Y. MUTO
KODAI MATH. J.
1 (1978), 329—338

RIEMANNIAN SUBMERSION AND THE LAPLACE-BELTRAMI OPERATOR

BY YOSIO MUTŌ

Introduction.

In the present paper we consider only Riemannian submersions $\pi:(\tilde{M}, \tilde{g})\to(B, {}^{B}g)$ such that fibers F are complete and connected and imbedded in $(\tilde{M}, \tilde{g})$ regularly as totally geodesic submanifolds.

It is well-known that, if φ is an eigenfunction of the Laplacian in $(B, {}^{B}g)$, the lift $\tilde{\varphi}=\varphi^{L}$ is also an eigenfunction of the Laplacian in $(\tilde{M}, \tilde{g})$ with the same eigenvalue [1]. The purpose of the present paper is to find corresponding relations in the case of p-forms. For p-forms we get a little more complicated result. If a p-form ω is an eigenelement of the Laplace-Beltrami operator $\varDelta$ in $(B, {}^{B}g)$, the horizontal lift $\tilde{\omega}=\omega^{L}$ is not always an eigenelement of the Laplace-Beltrami operator $\tilde{\varDelta}$ in $(\tilde{M}, \tilde{g})$. In order that $\tilde{\omega}$ be an eigenelement with the same eigenvalue as ω, ω must satisfy a necessary and sufficient condition which is obtained in § 4 of the present paper.

In § 1 we recall some properties of Riemannian submersions with totally geodesic fibers. There we use local coordinates adapted to the Riemannian submersion. In § 2 fundamental formulas in covariant differentiation are given. In § 3 a relation between $\tilde{\varDelta}\tilde{\omega}$ and $\varDelta\omega$ is obtained when $\tilde{\omega}=\omega^{L}$. In § 4 a necessary and sufficient condition to be satisfied by ω such that $\varDelta\omega=\lambda\omega$ is obtained in order that $\tilde{\omega}=\omega^{L}$ satisfy $\tilde{\varDelta}\tilde{\omega}=\lambda\tilde{\omega}$. A simple sufficient condition is also obtained. As an application harmonic forms are studied in some special case.

Remark. In the present paper lift always means horizontal lift.

§ 1. Riemannian submersions with totally geodesic fibers.

Riemannian submersions were studied extensively by the authors R. H. Escobales [2], S. Ishihara [3], S. Ishihara and M. Konishi [4], Y. Mutō [5], T. Nagano [7], B. O'Neill [8], K. Yano and S. Ishihara [10], [11] and others.

Riemannian submersions considered in the present paper are limited to those with totally geodesic fibers only, and this means that the tensor T of B. O'Neill vanishes [8]. Tensors in the total manifold $\tilde{M}$, in the base manifold B or in a

Received May 21, 1977.

fiber F are written in such letters as $\tilde{S}$, ${}^{B}S$ or ${}^{F}S$ respectively, but, if there is no possibility of confusion, tensors ${}^{B}S$ in B are written S for short. The Riemannian metrics on $\tilde{M}$, B and F are denoted respectively by $\tilde{g}$, ${}^{B}g$ and ${}^{F}g$.

Let $\tilde{W}$ be any vector field on $\tilde{M}$, $\tilde{E}$ any horizontal vector field on $\tilde{M}$ and $\tilde{X}$ any vertical vector field on $\tilde{M}$. Then for example, from any (1, 1)-tensor field $\tilde{S}$ on $\tilde{M}$, we get four (1, 1)-tensor fields $\tilde{S}_H{}^H$, $\tilde{S}_H{}^V$, $\tilde{S}_V{}^H$, $\tilde{S}_V{}^V$ such that

$$\tilde{S}=\tilde{S}_H{}^H+\tilde{S}_H{}^V+\tilde{S}_V{}^H+\tilde{S}_V{}^V,$$

$$\tilde{S}_H{}^H\tilde{X}=\tilde{S}_H{}^V\tilde{X}=\tilde{S}_V{}^H\tilde{E}=\tilde{S}_V{}^V\tilde{E}=0,$$

$$\tilde{g}(\tilde{S}_H{}^H\tilde{W}, \tilde{X})=0, \qquad \tilde{g}(\tilde{S}_V{}^H\tilde{W}, \tilde{X})=0,$$

$$\tilde{g}(\tilde{S}_H{}^V\tilde{W}, \tilde{E})=0, \qquad \tilde{g}(\tilde{S}_V{}^V\tilde{W}, \tilde{E})=0.$$

It is easy to see that such a decomposition of $\tilde{S}$ is unique. Similarly, if $\tilde{S}$ is a (0, 2)-tensor field, we have a unique decomposition

$$\tilde{S}=\tilde{S}_{HH}+\tilde{S}_{HV}+\tilde{S}_{VH}+\tilde{S}_{VV}.$$

The (0, 2)-tensor field and the (2, 0)-tensor field associated with the Riemannian metric $\tilde{g}$ are decomposed into $\tilde{g}_{HH}+\tilde{g}_{VV}$ and $\tilde{g}^{HH}+\tilde{g}^{VV}$ respectively since $\tilde{g}_{HV}$ and $\tilde{g}^{HV}$ vanish.

We define a tensor field $\tilde{R}$ with the following property.

$\tilde{R}$ has only one non-vanishing part, namely,

$$\tilde{R}=\tilde{R}_{HH}{}^V. \tag{1.1}$$

Let $\tilde{A}$ be the tensor field A in O'Neill's paper [8]. Let $\tilde{E}$, $\tilde{F}$ be any horizontal vector fields and $\tilde{X}$ any vertical vector field. Then $\tilde{R}$ satisfies

$$\tilde{A}_{\tilde{E}}\tilde{F}=-\frac{1}{2}\tilde{R}_{\tilde{E}}\tilde{F}, \qquad \tilde{g}(\tilde{A}_{\tilde{E}}\tilde{X}, \tilde{F})=\frac{1}{2}\tilde{g}(\tilde{R}_{\tilde{E}}\tilde{F}, \tilde{X}). \tag{1.2}$$

We assume that $\tilde{M}$ is covered by a set $\{V\}$ of coordinate neighborhoods with the following property. πV is a coordinate neighborhood of B and for any point $P\in V$ we have local coordinates $P\Leftrightarrow(x^1, \cdots, x^n, y^1, \cdots, y^m)=(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{n+m})$ such that $\pi P\Leftrightarrow(x^1, \cdots, x^n)$. If we use the natural frame attached to such a coordinate neighborhood V, the components $(\tilde{X}^1, \cdots, \tilde{X}^n, \tilde{X}^{n+1}, \cdots, \tilde{X}^{n+m})$ of a vertical vector $\tilde{X}$ satisfy $\tilde{X}^h=0$ where $h=1, \cdots, n$.

We use indices in the following ranges:

$$h, i, j, \cdots, r, s, t, \cdots=1, \cdots, n,$$

$$\kappa, \lambda, \mu, \cdots, \rho, \sigma, \tau, \cdots=n+1, \cdots, n+m,$$

$$A, B, C, \cdots, R, S, T, \cdots=1, \cdots, n+m.$$

Then the covariant components of the Riemannian metric $\tilde{g}$ are $\tilde{g}_{CB}$, or separately, $\tilde{g}_{ji}$, $\tilde{g}_{j\lambda}$, $\tilde{g}_{\mu\iota}$, $\tilde{g}_{\mu\lambda}$ where $\tilde{g}_{j\lambda}=\tilde{g}_{\lambda j}$. The covariant components ${}^F g_{\mu\lambda}$ of ${}^F g$ satisfy ${}^F g_{\mu\lambda}=\tilde{g}_{\mu\lambda}$. The inverse matrix of $({}^F g_{\mu\lambda})$ is denoted $({}^F g^{\mu\lambda})$.

Now we define Γ_i^κ by

$$\Gamma_i^\kappa={}^F g^{\kappa\tau}\tilde{g}_{i\tau}\,. \tag{1.3}$$

For any vector $\tilde{W}$ we have $\tilde{W}=\tilde{W}^H+\tilde{W}^V$. If $\tilde{W}^A$, namely $\tilde{W}^h$ and $\tilde{W}^\kappa$, are the components of $\tilde{W}$, and the components of $\tilde{W}^H$ and $\tilde{W}^V$ are denoted $(\tilde{W}^H)^A$ and $(\tilde{W}^V)^A$ respectively, then we have

$$\begin{aligned} (\tilde{W}^H)^h&=\tilde{W}^h\,, \qquad (\tilde{W}^H)^\kappa=-\Gamma_i^\kappa\tilde{W}^i\,,\\ (\tilde{W}^V)^h&=0\,, \qquad (\tilde{W}^V)^\kappa=\tilde{W}^\kappa+\Gamma_i^\kappa\tilde{W}^i\,. \end{aligned} \tag{1.4}$$

For any covariant vector $\tilde{U}$ we have $\tilde{U}=\tilde{U}_H+\tilde{U}_V$ and

$$\begin{aligned} (\tilde{U}_H)_h&=\tilde{U}_h-\Gamma_h^\kappa\tilde{U}_\kappa\,, \qquad (\tilde{U}_H)_\kappa=0\,,\\ (\tilde{U}_V)_h&=\Gamma_h^\kappa\tilde{U}_\kappa\,, \qquad (\tilde{U}_V)_\kappa=\tilde{U}_\kappa\,. \end{aligned} \tag{1.5}$$

Using such local coordinates and natural frames we can deduce that $\tilde{R}$ has components

$$\tilde{R}_{ji}{}^\kappa=(\tilde{R}_{HH}{}^V)_{ji}{}^\kappa=D_j\Gamma_i^\kappa-D_i\Gamma_j^\kappa$$

where

$$D_i=\partial_i-\Gamma_i^\lambda\partial_\lambda\,, \qquad \partial_i=\partial/\partial x^i\,, \qquad \partial_\lambda=\partial/\partial x^\lambda\,.$$

All other components of $\tilde{R}$ vanish and we shall write $R_{ji}{}^\kappa$ for the sake of convenience instead of $\tilde{R}_{ji}{}^\kappa$.

For the Riemannian metric ${}^B g$ on the base manifold B, we have

$${}^B g_{ji}=\tilde{g}_{ji}-\Gamma_j^\mu\Gamma_i^\lambda\tilde{g}_{\mu\lambda}\,, \qquad {}^B g^{ji}=\tilde{g}^{ji}\,.$$

It is easy to observe that ${}^B g_{ji}=(\tilde{g}_{HH})_{ji}$, ${}^B g^{ji}=(\tilde{g}^{HH})^{ji}$. Moreover we have

$${}^F g_{\mu\lambda}=(\tilde{g}_{VV})_{\mu\lambda}\,, \qquad {}^F g^{\mu\lambda}=(\tilde{g}^{VV})^{\mu\lambda}=\tilde{g}^{\mu\lambda}-\Gamma_t^\mu\Gamma_s^\lambda\tilde{g}^{ts}\,,$$

$$\tilde{g}^{j\lambda}=-\Gamma_t^\lambda\tilde{g}^{jt}\,.$$

As there is no possibility of confusion we shall write g_{ji}, g^{ji}, $g_{\mu\lambda}$, $g^{\mu\lambda}$ for ${}^B g_{ji}$, ${}^B g^{ji}$, ${}^F g_{\mu\lambda}$, ${}^F g^{\mu\lambda}$ respectively.

With the use of these components we can raise and lower indices of $\tilde{R}_{ji}{}^\kappa$ and get tensor fields such as $\tilde{R}_H{}^{HV}$, $\tilde{R}_{HHV}$ whose components are $R_j{}^{i\kappa}=R_{jt}{}^\kappa g^{ti}$, $R_{ji\kappa}=R_{ji}{}^\tau g_{\tau\kappa}$. $\tilde{R}^{HHV}$, $\tilde{R}^{HH}{}_V$ are defined similarly.

§ 2. Fundamental formulas in covariant differentiation.

Fundamental formulas of covariant differentiation have been obtained by B. O'Neill [8]. The following is only a translation into our terminology where $\tilde{W}$ is a vector field and $\tilde{U}$ a 1-form.

$$((\tilde{\nabla}\tilde{W})_H{}^H)_j{}^h = D_j(\tilde{W}^H)^h + {}^B\begin{Bmatrix} h \\ j\ t \end{Bmatrix}(\tilde{W}^H)^t + \frac{1}{2}R_j{}^h{}_\tau(\tilde{W}^V)^\tau,$$

$$((\tilde{\nabla}\tilde{W})_H{}^V)_j{}^\kappa = D_j(\tilde{W}^V)^\kappa + \partial_\tau\Gamma_j^\kappa(\tilde{W}^V)^\tau - \frac{1}{2}R_{jt}{}^\kappa(\tilde{W}^H)^t,$$

$$((\tilde{\nabla}\tilde{W})_V{}^H)_\mu{}^h = \partial_\mu(\tilde{W}^H)^h + \frac{1}{2}R_t{}^h{}_\mu(\tilde{W}^H)^t,$$

$$((\tilde{\nabla}\tilde{W})_V{}^V)_\mu{}^\kappa = \partial_\mu(\tilde{W}^V)^\kappa + {}^F\begin{Bmatrix} \kappa \\ \mu\ \tau \end{Bmatrix}(\tilde{W}^V)^\tau,$$

$$((\tilde{\nabla}\tilde{U})_{HH})_{ji} = D_j(\tilde{U}_H)_i - {}^B\begin{Bmatrix} t \\ j\ i \end{Bmatrix}(\tilde{U}_H)_t + \frac{1}{2}R_{ji}{}^\tau(\tilde{U}_V)_\tau,$$

$$((\tilde{\nabla}\tilde{U})_{HV})_{j\lambda} = D_j(\tilde{U}_V)_\lambda - \partial_\lambda\Gamma_j^\tau(\tilde{U}_V)_\tau - \frac{1}{2}R_j{}^t{}_\lambda(\tilde{U}_H)_t,$$

$$((\tilde{\nabla}\tilde{U})_{VH})_{\mu i} = \partial_\mu(\tilde{U}_H)_i - \frac{1}{2}R_i{}^t{}_\mu(\tilde{U}_H)_t,$$

$$((\tilde{\nabla}\tilde{U})_{VV})_{\mu\lambda} = \partial_\mu(\tilde{U}_V)_\lambda - {}^F\begin{Bmatrix} \tau \\ \mu\ \lambda \end{Bmatrix}(\tilde{U}_V)_\tau.$$

From these formulas we can get similar formulas of covariant differentiation of tensor fields.

From the tensor field $\tilde{R}$ we get two important tensor fields by covariant differentiation. They are $(\tilde{\nabla}\tilde{R})_{HHH}{}^V$ and $(\tilde{\nabla}\tilde{R})_{VHH}{}^V$ whose leading components are

$$R_{kji}{}^\kappa = ((\tilde{\nabla}\tilde{R})_{HHH}{}^V)_{kji}{}^\kappa, \qquad R_{\nu ji}{}^\kappa = ((\tilde{\nabla}\tilde{R})_{VHH}{}^V)_{\nu ji}{}^\kappa. \tag{2.1}$$

In view of (1.1) and the fundamental formulas given above we have

$$R_{kji}{}^\kappa = D_k R_{ji}{}^\kappa - {}^B\begin{Bmatrix} t \\ k\ j \end{Bmatrix} R_{ti}{}^\kappa - {}^B\begin{Bmatrix} t \\ k\ i \end{Bmatrix} R_{jt}{}^\kappa + \partial_\tau\Gamma_k^\kappa R_{ji}{}^\tau, \tag{2.2}$$

$$R_{\nu ji}{}^\kappa = \partial_\nu R_{ji}{}^\kappa + {}^F\begin{Bmatrix} \kappa \\ \nu\ \tau \end{Bmatrix} R_{ji}{}^\tau + \frac{1}{2}(R_j{}^t{}_\nu R_{it}{}^\kappa - R_i{}^t{}_\nu R_{jt}{}^\kappa). \tag{2.3}$$

We can raise and lower indices of such tensor fields and define for example

$$R_{kj}{}^{\iota\kappa} = R_{kjt}{}^\kappa g^{t\iota}, \qquad R_{\nu ji\kappa} = R_{\nu ji}{}^\tau g_{\tau\kappa}.$$

The following identities are obtained by direct computation or by applying identities satisfied by curvature tensors to formulas to be given a little later.

(2.4) $$R_{kji}{}^{\kappa}+R_{jik}{}^{\kappa}+R_{ikj}{}^{\kappa}=0\,,$$

(2.5) $$R_{\mu ji\lambda}+R_{\lambda ji\mu}=0\,.$$

Relations between the curvature tensors $\tilde{K}_{DCBA}$, ${}^{B}K_{kjih}$, ${}^{F}K_{\nu\mu\lambda\kappa}$ of $(\tilde{M}, \tilde{g})$, $(B, {}^{B}g)$, $(F, {}^{F}g)$ have been obtained by B. O'Neill [8]. In our terminology they are

$$(\tilde{K}_{HHHH})_{kjih}={}^{B}K_{kjih}-\frac{1}{4}(R_{ji}{}^{\tau}R_{kh\tau}-R_{ki}{}^{\tau}R_{jh\tau})+\frac{1}{2}R_{kj}{}^{\tau}R_{ih\tau}\,,$$

$$(\tilde{K}_{HHHV})_{kji\kappa}=\frac{1}{2}R_{ikj\kappa}\,,$$

$$(\tilde{K}_{HVHV})_{k\mu i\kappa}=\frac{1}{2}R_{\mu ki\kappa}-\frac{1}{4}R_{k}{}^{t}{}_{\mu}R_{it\kappa}\,,$$

$$(\tilde{K}_{VVHV})_{\nu\mu i\kappa}=0\,,$$

$$(\tilde{K}_{VVVV})_{\nu\mu\lambda\kappa}={}^{F}K_{\nu\mu\lambda\kappa}\,.$$

For the Ricci tensors $\widetilde{Ric}=\tilde{K}_{HH}+\tilde{K}_{HV}+\tilde{K}_{VH}+\tilde{K}_{VV}$, ${}^{B}Ric$, ${}^{F}Ric$ of $(\tilde{M}, \tilde{g})$, $(B, {}^{B}g)$, $(F, {}^{F}g)$ we have

$$(\tilde{K}_{HH})_{ji}={}^{B}K_{ji}-\frac{1}{2}R_{j}{}^{t\tau}R_{it\tau}\,,$$

$$(\tilde{K}_{HV})_{j\lambda}=\frac{1}{2}g^{ts}R_{tjs\lambda}\,,$$

$$(\tilde{K}_{VH})_{\mu i}=\frac{1}{2}g^{ts}R_{tis\mu}\,,$$

$$(\tilde{K}_{VV})_{\mu\lambda}={}^{F}K_{\mu\lambda}+\frac{1}{4}R^{ts}{}_{\mu}R_{ts\lambda}$$

where ${}^{B}K_{ji}$, ${}^{F}K_{\mu\lambda}$ are the components of ${}^{B}Ric$, ${}^{F}Ric$ respectively.

From the above formulas we get

(2.6) $$(\tilde{K}_{HH}{}^{HH})_{kj}{}^{ih}={}^{B}K_{kj}{}^{ih}+\frac{1}{2}R_{kj}{}^{\tau}R^{ih}{}_{\tau}+\frac{1}{4}(R_{k}{}^{i\tau}R_{j}{}^{h}{}_{\tau}-R_{j}{}^{i\tau}R_{k}{}^{h}{}_{\tau})\,,$$

(2.7) $$(\tilde{K}_{HV}{}^{HH})_{k\mu}{}^{ih}=\frac{1}{2}R_{k}{}^{ih}{}_{\mu}\,,$$

(2.8) $$(\tilde{K}_{VV}{}^{HH})_{\nu\mu}{}^{ih}=R_{\nu}{}^{ih}{}_{\mu}-\frac{1}{4}(R_{t}{}^{i}{}_{\nu}R^{th}{}_{\mu}-R_{t}{}^{i}{}_{\mu}R^{th}{}_{\nu})\,,$$

(2.9) $$(\tilde{K}_H{}^H)_j{}^i = {}^BK_j{}^i - \frac{1}{2} R_{tj}{}^\tau R^{ti}{}_\tau ,$$

(2.10) $$(\tilde{K}_V{}^H)_\mu{}^i = \frac{1}{2} R_t{}^{it}{}_\mu = -\frac{1}{2} R_t{}^{ti}{}_\mu .$$

§ 3. The Laplace-Beltrami operator in Riemannian submersion.

In $(B, {}^Bg)$ let $\omega = U_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ be a p-form and Δ the Laplace-Beltrami operator. Then we have

(3.1) $$(\Delta U)_{i_1 \cdots i_p} = -\nabla_t \nabla^t U_{i_1 \cdots i_p} + \sum_{a=1}^{p} {}^BK_{i_a}{}^t U_{i_1 \cdots t \cdots i_p} + \sum_{1 \leq a < b \leq p} {}^BK_{i_a i_b}{}^{ts} U_{i_1 \cdots t \cdots s \cdots i_p} .$$

Similarly in $(\tilde{M}, \tilde{g})$ we have for $\tilde{\omega} = \tilde{U}_{A_1 \cdots A_p} dx^{A_1} \wedge \cdots \wedge dx^{A_p}$

(3.2) $$(\tilde{\Delta} \tilde{U})_{A_1 \cdots A_p} = -\tilde{\nabla}_T \tilde{\nabla}^T \tilde{U}_{A_1 \cdots A_p} + \sum_{a=1}^{p} \tilde{K}_{A_a}{}^T \tilde{U}_{A_1 \cdots T \cdots A_p} + \sum_{1 \leq a < b \leq p} \tilde{K}_{A_a A_b}{}^{TS} \tilde{U}_{A_1 \cdots T \cdots S \cdots A_p} .$$

We now decompose $\tilde{\Delta} \tilde{U}$ into parts,

$$\tilde{\Delta} \tilde{U} = (\tilde{\Delta} \tilde{U})_{H \cdots HH} + (\tilde{\Delta} \tilde{U})_{H \cdots HV} + \cdots + (\tilde{\Delta} \tilde{U})_{V \cdots VV} .$$

But what we want to get is $\tilde{\Delta} \tilde{U}$ when $\tilde{U}$ is the lift, $\tilde{U} = U^L$. As $\tilde{U}$ satisfies $\tilde{U} = \tilde{U}_{H \cdots H}$ in this case, namely, any part such as $\tilde{U}_{H \cdots V \cdots H}$ vanishes, any part of $\tilde{\Delta} \tilde{U}$ where V appears more than twice in the subscript vanishes in view of fundamental formulas of differentiation. Hence we need only to calculate $(\tilde{\Delta} \tilde{U})_{H \cdots HHH}$, $(\tilde{\Delta} \tilde{U})_{H \cdots HHV}$ and $(\tilde{\Delta} \tilde{U})_{H \cdots HVV}$.

In order to obtain for example $(\tilde{\Delta} \tilde{U})_{H \cdots H}$ we first calculate the second derivative $\tilde{\nabla} \tilde{\nabla} \tilde{U}$ where we need only $(\tilde{\nabla} \tilde{\nabla} \tilde{U})_{HH, H \cdots H}$ and $(\tilde{\nabla} \tilde{\nabla} \tilde{U})_{VV, H \cdots H}$. As the required calculation is straightforward we give here only the result,

$$((\tilde{\nabla} \tilde{\nabla} U^L)_{HH, H \cdots H})_{kj, i_1 \cdots i_p} = \nabla_k \nabla_j U_{i_1 \cdots i_p} - \frac{1}{4} \sum_{a=1}^{p} R_{kj}{}^\tau R_{i_a}{}^t{}_\tau U_{i_1 \cdots t \cdots i_p} - \frac{1}{4} \sum_{a=1}^{p} R_{k i_a}{}^\tau R_j{}^t{}_\tau U_{i_1 \cdots t \cdots i_p} ,$$

$$((\tilde{\nabla} \tilde{\nabla} U^L)_{HH, H \cdots HV})_{kj, i_1 \cdots i_{p-1} \kappa} = -\frac{1}{2} R_{kj}{}^t{}_\kappa U_{i_1 \cdots i_{p-1} t} - \frac{1}{2} R_j{}^t{}_\kappa \nabla_k U_{i_1 \cdots i_{p-1} t} - \frac{1}{2} R_k{}^t{}_\kappa \nabla_j U_{i_1 \cdots i_{p-1} t} ,$$

$$((\tilde{\nabla}\tilde{\nabla}U^{L})_{HH,H\cdots HVV})_{kj,i_1\cdots i_{p-2}\kappa_1\kappa_2}$$

$$=\frac{1}{4}(R_k{}^t{}_{\kappa_1}R_j{}^s{}_{\kappa_2}-R_k{}^t{}_{\kappa_2}R_j{}^s{}_{\kappa_1})U_{i_1\cdots i_{p-2}ts},$$

$$((\tilde{\nabla}\tilde{\nabla}U^{L})_{VV,H\cdots H})_{\nu\mu,i_1\cdots i_p}$$

$$=-\frac{1}{2}\sum_{a=1}^{p}\left(R_{\nu i_a}{}^t{}_\mu+\frac{1}{2}R^{ts}{}_\nu R_{i_a s\mu}\right)U_{i_1\cdots t\cdots i_p}$$

$$+\frac{1}{4}\sum_{1\leq a<b\leq p}(R_{i_a}{}^t{}_\nu R_{i_b}{}^s{}_\mu+R_{i_a}{}^t{}_\mu R_{i_b}{}^s{}_\nu)U_{i_1\cdots t\cdots s\cdots i_p},$$

$$(\tilde{\nabla}\tilde{\nabla}U^{L})_{VV,H\cdots HV}=0,\qquad (\tilde{\nabla}\tilde{\nabla}U^{L})_{VV,H\cdots HVV}=0.$$

Let us define $\tilde{S}$ by

$$\tilde{\nabla}_T\tilde{\nabla}^T\tilde{U}_{A_1\cdots A_p}=\tilde{S}_{A_1\cdots A_p} \tag{3.3}$$

where $\tilde{U}=U^L$. Then we get from the foregoing result

$$\begin{aligned}(\tilde{S}_{H\cdots H})_{i_1\cdots i_p}&=g^{kj}((\tilde{\nabla}\tilde{\nabla}U^{L})_{HH,H\cdots H})_{kj,i_1\cdots i_p}\\&\quad+g^{\nu\mu}((\tilde{\nabla}\tilde{\nabla}U^{L})_{VV,H\cdots H})_{\nu\mu,i_1\cdots i_p}\\&=\nabla_t\nabla^tU_{i_1\cdots i_p}-\frac{1}{2}\sum_{a=1}^{p}R_{i_a s}{}^{\tau}R^{ts}{}_{\tau}U_{i_1\cdots t\cdots i_p}\\&\quad+\frac{1}{2}\sum_{1\leq a<b\leq p}R_{i_a}{}^{t\tau}R_{i_b}{}^s{}_\tau U_{i_1\cdots t\cdots s\cdots i_p},\end{aligned} \tag{3.4}$$

$$(\tilde{S}_{H\cdots HV})_{i_1\cdots i_{p-1}\kappa}=-\frac{1}{2}g^{kj}R_{kj}{}^t{}_\kappa U_{i_1\cdots i_{p-1}t}-R^{ts}{}_\kappa\nabla_tU_{i_1\cdots i_{p-1}s}, \tag{3.5}$$

$$(\tilde{S}_{H\cdots HVV})_{i_1\cdots i_{p-2}\kappa_1\kappa_2}=\frac{1}{2}R^{kt}{}_{\kappa_1}R_k{}^s{}_{\kappa_2}U_{i_1\cdots i_{p-2}ts}. \tag{3.6}$$

Substituting (2.6), (2.7), (2.8), (2.9), (2.10), (3.1), (3.3), (3.4), (3.5) and (3.6) into (3.2) we get

$$((\tilde{\Delta}U^{L})_{H\cdots H})_{i_1\cdots i_p}=(\Delta U)_{i_1\cdots i_p}+\frac{1}{2}\sum_{1\leq a<b\leq p}R_{i_a i_b}{}^{\tau}R^{ts}{}_{\tau}U_{i_1\cdots t\cdots s\cdots i_p}, \tag{3.7}$$

$$((\tilde{\Delta}U^{L})_{H\cdots HV})_{i_1\cdots i_{p-1}\kappa}=R^{ts}{}_\kappa\nabla_tU_{i_1\cdots i_{p-1}s}+\frac{1}{2}\sum_{a=1}^{p-1}R_{i_a}{}^{ts}{}_\kappa U_{i_1\cdots t\cdots i_{p-1}s}, \tag{3.8}$$

$$((\tilde{\Delta}U^{L})_{H\cdots HVV})_{i_1\cdots i_{p-2}\kappa_1\kappa_2}=(R_{\kappa_1}{}^{ts}{}_{\kappa_2}-R^{kt}{}_{\kappa_1}R_k{}^s{}_{\kappa_2})U_{i_1\cdots i_{p-2}ts}. \tag{3.9}$$

If $p=0$ we have only $\tilde{\Delta}\varphi^L=\Delta\varphi$. If $p=1$ we have only

$$((\tilde{\Delta}U^{L})_H)_i=(\nabla U)_i,\qquad ((\tilde{\Delta}U^{L})_V)_\kappa=R^{ts}{}_\kappa\nabla_tU_s.$$

§ 4. Eigenelement ω of the Laplace-Beltrami operator Δ in $(B, {}^Bg)$ such that the lift ω^L is also an eigenelement of $\tilde{\Delta}$ in $(\tilde{M}, \tilde{g})$.

From (3.7), (3.8) and (3.9) we get the following Main Theorem.

THEOREM 4.1. *Let $\omega = U_{\iota_1\cdots\iota_p} dx^{\iota_1}\wedge\cdots\wedge dx^{\iota_p}$ be an eigenelement of the Laplace-Beltrami operator Δ in the base manifold $(B, {}^Bg)$ with eigenvalue λ. A necessary and sufficient condition that $\tilde{\omega}=\omega^L$ be an eigenelement of the Laplace-Beltrami operator $\tilde{\Delta}$ in the total manifold $(\tilde{M}, \tilde{g})$ with the same eigenvalue λ is that ω satisfy the following equations,*

$$(\alpha)\qquad \sum_{1\leq a<b\leq p} R_{\iota_a\iota_b}{}^{\tau} R^{ts}{}_{\tau} U_{\iota_1\cdots t\cdots s\cdots\iota_p}=0\,,$$

$$(\beta)\qquad R^{ts}{}_{\kappa}\nabla_t U_{\iota_1\cdots\iota_{p-1}s}+\frac{1}{2}\sum_{a=1}^{p-1} R_{\iota_a}{}^{ts}{}_{\kappa} U_{\iota_1\cdots\iota\cdots\iota_{p-1}s}=0\,,$$

$$(\gamma)\qquad (R_{\kappa_1}{}^{ts}{}_{\kappa_2}-R^{kt}{}_{\kappa_1}R_k{}^s{}_{\kappa_2})U_{\iota_1\cdots\iota_{p-2}ts}=0\,.$$

From this theorem we get a simpler theorem,

THEOREM 4.2. *Let ω be an eigenelement of the Laplace-Beltrami operator Δ in $(B, {}^Bg)$. A sufficient condition that $\tilde{\omega}=\omega^L$ be an eigenelement of $\tilde{\Delta}$ in $(\tilde{M}, \tilde{g})$ is that ω satisfy the equations*

$$(\delta)\qquad R^{ts}{}_{\kappa} U_{\iota_1\cdots\iota_{p-2}ts}=0\,,$$

$$(\varepsilon)\qquad R^{ts}{}_{\kappa} V_{\iota_1\cdots\iota_{p-1}ts}=0$$

where $d\omega=V_{\iota_1\cdots\iota_{p+1}} dx^{\iota_1}\wedge\cdots\wedge dx^{\iota_{p+1}}$.

Proof. (α) is satisfied by (δ). From (δ) we get

$$R_{\iota_1}{}^{ts\kappa} U_{\iota_2\cdots\iota_{p-1}ts}+R^{ts\kappa}\nabla_{\iota_1} U_{\iota_2\cdots\iota_{p-1}ts}=0\,,$$

hence

$$\sum_{a=1}^{p-1} R_{\iota_a}{}^{ts\kappa} U_{\iota_1\cdots t\cdots\iota_{p-1}s}+R^{ts\kappa}\sum_{a=1}^{p-1}\nabla_{\iota_a} U_{\iota_1\cdots t\cdots\iota_{p-1}s}=0\,.$$

From (ε) we get

$$R^{ts\kappa}\sum_{a=1}^{p-1}\nabla_{\iota_a} U_{\iota_1\cdots t\cdots\iota_{p-1}s}-2R^{ts\kappa}\nabla_t U_{\iota_1\cdots\iota_{p-1}s}=0\,.$$

This proves that (β) is satisfied. From (δ) we also get

$$\left(\partial_{\kappa_1} R^{ts}{}_{\kappa_2}-\left\{{\lambda \atop \kappa_1\ \kappa_2}\right\} R^{ts}{}_{\lambda}\right) U_{\iota_1\cdots\iota_{p-2}ts}=0$$

which proves that (γ) is satisfied in view of (2.3).

Concerning (δ) and (ε) we get the following theorem.

THEOREM 4.3. *Let ω be a p-form satisfying $(\delta)_p$ and $(\varepsilon)_p$. Then $d\omega$ and $\delta\omega$ satisfy $(\delta)_{p+1}$, $(\varepsilon)_{p+1}$ and $(\delta)_{p-1}$, $(\varepsilon)_{p-1}$ respectively.*

Proof. That ω satisfies $(\varepsilon)_p$ is equivalent to that $d\omega$ satisfies $(\delta)_{p+1}$. Moreover $d\omega$ always satisfies $(\varepsilon)_{p+1}$ in view of $dd=0$. Thus Theorem 4.3 is proved for $d\omega$. From

$$R^{ts\kappa}U_{ji_1\cdots i_{p-3}ts}=0$$

we get

$$R^{jts\kappa}U_{ji_1\cdots i_{p-3}ts}+R^{ts\kappa}\nabla^j U_{ji_1\cdots i_{p-3}ts}=0\,.$$

As $R_{kji}{}^{\kappa}$ satisfies (2.4) we get

$$R^{ts\kappa}\nabla^j U_{ji_1\cdots i_{p-3}ts}=0$$

which proves that $\delta\omega$ satisfies $(\delta)_{p-1}$. As ω is an eigenelement of Δ we have $d\delta\omega=\lambda\omega-\delta d\omega$. As $d\omega$ satisfies $(\delta)_{p+1}$, $\delta d\omega$ satisfies $(\delta)_p$. Hence $d\delta\omega$ satisfies $(\delta)_p$. This proves that $\delta\omega$ satisfies $(\varepsilon)_{p-1}$ and completes the proof.

Applying the result obtained above to harmonic forms we get

THEOREM 4.4. *Let $\varphi=U_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ $(p\geqq 2)$ be a harmonic form of $(B, {}^Bg)$. Then φ^L is a harmonic form of $(\tilde{M}, \tilde{g})$ if φ satisfies (δ). The lift of any harmonic 1-form of $(B, {}^Bg)$ is a harmonic 1-form of $(\tilde{M}, \tilde{g})$.*

Let $(\tilde{M}, \tilde{g}, \tilde{\xi})$ be a Sasakian manifold [9]. Let there be a Riemannian submersion $\pi:(\tilde{M}, \tilde{g})\rightarrow(M, g)$ called a Sasakian submersion [6], [10], [11]. In this submersion fibers F are generated by the Killing vector field $\tilde{\xi}$. As $\dim F=1$ we can write R_{ji} instead of $R_{ji}{}^{\kappa}$. $\dfrac{1}{2}R_j{}^i=F_j{}^i$ represents a complex structure J such that (J, g) is a Kähler structure on M. Hence R_{ji} is a harmonic 2-form. This does not satisfy (α) since (α) assumes the form

$$(\alpha)_2 \qquad\qquad R_{ji}R^{ts}U_{ts}=0$$

if $m=1$, $p=2$. This gives an example of harmonic forms of $(B, {}^Bg)$ whose lift is not a harmonic form.

REFERENCES

[1] M. BERGER, P. GAUDUCHON, E. MAZET, Le spectre d'une variété Riemannienne, Springer-Verlag 1971.
[2] R. H. ESCOBALES, Riemannian submersions with totally geodesic fibers, J. Differential Geometry, 10 (1975), 253-276.
[3] S. ISHIHARA, Vector fields in fibred spaces with invariant Riemannian metric, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 163-178.

[4] S. ISHIHARA AND M. KONISHI, Differential geometry of fibred spaces, Publications of the study group of geometry, Tokyo, Vol. **8** (1973).

[5] Y. MUTŌ, On some properties of a fibred Riemannian manifold, Sci. Rep. Yokohama Nat. Univ., Sect. I, No. **1** (1952), 1-14.

[6] Y. MUTŌ, Riemannian submersions and critical Riemannian metrics, J. Math. Soc. Japan, **29** (1977), 493-511.

[7] T. NAGANO, On fibred Riemannian manifolds, Sci. Papers College Gen. Ed. Univ. Tokyo, **10** (1960), 17-27.

[8] B. O'NEILL, The fundamental equations of a submersion, Michigan Math. J., **13** (1966), 459-469.

[9] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures, J. Math. Soc. Japan, **14** (1962), 249-271.

[10] K. Yano AND S. ISHIHARA, Differential geometry of fibred spaces, Kōdai Math. Sem. Rep., **19** (1967), 257-288.

[11] K. YANO AND S. ISHIHARA, Fibred spaces with invariant Riemannian metric, Kōdai Math. Sem. Rep., **19** (1967), 317-360.

2262-150, TOMIOKA-CHO, KANAZAWA-KU,
YOKOHAMA 236, JAPAN