\leqq Q(\varepsilon)$, $R_0(\varepsilon)\leqq R(\varepsilon)$ for given positive quantities $P_0(\varepsilon)$, $Q_0(\varepsilon)$ and $R_0(\varepsilon)$ defined on $0\leqq\varepsilon\leqq\varepsilon_0$ and tending to zero with ε.

(II) $u_1(t;\varepsilon)$, $u_2(t;\varepsilon)$, $u_3(t;\varepsilon)$ are positive on $t_1\leqq t\leqq t_2$.

(III) $\boldsymbol{u}=\boldsymbol{u}(t;\varepsilon)$ satisfies the system (1) for $t_1\leqq t\leqq t_0$ and satisfies the system (2) for $t_0\leqq t\leqq t_2$.

The above-mentioned problem had to be solved in a paper [1], but the solution given in [1] was not complete. In the previous paper [2], we have given a proof of the existence of a solution of the above-mentioned problem for a special case where $t_1=t_0$. In this paper, we will treat the general case where $t_1<t_0<t_2$.

2. The solutions $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$ of two linear equations

$$(A_0-\varepsilon A_1)\boldsymbol{\omega}^{(1)}+\boldsymbol{\delta}_0(\varepsilon)-\varepsilon\boldsymbol{\delta}_1(\varepsilon)=\mathbf{0}$$

and

$$(A_0+\varepsilon A_1)\boldsymbol{\omega}^{(1)}+\boldsymbol{\delta}_0(\varepsilon)+\varepsilon\boldsymbol{\delta}_1(\varepsilon)=\mathbf{0}$$

coincide with each other, and hence let

$$\boldsymbol{\omega}(\varepsilon)=\begin{pmatrix}\omega_1(\varepsilon)\\ \omega_2(\varepsilon)\\ \omega_3(\varepsilon)\end{pmatrix}$$

be the solution of these equations, that is uniquely determined by virtue of the fact that

$$\det(A_0+A_1)=-\alpha(\beta+\gamma)(2\alpha+\beta-\gamma)\neq 0\,.$$

We see clearly $\omega_j(\varepsilon)\rightarrow 0\,(j=1,2,3)$ for $\varepsilon\rightarrow +0$.

Let $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$ be the solutions of equations

$$\varepsilon\frac{d\boldsymbol{v}^{(1)}}{dt}=(A_0-\varepsilon A_1)\boldsymbol{v}^{(1)} \tag{3}$$

and

$$\varepsilon\frac{d\boldsymbol{v}^{(2)}}{dt}=(A_0+\varepsilon A_1)\boldsymbol{v}^{(2)}\,. \tag{4}$$

Then the solutions $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ of the equations (1) and (2) can be written in the forms

$$\boldsymbol{u}^{(1)}=\boldsymbol{v}^{(1)}+\boldsymbol{\omega}(\varepsilon)\quad\text{and}\quad \boldsymbol{u}^{(2)}=\boldsymbol{v}^{(2)}+\boldsymbol{\omega}(\varepsilon)$$

Put

$$A^{(1)}=\frac{A_0}{\varepsilon}-A_1=\begin{pmatrix}-\alpha & -\alpha & -\alpha\\ \dfrac{\alpha}{\varepsilon} & -\dfrac{\beta}{\varepsilon} & \dfrac{\gamma}{\varepsilon}\\ -\dfrac{\alpha}{\varepsilon} & -\dfrac{\gamma}{\varepsilon} & \dfrac{\beta}{\varepsilon}\end{pmatrix}; \tag{5}$$

$$(6)\qquad A^{(2)}=\frac{A_0}{\varepsilon}+A_1=\begin{pmatrix} \alpha & \alpha & \alpha \\ \frac{\alpha}{\varepsilon} & -\frac{\beta}{\varepsilon} & \frac{\gamma}{\varepsilon} \\ -\frac{\alpha}{\varepsilon} & -\frac{\gamma}{\varepsilon} & \frac{\beta}{\varepsilon} \end{pmatrix},$$

then the characteristic equations of $A^{(1)}$ and $A^{(2)}$ are

$$(7)\qquad \varepsilon^2\lambda^3+\varepsilon^2\alpha\lambda^2-(\beta^2-\gamma^2)\lambda -\alpha\{(\beta^2-\gamma^2)+2\alpha(\beta+\gamma)\}=0$$

and

$$(8)\qquad \varepsilon^2\lambda^3-\varepsilon^2\alpha\lambda^2-(\beta^2-\gamma^2)\lambda +\alpha\{(\beta^2-\gamma^2)+2\alpha(\beta+\gamma)\}=0.$$

Since, if we replace λ by $-\lambda$ in the equation (7), we have the equation (8), the characteristic roots of $A^{(2)}$ are obtained, by changing the sign of the characteristic roots of $A^{(1)}$.

The roots $\rho_1^{(1)}$, $\rho_2^{(1)}$, $\rho_3^{(1)}$ of the equation (7) can be regarded as algebraic functions of ε, and hence we put

$$\lambda=a_0+a_1\varepsilon+\cdots,$$

or

$$\lambda=\frac{b_{-1}}{\varepsilon}+b_0+b_1\varepsilon+\cdots$$

in order to find these roots.

Substituting these series into (7) and determining the coefficients $a_0, a_1, \cdots$, or $b_{-1}, b_0, \cdots$, we get,

$$(9)\qquad \begin{cases} \rho_1^{(1)}=\rho_{10}^{(1)}+O(\varepsilon), \quad \rho_{10}^{(1)}=-\dfrac{2\alpha^2+\alpha(\beta-\gamma)}{\beta-\gamma}, \\ \rho_2^{(1)}=-\dfrac{\mu}{\varepsilon}+O(1), \quad \rho_3^{(1)}=\dfrac{\mu}{\varepsilon}+O(1), \end{cases}$$

where $\mu=\sqrt{\beta^2-\gamma^2}$.

Furthermore, as the characteristic roots $\rho_1^{(2)}$, $\rho_2^{(2)}$, $\rho_3^{(2)}$ of $A^{(2)}$, we have

$$(10)\qquad \begin{cases} \rho_1^{(2)}=-\rho_1^{(1)}=\rho_{10}^{(2)}+O(\varepsilon), \quad \rho_{10}^{(2)}=\dfrac{2\alpha^2+\alpha(\beta-\gamma)}{\beta-\gamma}, \\ \rho_2^{(2)}=-\rho_3^{(1)}=-\dfrac{\mu}{\varepsilon}+O(1), \\ \rho_3^{(2)}=-\rho_2^{(1)}=\dfrac{\mu}{\varepsilon}+O(1). \end{cases}$$

3. The normal forms of $A^{(1)}$ and $A^{(2)}$ are

$$\hat{A}^{(1)}=\begin{pmatrix} \rho_1^{(1)} & & \mathbf{0} \\ & \rho_2^{(1)} & \\ \mathbf{0} & & \rho_3^{(1)} \end{pmatrix}, \quad \hat{A}^{(2)}=\begin{pmatrix} \rho_1^{(2)} & & \mathbf{0} \\ & \rho_2^{(2)} & \\ \mathbf{0} & & \rho_3^{(2)} \end{pmatrix}.$$

Let $S^{(1)}(\varepsilon)$ and $S^{(2)}(\varepsilon)$ be the transforming matrices that transform $A^{(1)}$ into $\hat{A}^{(1)}$ and $A^{(2)}$ into $\hat{A}^{(2)}$, that is,

$$S^{(1)}(\varepsilon)^{-1}A^{(1)}S^{(1)}(\varepsilon)=\hat{A}^{(1)}, \qquad S^{(2)}(\varepsilon)^{-1}A^{(2)}S^{(2)}(\varepsilon)=\hat{A}^{(2)}.$$

The matrix $S^{(2)}(\varepsilon)$ is obtained by exchanging the second column with the third column in $S^{(1)}(\varepsilon)$ and by exchanging the second row with the third row in $S^{(1)}(\varepsilon)$. After all, $S^{(1)}(\varepsilon)$ and $S^{(2)}(\varepsilon)$ have respectively the following forms:

$$\text{(11)} \qquad S^{(1)}(\varepsilon)=(s_{ij}^{(1)}(\varepsilon)) = \begin{pmatrix} \beta-\gamma+O(\varepsilon) & O(\varepsilon) & O(\varepsilon) \\ \alpha+O(\varepsilon) & \beta+\mu+O(\varepsilon) & \beta-\mu+O(\varepsilon) \\ \alpha+O(\varepsilon) & \gamma+O(\varepsilon) & \gamma+O(\varepsilon) \end{pmatrix},$$

$$\text{(12)} \qquad S^{(2)}(\varepsilon)=(s_{ij}^{(2)}(\varepsilon)) = \begin{pmatrix} \beta-\gamma+O(\varepsilon) & O(\varepsilon) & O(\varepsilon) \\ \alpha+O(\varepsilon) & \gamma+O(\varepsilon) & \gamma+O(\varepsilon) \\ \alpha+O(\varepsilon) & \beta-\mu+O(\varepsilon) & \beta+\mu+O(\varepsilon) \end{pmatrix}.$$

By the transformation of unknowns $\boldsymbol{v}^{(1)}=S^{(1)}(\varepsilon)\hat{\boldsymbol{v}}^{(1)}$ and $\boldsymbol{v}^{(2)}=S^{(2)}(\varepsilon)\hat{\boldsymbol{v}}^{(2)}$, the systems (3) and (4) are changed into

$$\text{(13)} \qquad \frac{d\hat{\boldsymbol{v}}^{(1)}}{dt}=\hat{A}^{(1)}\hat{\boldsymbol{v}}^{(1)},$$

$$\text{(14)} \qquad \frac{d\hat{\boldsymbol{v}}^{(2)}}{dt}=\hat{A}^{(2)}\hat{\boldsymbol{v}}^{(2)}.$$

We will seek for the desired solution in the following form:

$$\text{(15)} \qquad \boldsymbol{u}^{(1)}(t;\ \varepsilon)=S^{(1)}(\varepsilon)\begin{pmatrix} C_1^{(1)}(\varepsilon)\,e^{\rho_1^{(1)}(t-t_0)} \\ C_2^{(1)}(\varepsilon)\,e^{\rho_2^{(1)}(t-t_1)} \\ C_3^{(1)}(\varepsilon)\,e^{\rho_3^{(1)}(t-t_0)} \end{pmatrix}+\boldsymbol{\omega}(\varepsilon)$$

for $t_1\leqq t\leqq t_0$ and

$$\boldsymbol{u}^{(2)}(t\,;\,\varepsilon)=S^{(2)}(\varepsilon)\begin{pmatrix} C_1^{(2)}(\varepsilon)e^{\rho_1^{(2)}(t-t_0)} \\ C_2^{(2)}(\varepsilon)e^{\rho_2^{(2)}(t-t_0)} \\ C_3^{(2)}(\varepsilon)e^{\rho_3^{(2)}(t-t_2)} \end{pmatrix}+\boldsymbol{\omega}(\varepsilon) \tag{16}$$

for $t_0\leqq t\leqq t_2$.

It is sufficient for us to determine the positive quantities $P(\varepsilon), Q(\varepsilon), R(\varepsilon)$ and then the coefficients $C_j^{(1)}(\varepsilon)$, $C_j^{(2)}(\varepsilon)$ $(j=1,2,3)$ in terms of the $P(\varepsilon), Q(\varepsilon), R(\varepsilon)$, so that the conditions

$$\begin{cases} u_1^{(1)}(t_0\,;\,\varepsilon)=P(\varepsilon)\,, \quad u_2^{(1)}(t_1\,;\,\varepsilon)=Q(\varepsilon)\,, \\ u_1^{(2)}(t_0\,;\,\varepsilon)=P(\varepsilon)\,, \quad u_3^{(2)}(t_2\,;\,\varepsilon)=R(\varepsilon)\,, \\ u_2^{(1)}(t_0\,;\,\varepsilon)=u_2^{(2)}(t_0\,;\,\varepsilon)\,, \quad u_3^{(1)}(t_0\,;\,\varepsilon)=u_3^{(2)}(t_0\,;\,\varepsilon) \end{cases} \tag{17}$$

are satisfied, and further $u_j^{(1)}(t\,;\,\varepsilon)$ and $u_j^{(2)}(t\,,\,\varepsilon)(j=1,2,3)$ are respectively positive on the intervals $t_1\leqq t\leqq t_0$ and $t_0\leqq t\leqq t_2$.

We denote $s_{ij}^{(k)}(\varepsilon)$, $C_j^{(k)}(\varepsilon)$ $(k=1,2\,;\,i,j=1,2,3)$ by $s_{ij}^{(k)}$, $C_j^{(k)}$ for brevity. Then, we can, by rearranging, write the conditions (17) explicitly as follows.

$$\begin{cases} s_{11}^{(1)}C_1^{(1)}+s_{12}^{(1)}C_2^{(1)}e^{\rho_2^{(1)}(t_0-t_1)}+s_{13}^{(1)}C_3^{(1)}=\hat{P}(\varepsilon) \\ s_{21}^{(1)}C_1^{(1)}e^{\rho_1^{(1)}(t_1-t_0)}+s_{22}^{(1)}C_2^{(1)}+s_{23}^{(1)}C_3^{(1)}e^{\rho_3^{(1)}(t_1-t_0)}=\hat{Q}(\varepsilon) \\ s_{31}^{(1)}C_1^{(1)}+s_{32}^{(1)}C_2^{(1)}e^{\rho_2^{(1)}(t_0-t_1)}+s_{33}^{(1)}C_3^{(1)} \\ \qquad\qquad -s_{31}^{(2)}C_1^{(2)}-s_{32}^{(2)}C_2^{(2)}-s_{33}^{(2)}C_3^{(2)}e^{\rho_3^{(2)}(t_0-t_2)}=0 \\ \qquad\qquad s_{11}^{(2)}C_1^{(2)}+s_{12}^{(2)}C_2^{(2)}+s_{13}^{(2)}C_3^{(2)}e^{\rho_3^{(2)}(t_0-t_2)}=\hat{P}(\varepsilon) \\ -s_{21}^{(1)}C_1^{(1)}-s_{22}^{(1)}C_2^{(1)}e^{\rho_2^{(1)}(t_0-t_1)}-s_{23}^{(1)}C_3^{(1)} \\ \qquad\qquad +s_{21}^{(2)}C_1^{(2)}+s_{22}^{(2)}C_2^{(2)}+s_{23}^{(2)}C_3^{(2)}e^{\rho_3^{(2)}(t_0-t_2)}=0 \\ s_{31}^{(2)}C_1^{(2)}e^{\rho_1^{(2)}(t_2-t_0)}+s_{32}^{(2)}C_2^{(2)}e^{\rho_2^{(2)}(t_2-t_0)}+s_{33}^{(2)}C_3^{(2)}=\hat{R}(\varepsilon) \end{cases} \tag{18}$$

where

$$\begin{cases} \hat{P}(\varepsilon)=P(\varepsilon)-\omega_1(\varepsilon)\,, \qquad \hat{Q}(\varepsilon)=Q(\varepsilon)-\omega_2(\varepsilon)\,, \\ \hat{R}(\varepsilon)=R(\varepsilon)-\omega_3(\varepsilon)\,. \end{cases} \tag{19}$$

4. We take a positive function $\eta(\varepsilon)$ defined on $0\leqq\varepsilon\leqq\varepsilon_0$ such that $\eta(\varepsilon)\to 0$ and $\eta(\varepsilon)/\varepsilon\to+\infty$ as $\varepsilon\to+0$. For example, we choose the function

$$\eta(\varepsilon)=\frac{M_0}{|\log\varepsilon|+1}\ (\varepsilon>0) \quad \text{and} \quad \eta(\varepsilon)=0\ (\varepsilon=0) \tag{20}$$

from now on, where M_0 is a positive constant.

We can assume that there exists a positive constant M such that

$$P_0(\varepsilon)\leqq M\,, \qquad Q_0(\varepsilon)\leqq M\,, \qquad R_0(\varepsilon)\leqq M\,, \tag{21}$$

(22) $$|\omega_j(\varepsilon)| \leqq M\ (j=1,2,3), \qquad \eta(\varepsilon)\leqq M$$

for $0\leqq\varepsilon\leqq\varepsilon_0$.

In the next number, it will be shown that it is sufficient for us to determine the $P(\varepsilon)$, $Q(\varepsilon)$, $R(\varepsilon)$ in the following manner.

Concerning the $P(\varepsilon)$, we first put

(23) $$\theta(\varepsilon)=\mathrm{Max}\left\{|\omega_1(\varepsilon)|,\ \frac{\beta-\gamma}{\alpha}|\omega_2(\varepsilon)|,\ \frac{\beta-\gamma}{\alpha}|\omega_3(\varepsilon)|\right\}$$

and we choose

(24) $$P(\varepsilon)=\mathrm{Max}\{P_0(\varepsilon),\ \eta(\varepsilon)+|\omega_1(\varepsilon)|+\theta(\varepsilon)\}.$$

Obviously $P_0(\varepsilon)\leqq P(\varepsilon)$ for $0\leqq\varepsilon\leqq\varepsilon_0$.

If we take a positive constant N_0 such that

$$\mathrm{Max}\left\{1,\ \frac{\beta-\gamma}{\alpha}\right\}\leqq N_0<+\infty,$$

then we see

(25) $$0\leqq\theta(\varepsilon)\leqq N_0M\ (0\leqq\varepsilon\leqq\varepsilon_0)$$

and further

(26) $$\eta(\varepsilon)+\theta(\varepsilon)\leqq\hat{P}(\varepsilon)=P(\varepsilon)-\omega_1(\varepsilon) \leqq(N_0+3)M,$$

(27) $$\lim_{\varepsilon\to+0}\frac{\hat{P}(\varepsilon)}{\varepsilon}=+\infty.$$

Concerning the $Q(\varepsilon)$, we first put

(28) $$N_1=\frac{\alpha e^{\rho_{10}^{(1)}(t_1-t_0)}}{\beta-\gamma}>0\ \left(\rho_{10}^{(1)}=-\frac{2\alpha^2+\alpha(\beta-\gamma)}{\beta-\gamma}\right).$$

Since it follows from the inequalities (22) and (26) that

$$N_1\hat{P}(\varepsilon)+\eta(\varepsilon)+|\omega_2(\varepsilon)|\leqq\{(N_0+3)N_1+2\}M,$$

we can determine the $Q(\varepsilon)$ so that

(29) $$\mathrm{Max}\{Q_0(\varepsilon),\ N_1\hat{P}(\varepsilon)+\eta(\varepsilon)+|\omega_2(\varepsilon)|\} \leqq Q(\varepsilon)\leqq\{(N_0+3)N_1+3\}M.$$

Then we see $Q_0(\varepsilon)\leqq Q(\varepsilon)$, and

(30) $$N_1\hat{P}(\varepsilon)+\eta(\varepsilon)\leqq\hat{Q}(\varepsilon)=Q(\varepsilon)-\omega_2(\varepsilon) \leqq\{(N_0+3)N_1+4\}M,$$

(31) $$\lim_{\varepsilon\to+0}\frac{\hat{Q}(\varepsilon)}{\varepsilon}=+\infty .$$

Concerning the $R(\varepsilon)$, we first put

(32) $$N_2=\frac{\alpha e^{\rho_{10}^{(2)}(t_2-t_0)}}{\beta-\gamma}>0\ \left(\rho_{10}^{(2)}=\frac{2\alpha^2+\alpha(\beta-\gamma)}{\beta-\gamma}\right)$$

and since, by virtue of the inequalities (22) and (26), the inequality

$$N_2\hat{P}(\varepsilon)+\eta(\varepsilon)+|\omega_3(\varepsilon)|\leqq\{(N_0+3)N_2+2\}M$$

holds, we can determine the $R(\varepsilon)$ so that

(33) $$\begin{aligned}&\mathrm{Max}\{R_0(\varepsilon),\ N_2\hat{P}(\varepsilon)+\eta(\varepsilon)+|\omega_3(\varepsilon)|\}\\&\qquad\leqq R(\varepsilon)\leqq\{(N_0+3)N_2+3\}M.\end{aligned}$$

Then we can verify $R_0(\varepsilon)\leqq R(\varepsilon)$, and

(34) $$\begin{aligned}&N_2\hat{P}(\varepsilon)+\eta(\varepsilon)\leqq\hat{R}(\varepsilon)=R(\varepsilon)-\omega_3(\varepsilon)\\&\qquad\leqq\{(N_0+3)N_2+4\}M,\end{aligned}$$

(35) $$\lim_{\varepsilon\to+0}\frac{\hat{R}(\varepsilon)}{\varepsilon}=+\infty .$$

5. In order to solve the equations (18) for $C_j^{(1)}$, $C_j^{(2)}(j=1, 2, 3)$, we put

$$\mathfrak{s}_1^{(1)}=\begin{pmatrix} s_{11}^{(1)}\\ s_{21}^{(1)}e^{\rho_1^{(1)}(t_1-t_0)}\\ s_{31}^{(1)}\\ 0\\ -s_{21}^{(1)}\\ 0\end{pmatrix},\quad \mathfrak{s}_2^{(1)}=\begin{pmatrix} s_{12}^{(1)}e^{\rho_2^{(1)}(t_0-t_1)}\\ s_{22}^{(1)}\\ s_{32}^{(1)}e^{\rho_2^{(1)}(t_0-t_1)}\\ 0\\ -s_{22}^{(1)}e^{\rho_2^{(1)}(t_0-t_1)}\\ 0\end{pmatrix},$$

$$\mathfrak{s}_3^{(1)}=\begin{pmatrix} s_{13}^{(1)}\\ s_{23}^{(1)}e^{\rho_3^{(1)}(t_1-t_0)}\\ s_{33}^{(1)}\\ 0\\ -s_{23}^{(1)}\\ 0\end{pmatrix},$$

$$\mathfrak{s}_1^{(2)}=\begin{pmatrix} 0 \\ 0 \\ -s_{31}^{(2)} \\ s_{11}^{(2)} \\ s_{21}^{(2)} \\ s_{31}^{(2)}e^{\rho_1^{(2)}(t_2-t_0)} \end{pmatrix}, \quad \mathfrak{s}_2^{(2)}=\begin{pmatrix} 0 \\ 0 \\ -s_{32}^{(2)} \\ s_{12}^{(2)} \\ s_{22}^{(2)} \\ s_{32}^{(2)}e^{\rho_2^{(2)}(t_2-t_0)} \end{pmatrix},$$

$$\mathfrak{s}_3^{(2)}=\begin{pmatrix} 0 \\ 0 \\ -s_{33}^{(2)}e^{\rho_3^{(2)}(t_0-t_2)} \\ s_{13}^{(2)}e^{\rho_3^{(2)}(t_0-t_2)} \\ s_{23}^{(2)}e^{\rho_3^{(2)}(t_0-t_2)} \\ s_{33}^{(2)} \end{pmatrix}, \quad \mathfrak{P}=\begin{pmatrix} \hat{P} \\ \hat{Q} \\ 0 \\ \hat{P} \\ 0 \\ \hat{R} \end{pmatrix},$$

where $\hat{P}$, $\hat{Q}$ and $\hat{R}$ denote $\hat{P}(\varepsilon)$, $\hat{Q}(\varepsilon)$ and $\hat{R}(\varepsilon)$.

Taking account of the fact that

$$s_{i2}^{(1)}e^{\rho_2^{(1)}(t_0-t_1)}=o(\varepsilon) \ (i=1, 2, 3),$$

$$s_{13}^{(1)}=O(\varepsilon), \qquad s_{23}^{(1)}e^{\rho_3^{(1)}(t_1-t_0)}=o(\varepsilon),$$

$$s_{12}^{(2)}=O(\varepsilon), \qquad s_{32}^{(2)}e^{\rho_2^{(2)}(t_2-t_0)}=o(\varepsilon),$$

$$s_{i3}^{(2)}e^{\rho_3^{(2)}(t_0-t_2)}=o(\varepsilon) \ (i=1, 2, 3)$$

hold and taking the inequalities (26), (30), (34) and the properties (27), (31), (35) of the $\hat{P}(\varepsilon)$, $\hat{Q}(\varepsilon)$, $\hat{R}(\varepsilon)$ into consideration, we have

$$\Delta(\varepsilon)=\det(\mathfrak{s}_1^{(1)}, \mathfrak{s}_2^{(1)}, \mathfrak{s}_3^{(1)}, \mathfrak{s}_1^{(2)}, \mathfrak{s}_2^{(2)}, \mathfrak{s}_3^{(2)})$$

$$=s_{11}^{(1)}s_{22}^{(1)}(s_{33}^{(1)}s_{22}^{(2)}-s_{23}^{(1)}s_{32}^{(2)})s_{11}^{(2)}s_{33}^{(2)}+O(\varepsilon)$$

$$\Delta_1^{(1)}(\varepsilon)=\det(\mathfrak{P}, \mathfrak{s}_2^{(1)}, \mathfrak{s}_3^{(1)}, \mathfrak{s}_1^{(2)}, \mathfrak{s}_2^{(2)}, \mathfrak{s}_3^{(2)})$$

$$=s_{22}^{(1)}(s_{33}^{(1)}s_{22}^{(2)}-s_{23}^{(1)}s_{32}^{(2)})s_{11}^{(2)}s_{33}^{(2)}\hat{P}+O(\varepsilon),$$

$$\Delta_2^{(1)}(\varepsilon)=\det(\mathfrak{s}_1^{(1)}, \mathfrak{P}, \mathfrak{s}_3^{(1)}, \mathfrak{s}_1^{(2)}, \mathfrak{s}_2^{(2)}, \mathfrak{s}_3^{(2)})$$

$$=(-s_{21}^{(1)}e^{\rho_1^{(1)}(t_1-t_0)}\hat{P}+s_{11}^{(1)}\hat{Q})$$

$$\times(s_{33}^{(1)}s_{22}^{(2)}-s_{23}^{(1)}s_{32}^{(2)})s_{11}^{(2)}s_{33}^{(2)}+O(\varepsilon)$$

$$\Delta_3^{(1)}(\varepsilon)=\det(\mathfrak{s}_1^{(1)},\ \mathfrak{s}_2^{(2)},\ \mathfrak{P},\ \mathfrak{s}_1^{(2)},\ \mathfrak{s}_2^{(2)},\ \mathfrak{s}_3^{(2)})$$
$$=s_{11}^{(1)}s_{22}^{(1)}(s_{31}^{(2)}s_{22}^{(2)}-s_{32}^{(2)}s_{21}^{(2)})s_{33}^{(2)}\hat{P}$$
$$-s_{22}^{(1)}s_{11}^{(2)}(s_{31}^{(1)}s_{22}^{(2)}-s_{32}^{(2)}s_{21}^{(1)})s_{33}^{(2)}\hat{P}+O(\varepsilon)\,.$$

$$\Delta_1^{(2)}(\varepsilon)=\det(\mathfrak{s}_1^{(1)},\ \mathfrak{s}_2^{(1)},\ \mathfrak{s}_3^{(1)},\ \mathfrak{P},\ \mathfrak{s}_2^{(2)},\ \mathfrak{s}_3^{(2)})$$
$$=s_{11}^{(1)}s_{22}^{(1)}(s_{33}^{(1)}s_{22}^{(2)}-s_{23}^{(1)}s_{32}^{(2)})s_{33}^{(2)}\hat{P}+O(\varepsilon)\,,$$

$$\Delta_2^{(2)}(\varepsilon)=\det(\mathfrak{s}_1^{(1)},\ \mathfrak{s}_2^{(1)},\ \mathfrak{s}_3^{(1)},\ \mathfrak{s}_1^{(2)},\ \mathfrak{P},\ \mathfrak{s}_3^{(2)})$$
$$=s_{11}^{(1)}s_{22}^{(1)}(s_{31}^{(2)}s_{23}^{(1)}-s_{33}^{(1)}s_{21}^{(2)})s_{33}^{(2)}\hat{P}$$
$$-s_{22}^{(1)}s_{11}^{(2)}(s_{31}^{(1)}s_{23}^{(1)}-s_{33}^{(1)}s_{21}^{(1)})s_{33}^{(2)}\hat{P}+O(\varepsilon)\,,$$

$$\Delta_3^{(2)}(\varepsilon)=\det(\mathfrak{s}_1^{(1)},\ \mathfrak{s}_2^{(1)},\ \mathfrak{s}_3^{(1)},\ \mathfrak{s}_1^{(2)},\ \mathfrak{s}_2^{(2)},\ \mathfrak{P})$$
$$=s_{11}^{(1)}s_{22}^{(1)}(s_{33}^{(1)}s_{22}^{(2)}-s_{23}^{(1)}s_{32}^{(2)})$$
$$\times(-s_{31}^{(2)}e^{\rho_1^{(2)}(t_2-t_0)}\hat{P}+s_{11}^{(2)}\hat{R})+O(\varepsilon)\,.$$

Thus, we get

$$C_1^{(1)}(\varepsilon)=\frac{\Delta_1^{(1)}(\varepsilon)}{\Delta(\varepsilon)}=\frac{\hat{P}}{s_{11}^{(1)}}+O(\varepsilon)=\frac{\hat{P}(\varepsilon)}{\beta-\gamma}+O(\varepsilon)\,, \tag{36}$$

$$C_2^{(1)}(\varepsilon)=\frac{\Delta_2^{(1)}(\varepsilon)}{\Delta(\varepsilon)}=\frac{-s_{21}^{(1)}e^{\rho_1^{(1)}(t_1-t_0)}\hat{P}+s_{11}^{(1)}\hat{Q}}{s_{11}^{(1)}s_{22}^{(1)}}+O(\varepsilon) \tag{37}$$
$$=\frac{1}{\beta+\sqrt{\beta^2-\gamma^2}}\Big(-\frac{\alpha e^{\rho_{10}^{(1)}(t_1-t_0)}}{\beta-\gamma}\hat{P}(\varepsilon)+\hat{Q}(\varepsilon)\Big)+O(\varepsilon)\,.$$

where $\rho_1^{(1)}=\rho_{10}^{(1)}+O(\varepsilon)$, $\rho_{10}^{(1)}=-\dfrac{2\alpha^2+\alpha(\beta-\gamma)}{\beta-\gamma}$.

By virtue of $s_{11}^{(1)}=s_{11}^{(2)}$, $s_{21}^{(1)}=s_{21}^{(2)}$, $s_{31}^{(1)}=s_{31}^{(2)}$, we have

$$C_3^{(1)}(\varepsilon)=\frac{\Delta_3^{(1)}(\varepsilon)}{\Delta(\varepsilon)}=O(\varepsilon)\,. \tag{38}$$

Furthermore we obtain

$$C_1^{(2)}(\varepsilon)=\frac{\Delta_1^{(2)}(\varepsilon)}{\Delta(\varepsilon)}=\frac{\hat{P}}{s_{11}^{(2)}}+O(\varepsilon)=\frac{\hat{P}(\varepsilon)}{\beta-\gamma}+O(\varepsilon)\,. \tag{39}$$

By virtue of $s_{11}^{(1)}=s_{11}^{(2)}$, $s_{21}^{(1)}=s_{21}^{(2)}$, $s_{31}^{(1)}=s_{31}^{(2)}$, we get

$$C_2^{(2)}(\varepsilon)=\frac{\Delta_2^{(2)}(\varepsilon)}{\Delta(\varepsilon)}=O(\varepsilon)\,. \tag{40}$$

And we have

(41) $$C_3^{(2)}(\varepsilon)=\frac{\Delta_3^{(2)}(\varepsilon)}{\Delta(\varepsilon)}=\frac{-s_{31}^{(2)}e^{\rho_1^{(2)}(t_2-t_0)}\hat{P}+s_{11}^{(2)}\hat{R}}{s_{11}^{(2)}s_{33}^{(2)}}+O(\varepsilon)$$
$$=\frac{1}{\beta+\sqrt{\beta^2-\gamma^2}}\Big(-\frac{\alpha e^{\rho_{10}^{(2)}(t_2-t_0)}}{\beta-\gamma}\hat{P}(\varepsilon)+\hat{R}(\varepsilon)\Big)+O(\varepsilon),$$

where $\rho_1^{(2)}=\rho_{10}^{(2)}+O(\varepsilon)$, $\rho_{10}^{(2)}=\dfrac{2\alpha^2+\alpha(\beta-\gamma)}{\beta-\gamma}$.

We denote generically by ε_1, a sufficiently small positive number such that $0<\varepsilon_1\leqq\varepsilon_0$.

By virtue of (11) and (15), we can write for $u_1^{(1)}(t;\varepsilon)$

$$u_1^{(1)}(t;\varepsilon)=s_{11}^{(1)}C_1^{(1)}e^{\rho_1^{(1)}(t-t_0)}+s_{12}^{(1)}C_2^{(1)}e^{\rho_2^{(1)}(t-t_1)}$$
$$+s_{13}^{(1)}C_3^{(1)}e^{\rho_3^{(1)}(t-t_0)}+\omega_1(\varepsilon)$$
$$=(\hat{P}(\varepsilon)+\omega_1(\varepsilon)e^{-\rho_{10}^{(1)}(t-t_0)})e^{\rho_{10}^{(1)}(t-t_0)}+O(\varepsilon).$$

The definition (24) of the $P(\varepsilon)$ and the inequality (26) imply

$$\hat{P}(\varepsilon)+\omega_1(\varepsilon)e^{-\rho_{10}^{(1)}(t-t_0)}\geqq\eta(\varepsilon)>0$$
$$(0<\varepsilon\leqq\varepsilon_1,\ t_1\leqq t\leqq t_0)$$

because $e^{-\rho_{10}^{(1)}(t-t_0)}\leqq 1$ $(t_1\leqq t\leqq t_0)$ and hence we see

$$u_1^{(1)}(t;\varepsilon)>0 \qquad (0<\varepsilon\leqq\varepsilon_1,\ t_1\leqq t\leqq t_0).$$

Next, by virtue of (11) and (15), we have

$$u_2^{(1)}(t;\varepsilon)=s_{21}^{(1)}C_1^{(1)}e^{\rho_1^{(1)}(t-t_0)}+s_{22}^{(1)}C_2^{(1)}e^{\rho_2^{(1)}(t-t_1)}$$
$$+s_{23}^{(1)}C_3^{(1)}e^{\rho_3^{(1)}(t-t_0)}+\omega_2(\varepsilon)$$
$$=\Big(\frac{\alpha}{\beta-\gamma}\hat{P}(\varepsilon)+\omega_2(\varepsilon)e^{-\rho_{10}^{(1)}(t-t_0)}\Big)e^{\rho_1^{(1)}(t-t_0)}$$
$$+\Big(-\frac{\alpha e^{\rho_{10}^{(1)}(t_1-t_0)}}{\beta-\gamma}\hat{P}(\varepsilon)+\hat{Q}(\varepsilon)\Big)e^{\rho_2^{(1)}(t-t_1)}+O(\varepsilon).$$

It follows from (24) and (26) that

$$\frac{\alpha}{\beta-\gamma}\hat{P}(\varepsilon)+\omega_2(\varepsilon)e^{-\rho_{10}^{(1)}(t-t_0)}\geqq\frac{\alpha}{\beta-\gamma}\eta(\varepsilon)$$
$$(0<\varepsilon\leqq\varepsilon_1,\ t_1\leqq t\leqq t_0),$$

and the definition (28) of N_1 and the inequality (30) imply

$$-\frac{\alpha e^{\rho_{10}^{(1)}(t_1-t_0)}}{\beta-\gamma}\hat{P}(\varepsilon)+\hat{Q}(\varepsilon)\geqq\eta(\varepsilon)$$
$$(0<\varepsilon\leqq\varepsilon_1,\ t_1\leqq t\leqq t_0).$$

Hence we see

$$u_2^{(1)}(t;\varepsilon)>0 \qquad (0<\varepsilon\leqq\varepsilon_1,\ t_1\leqq t\leqq t_0).$$

Moreover, we get

$$\begin{aligned}u_3^{(1)}(t;\varepsilon)&=s_{31}^{(1)}C_1^{(1)}e^{\rho_1^{(1)}(t-t_0)}+s_{32}^{(1)}C_2^{(1)}e^{\rho_2^{(1)}(t-t_1)}\\&\quad+s_{33}^{(1)}C_3^{(1)}e^{\rho_3^{(1)}(t-t_0)}+\omega_3(\varepsilon)\\&=\Big(\frac{\alpha}{\beta-\gamma}\hat{P}(\varepsilon)+\omega_3(\varepsilon)e^{-\rho_{10}^{(1)}(t-t_0)}\Big)e^{\rho_1^{(1)}(t-t_0)}\\&\quad+\frac{\gamma}{\beta+\sqrt{\beta^2-\gamma^2}}\Big(-\frac{\alpha e^{\rho_{10}^{(1)}(t_1-t_0)}}{\beta-\gamma}\hat{P}(\varepsilon)+\hat{Q}(\varepsilon)\Big)e^{\rho_2^{(1)}(t-t_1)}+O(\varepsilon),\end{aligned}$$

and therefore, in the same way as for $u_2^{(1)}(t;\varepsilon)$, we can verify

$$u_3^{(1)}(t;\varepsilon)>0 \qquad (0<\varepsilon\leqq\varepsilon_1,\ t_1\leqq t\leqq t_0).$$

Similarly we can show, by using (23), (24) and (34), that

$$u_j^{(2)}(t;\varepsilon)>0 \qquad (j=1,2,3)$$
$$(0<\varepsilon\leqq\varepsilon_1,\ t_0\leqq t\leqq t_2).$$

Thus, we have accomplished our purpose.

Remark. 1. It is obvious that the solution $\boldsymbol{u}=\boldsymbol{u}(t;\varepsilon)$ obtained in this paper, tends to the zero vector $\mathbf{0}$ on $t_1\leqq t\leqq t_2$ with ε.

Remark 2. If we choose sufficiently small positive constants Q, R instead of $Q(\varepsilon)$, $R(\varepsilon)$, we obtain the desired solution. In this case, the solution $\boldsymbol{u}=\boldsymbol{u}(t;\varepsilon)$ tends to the zero vector $\mathbf{0}$ on $t_1<t<t_2$ with ε.

REFERENCES

[1] HIRASAWA, Y., On singular perturbation problems of non-linear systems of differential equations, III, Comment. Math. Univ. Sancti. Pauli, 4 (1955) 93-104.

[2] HIRASAWA, Y., On a singular perturbation problem for linear systems of ordinary differential equations, I, Kodai Math. J. 1 (1978), 85-88.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.