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A NOTE ON ASYMPTOTIC UNBIASEDNESS OF ESTIMATES

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§1. Introduction.

In estimation theory the concept of asymptotic unbiasedness of estimates is an important extension of ordinary unbiasedness. An estimator is intuitively called asymptotic ally unbiased when the bias function converges to zero with the smaller order than that of the variance for each parameter. This definition, however, admits so called super-efficient estimator whose asymptotic variance never exceeds the Cramér-Rao lower bound and is below it at some parameter ([10], [13]). Walker [12] showed that if the derivative of the bias function also converges to zero, such estimator is excluded. Another approach by Rao [7], [8] and Schmetterer [9] is that if an estimator is asymptotic normal and its convergence is uniform in the compact neighbourhood of each parameter, it is not super-efficient.

In this paper combining these two approaches we will propose a new definition of asymptotic unbiasedness and show that the Cramér-Rao inequality asymptotically holds in a general framework. Although the proposed class of estimates is wider, it is easier to verify the conditions because the definition depends only on how the bias function converges to zero and no asymptotic distribution or derivative of the bias function is needed.

§2. Asymptotic unbiasedness.

Let $X_n=(x_1, x_2, \cdots, x_n)$ be a sequence of n random variables distributed according to the joint probability density function $f(X_n, \theta)$ with respect to a σ-finite measure $\mu(dX_n)$ on R^n, where θ is a parameter which can take any value in an open interval Θ. We assume the following regularity condition on the density $f(X_n, \theta)$.

CONDITION 1. *For each $\theta\in\Theta$ there exists a positive sequence c_n (which may depend on θ) such that*

$$(2.1)\qquad \bar{I}(\theta)=\overline{\lim_{h\to+0}}\frac{1}{h^2}\overline{\lim_{n\to\infty}}E_\theta\left(\frac{f(X_n, \theta+h/c_n)-f(X_n, \theta)}{f(X_n, \theta)}\right)^2$$

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is positive finite, where the expectation E_θ is taken on the support of $f(X_n, \theta)$.

As is well known, with some reguralty conditions $\bar{I}(\theta)$ is represented as

$$\bar{I}(\theta)=\overline{\lim_{h\to+0}}\frac{1}{h^2}\overline{\lim_{n\to\infty}} I_n(\theta, \theta+h/c_n)$$

by the Kullback Leibler information

$$I_n(\theta_1, \theta_2)=E_{\theta_1}\left\{\log\frac{f(X_n, \theta_1)}{f(X_n, \theta_2)}\right\} \qquad \theta_1, \theta_2\in\Theta$$

(see [4]).

Condition 1 is satisfied if $x_1, x_2, \cdots, x_n$ are independent, identically distributed and the Fisher information

$$I(\theta)=E_\theta\left(\frac{\partial}{\partial\theta}\log f(X_1, \theta)\right)^2$$

is positive finite, in fact putting $c_n=\sqrt{n}$ we have

$$\bar{I}(\theta)=\overline{\lim_{h\to+0}}\frac{1}{h^2}\overline{\lim_{n\to\infty}}\left\{\left(E_\theta\left(\frac{f(X_1, \theta+h/\sqrt{n})-f(X_1, \theta)}{f(X_1, \theta)}\right)^2+1\right)^n-1\right\}$$

$$=\lim_{h\to0}\frac{1}{h^2}\left[\lim_{n\to\infty}\left\{I(\theta)\frac{h^2}{n}+1\right\}^n-1\right]$$

$$=\lim_{h\to0}\frac{1}{h^2}[e^{I(\theta)h^2}-1]=I(\theta).$$

In this paper we will consider estimation of the parameter θ. Condition 1 implies that $|c_n(T_n-\theta)|$ is bounded away from zero in probability for any estimator $T_n(X_n)$ of θ ([11]). That is, the possible highest order of consistency is $O(c_n)$. Assuming the existence of the bias function $b_n(\theta)=E_\theta(T_n-\theta)$ we will define asymptotic unbiasedness of T_n in such a general framework.

DEFINITION. An estimator T_n of θ is said to be asymptotically unbiased (from the right) at θ if

(2.2) $$c_nb_n(\theta)\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty,$$

and one of the following conditions is satisfied

(1) for any small $h>0$

(2.3) $$c_nb_n(\theta+h/c_n)\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty,$$

(2) there exists a positive sequence δ_n which converges to zero with $o(1/c_n)$ and

(2.4) $$(b_n(\theta+\delta_n)-b_n(\theta))/\delta_n\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty.$$

The asymptotic unbiasedness from the left is similarly defined and the discussion is the same. The condition (2.2) requires that the bias of T_n vanishes as $n \to \infty$ with the smaller order than that of variance and the condition (2.3) or (2.4) requires some uniformity of the convergence.

PROPOSITION 1. *If one of the following conditions is satisfied, then T_n is asymptotic ally unbiased,*

a) *both $c_n b_n(\theta)$ and the derivative $b_n'(\theta)$ converge to zero as $n \to \infty$ (Walker [12]),*

b) *the distribution function $F_n(t, \theta)$ of $c_n(T_n-\theta)$ converges to a normal distribution function uniformly in the compact neighbourhood of θ for each fixed t and the $(1+\delta)$th absolute moment of $F_n(t, \theta)$ is bounded for some $\delta>0$ (Rao and Schmetterer [7], [8], [9]),*

c) *$c_n b_n(\theta)$ converges to zero as $n \to \infty$ uniformly in the compact neighbourhood of θ (Ibragimov and Khas'minskii [5]),*

d) *$c_n b_n(\theta)$ converges to zero as $n \to \infty$ and is equicontinuous.*

Proof. If a) holds, we can choose a positive sequence $\delta_n = O(1/c_n)$ such that for any $\varepsilon>0$ there exists a positive integer N and

$$\left|\frac{b_n(\theta+\delta_n)-b_n(\theta)}{\delta_n} - b_n'(\theta)\right| < \varepsilon\,, \qquad n \geqq N\,,$$

so that (2.4) follows. The condition b) implies c), and (2.2) and (2.3) follow from c) or d).

Remark. The moment condition in b) is not so strong since our main concern is in the mean squared error and it is sufficient to consider only estimates whose second moments are bounded.

The above proposition shows that our definition is very wide. Next we will show that the Cramér-Rao inequality holds for such estimates.

THEOREM. *Assume Condition 1. If T_n is asymptotically unbiased then the following inequality holds,*

$$\varliminf_{n\to\infty} E_\theta(c_n(T_n-\theta))^2 \geqq 1/\bar{I}(\theta)\,.$$

Proof. From the Chapman-Robbins inequality [3], we have

$$E_\theta(c_n(T_n-\theta))^2 \geqq \frac{c_n^2(b_n(\theta+\alpha)-b_n(\theta)+\alpha)^2}{E_\theta\left(\dfrac{f(X_n, \theta+\alpha)-f(X_n, \theta)}{f(X_n, \theta)}\right)^2} \tag{2.5}$$

for any $\theta, \theta+\alpha \in \Theta$, where $\alpha>0$. If T_n satisfies (2.2) and (2.3), putting $\alpha = h/c_n$ and dividing the denominator and the numerator of the right side of (2.5) by h^2, we have the desired result from Condition 1. We can obtain the result putting $\alpha=\delta_n$ in the same manner if T_n satisfies (2.2) and (2.4).

Example. Let x_i have the mean

$$\mu_i(\theta)=E_\theta x_i=\begin{cases}\theta-(1-\theta)\theta^{i-1} & \text{if } |\theta|\leqq 1\\ \theta-(1-\theta)\theta^{-i} & \text{otherwise},\end{cases}$$

where $-\infty<\theta<\infty$. Consider an estimator $T_n=\frac{1}{n}\sum_{i=1}^{n}x_i$ of θ, then the bias function of T_n becomes

$$b_n(\theta)=\begin{cases}\frac{1}{n}(\theta^n-1) & \text{if } |\theta|\leqq 1\\ \frac{1}{n}(1-\theta^{-n}) & \text{otherwise}\end{cases}$$

and the derivative is

$$b_n'(\theta)=\begin{cases}\theta^{n-1} & \text{if } |\theta|\leqq 1\\ \theta^{-n-1} & \text{otherwise}.\end{cases}$$

Since $b_n'(1)=1$ and $b_n'(-1)=(-1)^{n-1}$, the condition a) in Proposition 1 is not satisfied at $\theta=1$ or -1. Therefore we can not apply Walker's result. Furthermore if x_i satisfies the following linear equation,

(2.6) $$x_i=x_1+\mu_i(\theta)-\mu_1(\theta)$$

then

$$T_n-\theta=x_1+b_n(\theta)-\mu_1(\theta).$$

Unless x_1 is normal, T_n is not asymptotically unbiased in the sense of Rao and Schmetterer since T_n is not asymptotically normal.

However (2.2) and (2.4) hold for any positive sequence δ_n such as $n\delta_n\to\infty$ and $\delta_n c_n\to 0$ when the Condition 1 is satisfied and $c_n=o(n)$, and T_n is asymptotically unbiased in our sense.

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