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CONFORMAL FOLIATIONS*

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Differential Geometry already suggested a large number of interesting classes of foliations. Particularly interesting were the foliations with bundle-like metric [6] also called Riemannian foliations [5].

Our aim in this Note is to introduce a new class of foliations which generalize the Riemannian foliations and will be called *conformal foliations*. The definition is suggested by natural geometric reasons and the conformal foliations seem to be worthy of attention because there are enough such foliations. E.g., the differentiable foliations of codimension 1 and the complex analytic foliations of complex codimension 1 are all conformal.

We shall obtain characteristic properties of the conformal foliations. These introduce an important 1-form which we shall use in defining and studying some particular classes of conformal foliations.

All our considerations will be in the C^∞-category.

§1. Characteristic properties of conformal foliations.

We begin by defining conformal foliations, via Haefliger cocycles, following the definition of the Riemannian foliations in [5].

If we take as objects (units) the germs of Riemann metrics over R^q and as morphisms the germs of the local diffeomorphisms of R^q which are conformal transformations between the respective germs of Riemann metrics, we get a small category, which is a topological groupoid $C\Gamma_q$.

A *conformal Haefliger structure* of codimension q on a topological space X is defined as a $C\Gamma_q$-structure. Such a structure is represented by a *cocycle,* which consists of an open covering $\{U_\alpha\}$ of X and a system of continuous maps $\gamma_{\alpha\beta}$: $U_\alpha \cap U_\beta \to C\Gamma_q$ such that

$$\gamma_{\lambda\mu}(x)=\gamma_{\lambda\nu}(x)\gamma_{\nu\mu}(x) \qquad (x\in U_\lambda\cap U_\nu\cap U_\mu)\,. \tag{1.1}$$

Two cocycles define the same structure iff their union is again a cocycle.

Because of (1.1), $\gamma_{\alpha\alpha}$: $U_\alpha \to C\Gamma_q$ define continuous maps into the space of the

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units of $C\Gamma_q$ which obviously induce *some* continuous maps $f_\alpha: U_\alpha \to R^q$. If X is a differentiable manifold and the functions f_α are submersions, the considered Haefliger structure is said to be a *conformal foliation* on X.

If the values of $\gamma_{\alpha\beta}$ above are germs of isometries we obtain a Riemannian Haefliger structure or foliation [5]. Hence, these are particular conformal structures or foliations.

A conformal structure (foliation) obviously has an underlying Γ_q-structure (foliation) [1], which allows using the classical theory.

Hereafter, we shall consider only conformal foliations and, in this section, we shall give some characteristic properties of such foliations. We always denote by M an n-dimensional differentiable manifold and by E a differentiable foliation of codimension q on M. $\nu=T(M)/T(E)$ is the *transverse bundle* of E and $P(\nu)$ is its principal frame bundle. We use the labeling *foliate* for all the elements (functions, forms, bundles, etc.) which are constant along the leaves of E [8].

Like in [8], if g is a Riemann metric on M, ν may be identified with the normal bundle of $T(E)$. The foliation E is characterized by an *adapted atlas* $\{U_\alpha;\ x_\alpha^a, x_\alpha^u\}$, where $a=1, \cdots, q$, $u=q+1, \cdots, n$, and the leaves of E are given by $x^a=$const. The transition functions of this atlas have the local form $x_\beta^a=x_\beta^a(x_\alpha^b)$, $x_\beta^u=x_\beta^u(x_\alpha^a, x_\alpha^v)$. The foliate elements depend locally on x^a only. $T(E)$ has the local bases $X_u=\partial/\partial x^u$ and ν has local bases of the form $X_a=\partial/\partial x^a-\sum_u t_a^u(\partial/\partial x^u)$. (The usual summation convention will also be used.) The dual cobases are dx^a, $\theta^u=dx^u+t_a^u dx^a$.

Furthermore, all the components of the tensors and forms will be with respect to the previous bases and cobases, The convention for indices will always be: $a, b, c, \cdots=1, \cdots, q$, $u, v, w, \cdots=q+1, \cdots, n$. The mentioned two parts of the cobases give a natural definition of *forms of the type* (p, q) and this leads to a decomposition $d=d'+d''+\partial$, where d is the exterior differential and the components have respectively the type $(1, 0)$, $(0, 1)$, $(2, -1)$. The expression and properties of these operators are given in [8]. One has $d''^2=0$ and we may speak of a *d''-cohomology*. One also has $d'd''=-d''d'$.

The main characteristic property of the conformal foliations is given by

PROPOSITION 1.1. *The foliation E on M is conformal iff its transverse bundle ν has a Riemann metric which is locally conformal with a foliate metric. Equivalently, E is conformall iff M has a Riemann metric which is locally conformal with a bundle-like metric* [6].

Proof. Suppose E is conformal and take a representative cocycle of E with the already introduced notation. Then, for every $x\in U_\alpha$, $\gamma_{\alpha\alpha}(x)=[g_\alpha]_{f_\alpha(x)}$ is some germ of a Riemann metric on R^q and $h_\alpha=f_\alpha^* g_\alpha$ gives a germ of a Riemann metric of ν at x. (Recall that $\nu|_{U_\alpha}=f_\alpha^{-1}(TR^q)$.) Because of the continuity of $\gamma_{\alpha\alpha}$, these germs fit over U_α, and we get a Riemann foliate metric h_α of $\nu|_{U_\alpha}$.

Now, from the definition of $C\Gamma_q$ and from (1.1), we get over $U_\alpha\cap U_\beta$

$$h_\alpha=\phi_{\alpha\beta}h_\beta, \tag{1.2}$$

where $\phi_{\alpha\beta}: U_\alpha \cap U_\beta \to R$ are differentiable positive foliate functions, such that $\{ln\ \phi_{\alpha\beta}\}$ is a 1-cocycle of $\{U_\alpha\}$ with values in the sheaf of germs of differentiable functions on M.

Since the mentioned sheaf is fine, we have

$$\ln \phi_{\alpha\beta} = \psi_\beta - \psi_\alpha\,, \tag{1.3}$$

where $\psi_\alpha : U_\alpha \to R$ are some (generally non-foliate) differentiable functions, defined up to the addition of the restrictions of a globally defined function $F: M \to R$ to U_α.

In view of (1.3), (1.2) becomes

$$e^{\psi_\alpha} h_\alpha = e^{\psi_\beta} h_\beta\,, \tag{1.4}$$

which gives a global Riemann metric h of ν, defined up to a global positive factor and which is locally conformal to the foliate metrics h_α.

Conversely, we may choose a representative cocycle of E attached to a covering $\{U_\alpha\}$, for which $\nu|_{U_\alpha}$ has a foliate conformal metric. The existence of these foliate local metrics is just the conformal structure of E.

The second part of proposition 1.1 is now obvious.

The Riemann metrics of proposition 1.1 will be called *locally conformal foliate* (*l. c. f.*). A manifold M with a conformal foliation E and a fixed locally conformal bundle-like (*l. c. b. —l.*) Riemann metric γ will be called a *locally conformal Reinhart* (*l. c. R.*) *space*.

COROLLARY 1.2. *Every differentiable foliation of codimension* 1 *is conformal. Every complex analytic foliation of complex codimension* 1 *is conformal.*

Proof. In the first case, an arbitrary Riemann metric of ν has the local form $h(dx^1)^2$. In the second case we use similarly a Hermitian metric, which is locally $hdz^1 d\bar{z}^1$.

Now, in order to give a second characteristic property, let us recall that the *conformal group* $C(q)$ is $R' \times 0(q)$, where $0(q)$ is the real orthogonal group and R' is the multiplicative group of the real positive numbers. Then, we have

PROPOSITION 1.3. *The foliation E of M is conformal iff there exists a foliate reduction of the structure group of its transverse bundle ν to $C(q)$.*

Proof. Take a conformal E and a *l. c. f.* metric of $\nu(E)$. Then, we have a convenient open covering $\{U_\alpha\}$ of M, such that $\nu|_{U_\alpha}$ are trivial and have the foliate metrics h_α. Now, if we take the subset of the principal frame bundle $P(\nu)$ consisting of frames $\{e_a\}$ whose vectors e_a are pairwise orthogonal and have the same length with respect to the metrics h_α, we clearly get a foliate principal subbundle $P'(\nu)$, whose structure group is $C(q)$. This gives the desired reduction of the structure group of ν.

Conversely, if the mentioned reduction exists, we have the $C(q)$-principal subbundle $P'(\nu)$ of $P(\nu)$, which is foliate, and whose local cross-sections clearly allow the construction of the local foliate metrics h_α. The conformal structure of E is thereby defined.

Finally, we may give a third characterization of the conformal foliations by generalizing Morgan's characterization of the Riemannian foliations. Recall that $\nu(E)$ has always *adapted connections* [1, 3], which may be defined by the condition that parallel translation along paths on the leaves of E coincides with the "vector equipolence", i. e. equality of the coordinates with respect to the bases X_a. The mentioned characterization is then

PROPOSITION 1.4. *The foliation E on M is conformal iff ν has an adapted connection, whose holonomy group at some point x_0 is a subgroup of the conformal group $C(q)$.*

Proof. Consider an arbitrary foliation E and an arbitrary Riemann metric γ on M. An arbitrary connection ∇ on $\nu(E)$ has the local equations

$$dX_a=\omega_a^b X_b\,, \tag{1.5}$$

where

$$\omega_a^b=\Gamma^b_{ac}dx^c+\Gamma^b_{au}\theta^u\,. \tag{1.6}$$

This connection is adapted iff $\Gamma^b_{au}=0$, because in this case $\xi^a X_a$ is parallel along a line with $x^a=$const. iff $X_u\xi^a=\partial\xi^a/\partial x^u=0$.

Also, we shall say that ∇ *has no torsion*, iff $\Gamma^b_{ac}=\Gamma^b_{ca}$ (the geometric character of this equality may easily be established) and that ∇ in *transversally metric* if one has for the transversal part h of γ :

$$dh_{ab}-\omega_a^c h_{cb}-\omega_b^c h_{ac}=0 \text{ (modulo } \theta^u=0)\,. \tag{1.7}$$

But then Γ^a_{bc} may be uniquely determined by the classical computation leading to the Christoffel symbols of a Riemann metric [7]. I. e. there is always a unique adapted transversally metric connection without torsion on ν, which we'll call the *transversal Levi-Civita connection* [8].

Now, let E be conformal and γ a *l. c. b.* $-l.$ metric on M. Then, if ∇ is the corresponding transversal Levi-Civita connection, (1.7) implies

$$(\nabla_X h)(\xi,\,\eta)=\lambda(X)h(\xi,\,\eta)\,, \tag{1.8}$$

where X is a tangent field to M, ξ, η, are sections of ν and λ is a 1-form on M given locally by $\lambda=d''\phi_\alpha$, where ϕ_α are the functions entering in (1.4); formula (1.3) shows that λ is, actually, globally defined. Indeed, it suffices to verify (1.8) for $\xi=X_a$, $\eta=X_b$ and $X=X_c$, X_u. In the first case we get by (1.7) 0=0 and in the second case (1.8) follows directly from $h=e^{\phi_\alpha}h_\alpha$.

But (1.8) clearly implies that the holonomy group of ∇ is a subgroup of $C(q)$, thereby proving the necessity part of proposition 1.4.

As for the sufficiency part, if the mentioned connection exists, then, by a well-known result of the connection theory, $P(\nu)$ admits a $C(q)$-reduction. In our case, this reduction must be foliate because, ∇ being adapted, loops on a slice of E define trivial elements of the holonomy group and this shows that the considered reduction is locally constant on the leaves of E.

Hence, the proof of proposition 1.4 is completed by using proposition 1.3.

The 1-form λ defined above will be very important. First, note that, by taking in (1.8) $\xi=X_a$, $\eta=X_b$, $X=X_u$, we get

$$X_u h_{ab}=\lambda_u h_{ab}\,, \tag{1.9}$$

where $\lambda=\lambda_u\theta^u$. (By its definition, λ is of the type (0, 1).) Hence

$$\lambda_u=\frac{1}{q}h^{ab}X_u h_{ab}=\frac{1}{q}X_u[\ln \det(h_{ab})]\,. \tag{1.10}$$

where $h_{ab}h^{ac}=\delta^c_b$.

I. e., we have

$$\lambda=\frac{1}{q}d''\ln\det(h_{ab})\,, \tag{1.11}$$

and this formula may be used to define a 1-form λ of type (0, 1) for every foliation E and every metric γ (which may be not $l.c.b.-l.$). λ is globally defined, because of the transformation law of the components h_{ab}, and it depends on the whole metric γ and not on its transversal part only. We shall call λ the *complementary form* of (X, E, γ) and this is clearly a d''-closed form.

Now, we may use λ to give a characterization of the locally conformal Reinhart spaces:

PROPOSITION 1.5. *(M, E, γ) is a locally conformal Reinhart space iff*

$$(\nabla_X h)(\xi, \eta)=\lambda(X)h(\xi, \eta)\,, \tag{1.12}$$

where h is the transversal part of γ, ∇ is the transversal Levi-Civita connection, λ is the complementary form, X is any vector field tangent to E and ξ, η are normal fields of E. Moreover, the considered space is globally conformal Reinhart iff the supplementary condition that λ be a d''-exact form also holds.

Proof. If (M, E, γ) is locally conformal Reinhart, (1-12) follows from the already proven relation (1.8). Moreover, if it is globally conformal Reinhart, λ is d''-exact because we may define a global function ϕ.

Conversely, if (1.12) holds, and on $U\subseteq X$, $\lambda=d''\phi$ (such U clearly exist and by (1.11) we may take $\phi=(1/q)\ln\det(h_{ab})$), then, taking $\xi=X_a$, $\eta=X_b$, $X=X_u$, we get from (1.12)

$$d''h_{ab}=(d''\phi)h_{ab}\,,$$

which is equivalent to

$$d''(e^{-\phi}h_{ab})=0\,. \tag{1.13}$$

Hence, $e^{-\phi}h$ is a bundle-like metric on U and we see that γ is a $l.c.b.-l.$ metric.

If λ is d''-exact, we may take $U=X$ and proposition 1.5 will be completely proven.

§ 2. Some particular classes of conformal foliations.

We defined in section 1 the complementary form λ, which was used to characterize the $l.c.R.$ spaces. In this section, we shall see that the properties of λ influence much the properties of the space.

Let us begin by proving some new facts about λ.

First, we know by (1.11) that λ is a d''-closed form, hence it defines a d''-cohomology class Λ. If we interpret d'' as the differential in the exterior algebra of the dual bundle $T^*(E)$, this will be an operator which depends only on E and is independent on the metric. Namely, d'' is the exterior differential along the leaves. Then, using the transformation law of h_{ab}, we get easily from (1.11) that Λ does not depend on the metric. Hence, the de Rham theorem implies

PROPOSITION 2.1. *Every foliation E has a well defined associated family of cohomology classes Λ of $H^1(F, R)$, where F denotes a generic leaf of E.*

Another interesting fact is that λ is related with the so-called *Atiyah classes* defined by Molino [3].

Namely, let ν be, as usual, the transversal bundle of E, and consider the line bundle $\alpha=(\wedge^q\nu)^{1/q}$, where $\wedge^q$ denotes q-th exterior power and the exponent $1/q$ refers to tensor product of line bundles. α will be called the *characteristic line bundle* of E. If q is even α is defined only when E is transversally orientable and, in the rest of this paper, this condition is assumed to hold whenever α is used.

PROPOSITION 2.2. *The Atiyah class of the characteristic line bundle $\alpha(E)$ is the d''-cohomology class of the form $(1/2)d'\lambda$, where $d'\lambda$ is calculated by the help of an arbitrary Riemann metric γ.*

Proof. We refer to [3] for the construction of the Atiyah class which we use in the sequel. Our notation is always like in section 1. α is clearly a foliate bundle and it is easy to see that $(\det h_{ab})^{-1/q}$, where h is the transverse part of γ, is the local component of a Riemann metric of α. Then, using the known transformation law of the local forms of a connection, we see that

$$\pi=d\ln[\det(h_{ab}))^{-1/2q}]$$

defines a connection on α. Next, it follows by (1.11), that $\theta=\pi+(1/2)\lambda$ defines an *adapted connection* of α. (The notion of an adapted connection may be defined for every foliate vector bundle in the same way like for ν and such connections always exist [3].)

By its very definition, the Atiyah class of α is the d''-cohomology class of the globally defined form $d''\theta$. Because $d''\lambda=0$ and because of $d'd''=-d''d'$, we get that this form is just $(1/2)d'\lambda$ as announced.

We might give a similar interpretation using $\wedge^q\nu$, but it is α which will be used in the sequel. Also, recall that the significancy of the Atiyah class lies in the fact that its vanishing is equivalent to the existence of an adapted foliate connection [3].

Now, before considering particular classes of conformal foliations, let us consider some other general results.

If E is a conformal foliation, the foliate bundle $\bar{\nu}=\alpha\otimes\nu$ will be called the *conformal transverse bundle* of E and we have

PROPOSITION 2.3. *The conformal transverse bundle $\bar{\nu}$ of a conformal foliation E has a foliate Riemann metric.*

Proof. Take the covering $\{U_\alpha\}$ of M endowed with the local foliate metrics h_α of $\nu|_{U_\alpha}$. Define over each U_α the form of T. Y. Thomas [7]

$$\bar{h}_\alpha=(\det h_\alpha)^{-1/q}h_\alpha\,. \tag{2.1}$$

It is easy to see that this gives a Riemann foliate metric for $\bar{\nu}|_{U_\alpha}$ and that $\bar{h}_\alpha=\bar{h}_\beta$ over each intersection $U_\alpha\cap U_\beta$. Hence (2.1) gives the desired metric $\bar{h}$ of $\bar{\nu}$.

It is obvious that the principal bundle $P(\bar{\nu})$ of the frames of $\bar{\nu}$ is foliate in the sense of [2] as well. Proposition 2.3 shows that $P(\bar{\nu})$ has an $SO(q)$-principal foliate subbundle $P'(\bar{\nu})$, whence in view of the results of Kamber-Tondeur [2, p. 73, 75] we have

COROLLARY 2.4. *The usual secondary characteristic classes of the foliate bundle $P(\bar{\nu})$ reduce to the principal characteristic classes of $P'(\bar{\nu})$. In exchange, a secondary characteristic homomorphism*

$$\Delta_*: H(W(so(q), G)_q) \longrightarrow H_{DR}(M) \tag{2.2}$$

appears for every G-reduction of $P'(\bar{\nu})$.

The notation here is like in [2], i.e. $so(q)$ denotes the Lie algebra of $SO(q)$, G is an arbitrary closed subgroup of $SO(q)$, $W(\cdot,\cdot)_q$ is the corresponding truncated Weil algebra, H is cohomology and H_{DR} is the de Rham cohomology.

We see thereby that *the conformal foliations enjoy of their own theory of secondary characteristic classes.*

We go over finally to the announced particular classes of foliations. Namely, a conformal foliation for which a *l. c. b. — l.* metric exists such that: A) $d\lambda=0$, B) $\lambda\neq0$ at every point, C) $\lambda\neq0$ at every point and $d\lambda=0$, D) $\lambda\neq0$ at every point and $d'\lambda=0$, E) λ is d''-exact, F) $d'\lambda$ is d''-exact, is called respectively a *foliation of the type A, B, C, D, E, F.* In this case, the foliation and the metric give an *l. c. R. space of type A—F.*

The following relations are obvious:

$$C \longrightarrow D \longrightarrow F,\ E \longrightarrow F,\ A \longrightarrow F. \tag{2.3}$$

The foliations of type E were already considered and they are characterized by the existence of a globally $c.b.-l.$ metric, i.e. they are just the Riemannian foliations. We want to prove next an interesting property of the foliations of type F. By (2.3) this will hold for all the considered types of foliations, except B.

PROPOSITION 2.5. *Any conformal foliation of type F has a transverse projectable connection.*

Proof. A transverse projectable connection is defined as a foliate connection on $\nu(E)$ [3].

Suppose E satisfies the hypotheses. We shall prove then that $\bar{\nu}(E)$ has a foliate connection.

Let e be a local basis of α and $\bar{X}_a = e\otimes X_a$ the corresponding natural basis of $\bar{\nu}$. Then, a connection of $\bar{\nu}$ has the local equations

$$d\bar{X}_a = \bar{\omega}_a^b \bar{X}_b, \tag{2.4}$$

where

$$\bar{\omega}_a^b = \Gamma_{ac}^b dx^c + \Gamma_{au}^b \theta^u. \tag{2.5}$$

Like for ν, this connection is adapted iff $\Gamma_{au}^b = 0$ and it is foliate iff, moreover, Γ_{ac}^b are foliate functions.

The Riemann metric $\bar{h}$ of proposition 2.3 has some local components $\bar{h}_{ab} = \bar{h}(\bar{X}_a, \bar{X}_b)$ and the connection (2.4) preserves this metric iff

$$d\bar{h}_{ab} - \bar{\omega}_a^c \bar{h}_{cb} - \bar{\omega}_b^c \bar{h}_{ac} = 0. \tag{2.6}$$

We shall look for the desired connection of $\bar{\nu}$ by looking for an adapted metric foliate connection. Because $\bar{h}_{ab}$ are foliate functions, one sees, by the classical computation which gives the Christoffel symbols of a metric [7], that the Γ_{ac}^b are defined by (2.6) if one knows the quantities $T_{ac}^b = \Gamma_{ac}^b - \Gamma_{ca}^b$.

By the transformation law of the connection forms $\bar{\omega}_a^b$, it follows that T_{ac}^b are the local components of a geometric object on M, obeying to the transformation law

$$\begin{aligned} T'^{a'}_{b'c'} &= \frac{\partial x'^{a'}}{\partial x^a} \frac{\partial x^b}{\partial x'^{b'}} \frac{\partial x^c}{\partial x'^{c'}} T^a_{bc} \\ &+ \Delta^{\frac{1}{q}} \left(\delta^{a'}_{b'} \frac{\partial \Delta^{-\frac{1}{q}}}{\partial x'^{c'}} - \delta^{a'}_{c'} \frac{\partial \Delta^{-\frac{1}{q}}}{\partial x'^{b'}} \right), \end{aligned} \tag{2.7}$$

where $(x'^{a'}, x'^{u'})$ and (x^a, x^u) are two adapted systems of local coordinates and $\Delta = \det(\partial x'^{a'}/\partial x^a)$.

Hence, if the local foliate functions T^a_{bc} satisfying (2.7) exist, $\bar{\nu}$ has a foliate connection.

But, because E is of type F, it follows from proposition 2.2 that α has a foliate connection. This will be defined by a local equation $de=\eta e$, where η is a local foliate 1-form with the transformation law

$$\eta'=\eta+d\ln\Delta^{-\frac{1}{q}}. \tag{2.8}$$

If we denote $\eta=\eta_a dx^a$, it follows by a straightforward computation that

$$T^a_{bc}=\delta^a_b\eta_c-\delta^a_c\eta_b \tag{2.9}$$

satisfy (2.7).

Hence, $\bar{\nu}$ has a foliate connection.

Now, since by the definition of $\bar{\nu}$ we clearly get $\nu=\alpha^{-1}\otimes\bar{\nu}$, it follows that ν has a foliate connection whose local matrices are given by

$$\tau=\bar{\omega}-\eta I, \tag{2.10}$$

where $\bar{\omega}$ is the foliate connection of $\bar{\nu}$, η is the foliate connection of α and I is the unit q-matrix.

COROLLARY 2.6. *If E is a conformal foliation of one of the types A, C, D, E, F, we have necessarily*

$$Pont^{k}\nu=0 \quad for \quad k>q, \tag{2.11}$$

where $Pont^{*}\nu$ denotes the ring of the real Pontryagin classes of ν.

This follows from proposition 2.5 in view of the results in [1] and [3].

We shall list in the sequel a few other properties of the considered types of conformal foliations.

PROPOSITION 2.7. *If E is a conformal foliation of the type A, the lift $\tilde{E}$ of E to the universal covering $\tilde{M}$ of M (M-connected) is a Riemannian foliation.*

Proof. Indeed, if we also take the lift of the metric attached to E and satisfying $d\lambda=0$, we get on $\tilde{M}$ a structure of an $l.c.R.$ space of type A. Since $\tilde{M}$ is simply connected, we have for this structure $\tilde{\lambda}=df$ and because $\tilde{\lambda}$ is of type (0, 1), $\tilde{\lambda}=d''f$. I.e. $\tilde{E}$ is of type E, whence the announced result.

PROPOSITION 2.8. *If E is a conformal foliation of type B, E admits a subfoliation of codimension* 1, *whose restriction to any leaf of E is transversally parallelizable. Moreover, if E is of type C or D, M has a bundle-like metric with respect to this subfoliation.*

Proof. The first assertion of this proposition means that there is a foliation E' of codimension $q+1$ on M, whose tangent bundle is a subbundle of $T(E)$. And, as a matter of fact, E' exists and it has the local equations

$$dx^a_\alpha=0\,,\ d\phi_\alpha=0\,, \tag{2.12}$$

over the coordinate neighbourhood U_α of M. The fact that this is a regular foliation follows from the condition $\lambda \neq 0$ at every point and from the relation between λ and ϕ_α.

The equation of the restriction of E' to a leaf of E is just $\lambda=0$, where λ is closed on the leaf. This means that the restriction is transversally parallelizable. In this case, one knows, for instance, that if we have a compact leaf S of E its universal covering manifold is of the form $S' \times R$, where S' is the universal covering of a leaf of $E'|_S$.

Next, it is clear that the transverse bundle of E' is given by

$$\nu(E')=\nu(E)\oplus n\,, \tag{2.13}$$

where n is the line bundle defined by the normals of E' with respect to E.

Here $\nu(E)$ is foliate with respect to E' as well, and it has the E'-foliate metric $h=e^{\phi_\alpha}h_\alpha$ (which is the $l.c.f.$ metric of E, satisfying C or D).

Hence, we get for $\nu(E')$ the metric

$$h'=h+\lambda\otimes\lambda\,, \tag{2.14}$$

which has the local expression

$$h'=e^{\phi_\alpha}h_\alpha+d'\phi_\alpha^2-2d'\phi_\alpha d\phi_\alpha+d\phi_\alpha^2\,. \tag{2.15}$$

But, both in C and D, we have

$$d'\lambda'=d'd''\phi_\alpha=-d''d'\phi_\alpha=0\,.$$

Hence $d'\phi_\alpha$ are foliate forms and we see that (2.15) may be extended to a bundle-like metric for E'. Q. e. d.

As an application, let us consider the case of a transversally orientable foliation E of codimension 1, which is always conformal by corollary 1.2.

Then, E has a global Pfaff equation

$$w=0\,, \tag{2.16}$$

where

$$dw=\theta\wedge w\,, \tag{2.17}$$

θ being a second Pfaff form on M, which we may choose of the type (0, 1).

From (2.17), it follows that we must have locally $w=adx$ and, in this case, a^2dx^2 is a $l.c.f.$ metric on the transverse bundle of E. Hence, with our usual notation, $e^\phi=a^2$, i. e. $\phi=2\ln a$ and $\lambda=2d''\ln a=2\theta$.

Hence, E is of type A iff $d\theta=0$. Actually, this last condition is equivalent also with $d'\theta=0$, because $d''\theta=0$ and $\partial\theta=0$ as a form of type (2,0) for a codimension 1 foliation. E is of type B iff $dw\neq 0$ at every point and in this case E has a subfoliation E' of codimension 1. E is simultaneously of types C and D and this happens iff $dw\neq 0$ at every point and $d\theta=0$. In this case, E' is a transversally parallelizable foliation of codimension 2 on M. Finally, E is of type E iff θ is d''-exact and of type F iff $d\theta$ is d''-exact.

Added on May 28, 1978. 1) It is worth while to note that, in the case of a transversally orientable foliation, the line bundle α used in this paper is differentially trivial but, generally, it is not foliately trivial. Similarly, $\bar{\nu}$ is then differentially but not foliately isomorphic to the transverse bundle ν.

2) We learned later that conformal foiations were also studied by S. Nishikawa and H. Sato (*On characteristic classes of riemannian conformal and projective foliations*, J. Math. Soc. Japan **28** (1976), 223-241). Namely, by using Cartan connections instead of linear connections, these authors showed that Corollary 2.6 actually holds for general conformal foliations. They also gave examples of conformal foliations of an arbitrary codimension, which are not riemannian. Other results on the characteristic classes of conformal foliations were proved by S. Morita and K. Yamato (to appear).

Conformal foliations

Summary

The conformal foliations are a generalization of the Riemannian foliations. They are characterized by the existence of a global Riemann metric which is locally conformal with bundle-like metrics. They are also characterized by the existence of a foliate reduction of the structure group of the transverse bundle to the conformal group, and by the existence of an adapted connection whose holonomy group at a point is a conformal group. A relevant 1-form λ may be attached to such foliations. The conformal foliations have their own theory of secondary characteristic classes. Several particular classes of conformal foliations are considered. For these classes we show the existence of a transversally projectable connection and we obtain a number of different other geometric properties. Note that all the differentiable foliations of codimension 1 and all the complex analytic foliations of complex codimension 1 are conformal foliations.

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