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MOMENT INEQUALITIES FOR MIXING SEQUENCES

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1. Introduction. Let $\{\xi_j, -\infty<j<\infty\}$ be a sequence of random variables which satisfy one of the following mixing conditions;
(I) ϕ-mixing condition, i. e.,

$$\phi(n)=\sup_k \sup_{A\in M_{-\infty}^k,\, B\in M_{k+n}^\infty} \frac{1}{P(A)} |P(A\cap B)-P(A)P(B)| \downarrow 0\ (n\to\infty) \tag{1}$$

or
(II) the strong mixing (s. m.) condition, i. e.,

$$\alpha(n)=\sup_k \sup_{A\in M_{-\infty}^k,\, B\in M_{k+n}^\infty} |P(A\cap B)-P(A)P(B)| \downarrow 0\ (n\to\infty) \tag{2}$$

where M_a^b denotes the σ-algebra generated by $\xi_a, \cdots, \xi_b (a\leqq b)$.

In this paper, firstly we shall prove some moment inequalities for mixing sequences. Secondly, using these inequalities we shall find sufficient conditions for the almost everywhere convergence of series $\sum_{j=1}^\infty a_j\xi_j$ and obtain the convergence rates of the strong laws of large numbers, and the functional central limit theorem for sums of (not necessarily strictly stationary) mixing sequences.

2. Preparatory lemmas.

LEMMA A (Theorem 17.2.3 in [3]). *Suppose that condition (I) is satisfied and that ξ and η are measurable over $M_{-\infty}^k$ and M_{k+n}^∞ respectively. If $E|\xi|^p<\infty$ and $E|\eta|^q<\infty$ with $p>1$, $q>1$, $p^{-1}+q^{-1}=1$, then*

$$|E\xi\eta-E\xi E\eta| \leqq 2\{\phi(n)\}^{p^{-1}}\{E|\xi|^p\}^{p^{-1}}\{E|\eta|^q\}^{q^{-1}}. \tag{3}$$

LEMMA B (Lemma 2.1 in [2]). *Suppose that condition (II) is satisfied and that ξ and η are measurable over $M_{-\infty}^k$ and M_{k+n}^∞ respectively. If $E|\xi|^p<\infty$ and $E|\eta|^q<\infty$ with $p>1$, $q>1$, $p^{-1}+q^{-1}<1$, then*

$$|E\xi\eta-E\xi E\eta| \leqq 12\{E|\xi|^p\}^{p^{-1}}\{E|\eta|^q\}^{q^{-1}}\{\alpha(n)\}^{1-p^{-1}-q^{-1}} \tag{4}$$

3. Moment inequalities for sums of s. m. sequences. In what follows, we shall agree that K denotes some absolute constant.

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THEOREM 1. *Let $\{\xi_i\}$ be ϕ-mixing with $\phi(n)$. We assume that for an even integer m $(\geqq 2)$*

$$\text{(i)} \qquad E\xi_i=0 \quad \text{and} \quad E|\xi_i|^m \leqq M(i=1, 2, \cdots),$$

and

$$\text{(ii)} \qquad \sum_{i=1}^{\infty}(i+1)^{\frac{m}{2}-1}\{\phi(i)\}^{\frac{1}{m}}<\infty .$$

Then, for every sequence $\{a_k\}$ and for every integer n, we have

$$E\Big(\sum_{i=b+1}^{b+n} a_i\xi_i\Big)^m \leqq c_m A_{b,n}^m \qquad (\text{all } b\geqq 0,\ n\geqq 1) \tag{5}$$

where c_m is an absolute constant depending only on m and

$$A_{b,n}^2=\sum_{i=b+1}^{b+n} a_i^2 . \tag{6}$$

Proof of Theorem 1. (5) is easily proved in the case $m=2$, and so is omitted (cf. the proof of Theorem 3).

For simplicity of the proofs, we explicitely consider the case where $m=4$ and $b=0$; an essentially same but more laborious proof holds for more general $m(\geqq 6)$. Put $A_{0,n}^2=A_n^2$. We note that

$$\begin{aligned} E\Big(\sum_{i=1}^{n} a_i\xi_i\Big)^4 = \sum_{i=1}^{n} a_i^4 E\xi_i^4 + \sum_{i\neq j} a_i^2a_j^2 E\xi_i^2\xi_j^2 + \sum_{i\neq j} a_i^3 a_j E\xi_i^3\xi_j \\ + \sum_{i\neq j\neq k} a_i^2 a_j a_k E\xi_i^2\xi_j\xi_k + \sum_{i\neq j\neq k\neq l} a_i a_j a_k a_l E\xi_i\xi_j\xi_k\xi_l . \end{aligned} \tag{7}$$

From Hölder's inequality

$$\sum_{i\neq j} a_i^2a_j^2 E\xi_i^2\xi_j^2 \leqq K\sum_{i\neq j} a_i^2a_j^2 \leqq KA_n^4 . \tag{8}$$

By Lemma A

$$\begin{aligned} &\Big|\sum_{i<j} a_i^3 a_j E\xi_i^3\xi_j\Big| \leqq K\sum_{i<j}|a_i^3|\,|a_j|\,\{\phi(j-i)\}^{3/4} \\ &\leqq K\sum_{i<j}(a_i^4+a_i^2a_j^2)\{\phi(j-i)\}^{3/4} \\ &\leqq K\Big[\sum_{i=1}^{n-1} a_i^4 \sum_{j=i+1}^{n}\{\phi(j-i)\}^{3/4} + \sum_{i<j} a_i^2a_j^2\{\phi(j-i)\}^{3/4}\Big] \\ &\leqq K\Big[\sum_{i=1}^{n} a_i^4 + 2\sum_{i<j} a_i^2a_j^2\Big\} = KA_n^4 \end{aligned} \tag{9}$$

and similarly

$$\Big|\sum_{i<j} a_i a_j^3 E\xi_i\xi_j^3\Big| \leqq K\sum_{i<j}|a_i|\,|a_j|^3\{\phi(j-i)\}^{1/4} \leqq KA_n^4 . \tag{10}$$

Now, we shall show

(11) $$|\sum_{i<j<k} a_i^2 a_j a_k E\xi_i^2\xi_j\xi_k| \leqq KA_n^4 .$$

Since $(E|\xi_i|^2)^2 \leqq E|\xi_i|^4 \leqq M < \infty$ and $E\xi_i = 0$, so using Lemma A and Hölder's inequality, we have the followings:

$$|\sum_{\substack{i<j<k \\ j-i<k-j}} a_i^2 a_j a_k E\xi_i^2\xi_j\xi_k|$$

$$\leqq 2\sum_{\substack{i<j<k \\ j-i<k-j}} a_i^2|a_j||a_k|\{E|\xi_i^2\xi_j|^{4/3}\}^{3/4}\{E|\xi_k|^4\}^{1/4}\{\phi(k-j)\}^{3/4}$$

$$\leqq K\sum_{i=1}^{n-2}\sum_{q=2}^{n-i-1}\sum_{p=1}^{q}\{a_i^2a_{i+p}^2 + a_i^2a_{i+p+q}^2\}\{\phi(q)\}^{3/4}$$

$$\leqq K\sum_{i=1}^{n-2}\{\sum_{p=1}^{n-i-1} a_i^2a_{i+p}^2 + (\sum_{p=1}^{n} a_p^2)a_i^2\}\sum_{q=1}^{\infty}\{\phi(q)\}^{3/4} \leqq KA_n^4$$

$$|\sum_{\substack{i<j<k \\ j-i\geqq k-j}} a_i^2 a_j a_k E\xi_i^2\xi_j\xi_k|$$

$$\leqq \sum_{\substack{i<j<k \\ j-i\geqq k-j}} a_i^2|a_j||a_k|[E\xi_i^2(E|\xi_j|^4)^{1/4}(E|\xi_k|^{4/3})^{3/4}\{\phi(k-j)\}^{1/4}$$

$$+2\{E|\xi_i|^4\}^{1/2}\{E|\xi_j\xi_k|^2\}^{1/2}\{\phi(j-i)\}^{1/2}]$$

$$\leqq K\sum_{\substack{i<j<k \\ j-i\geqq k-j}} (a_i^2a_j^2 + a_i^2a_k^2)[\{\phi(k-j)\}^{1/4} + \{\phi(j-i)\}^{1/2}]$$

$$\leqq K\sum_{i=1}^{n-2}\sum_{p=1}^{n-i-1}\sum_{q=1}^{p}(a_i^2a_{i+p}^2 + a_i^2a_{i+p+q}^2)[\{\phi(q)\}^{1/4} + \{\phi(p)\}^{1/2}]$$

$$\leqq K[\{\sum_{i=1}^{n-2}\sum_{p=1}^{n-i-1} a_i^2a_{i+p}^2\sum_{q=1}^{\infty}\{\phi(q)\}^{1/4} + \sum_{i=1}^{n-2}\sum_{p=1}^{n-i-1} a_i^2a_{i+p}^2 p\{\phi(p)\}^{1/2}\}$$

$$+\{(\sum_{p=1}^{n} a_p^2)\sum_{i=1}^{n-2}\sum_{q=1}^{n} a_i^2\{\phi(q)\}^{1/4} + (\sum_{q=1}^{n} a_q^2)\sum_{i=1}^{n}\sum_{p=1}^{n} a_i^2\{\phi(p)\}^{1/2}\}]$$

$$\leqq KA_n^4 .$$

Hence, we have (11). Similarly, we have

(12) $$|\sum_{i<j<k} a_i a_j^2 a_k E\xi_i\xi_j^2\xi_k| \leqq KA_n^4 ,$$

(13) $$|\sum_{i<j<k} a_i a_j a_k^2 E\xi_i\xi_j\xi_k^2| \leqq KA_n^4 .$$

Next, we shall prove

(14) $$|\sum_{i<j<k<l} a_i a_j a_k a_l E\xi_i\xi_j\xi_k\xi_l| = KA_n^4 .$$

For fixed i, let $\sum_i^{(1)}$, $\sum_i^{(2)}$ and $\sum_i^{(3)}$ be respectively the components of the summation $\sum_{i<j<k<l}$ for $j-i\geqq(k-j, l-k)$, $k-j\geqq(j-i, l-k)$ and $l-k\geqq(j-i, k-j)$. From

Lemma A

$$\sum_{i=1}^{n-3}\sum_i^{(1)}|a_ia_ja_ka_l|\,|E\xi_i\xi_j\xi_k\xi_l|$$

$$\leqq K\sum_{i=1}^{n-3}\sum_i^{(1)}\{a_i^2a_j^2+a_k^2a_l^2\}\{\phi(j-i)\}^{1/4}$$

$$\leqq K\sum_{i=1}^{n-3}\Big[\sum_{p=1}^{n-i-2}\sum_{q=1}^{p}\sum_{r=1}^{p}\big[a_i^2a_{i+p}^2\{\phi(p)\}^{1/4}$$

$$+a_{i+p+q}^2a_{i+p+q+r}^2\{\phi(p)\}^{1/4}\big]$$

$$\leqq K\sum_{i=1}^{n-3}\Big[\sum_{p=1}^{n-i-2}a_i^2a_{i+p}^2p^2\{\phi(p)\}^{1/4}$$

$$+\Big(\sum_{r=1}^{n}a_r^2\Big)\sum_{p=1}^{n-i-2}\sum_{q=1}^{p}|a_{i+p+q}|^2\{\phi(p)\}^{1/4}\Big]$$

$$\leqq K\Big[\sum_{i=1}^{n-3}\sum_{p=1}^{n-i-2}a_i^2a_{i+p}^2+A_n^2\sum_{p=1}^{n-3}\sum_{q=1}^{p}\sum_{i=1}^{n-p-q-1}a_{i+p+q}^2\{\phi(p)\}^{1/4}\Big]$$

$$\leqq KA_n^4\{1+\sum_{p=1}^{n-3}p\{\phi(p)\}^{1/4}\}\leqq KA_n^4\,.$$

Similarly, we have

$$\sum_{i=1}^{n-3}\sum_i^{(2)}|a_ia_ja_ka_l|\,|E\xi_i\xi_j\xi_k\xi_l|$$

$$\leqq K\sum_{i=1}^{n-3}\sum_i^{(2)}\{a_i^2a_k^2+a_j^2a_l^2\}\big[\{\phi(j-i)\}^{1/2}\{\phi(l-k)\}^{1/2}+\{\phi(k-j)\}^{1/2}\big]$$

$$\leqq KA_n^4$$

and

$$\sum_{i=1}^{n-3}\sum_i^{(3)}|a_ia_ja_ka_l|\,|E\xi_i\xi_j\xi_k\xi_l|$$

$$\leqq K\sum_{i=1}^{n-3}\sum_i^{(3)}\{a_i^2a_j^2+a_k^2a_l^2\}\{\phi(l-k)\}^{3/4}\leqq KA_n^4\,.$$

So, we have (14). Hence, from (7)-(14), we have (5) in the case where $m=4$ and $b=0$.

From Theorem F in [4] and Theorem 1, we have the following conclusion (cf. [1, p. 102], [9, p. 83] and [11])

THEOREM 2. *Let the conditions of Theorem* 1 *is satisfied for some even integer. If* $m=2$, *then*

$$E\Big(\max_{1\leqq j\leqq n}\Big|\sum_{i=b+1}^{b+j}a_i\xi_i\Big|^2\Big)\leqq c_2A_{b,n}^2(\log^2 2n)\qquad (all\ b\geqq 0,\ n\geqq 1) \tag{15}$$

and if $m\geqq 4$, *then*

(16) $$E(\max_{1\leqq j\leqq n}|\sum_{i=b+1}^{b+j}a_i\xi_i|^m)\leqq c_m A_{b,n}^m \qquad (all\ b\geqq 0,\ n\geqq 1)$$

Here, $c_m (m=2, 4, \cdots)$ are constants defined in Theorem 1.

4. Moment inequalities for sums of s. m. sequences.

THEOREM 3. Let $\{\xi_i\}$ be a s. m. sequence with coefficient $\alpha(n)$. We assume that for some $\delta>0$ and for an even integer $m(\geqq 2)$

(i) $$E\xi_i=0 \quad \text{and} \quad E|\xi_i|^{m+\delta}\leqq M<\infty \qquad (i=1, 2, \cdots),$$

and

(ii) $$\sum_{i=1}^{\infty}(i+1)^{m/2-1}\{\alpha(i)\}^{\delta/(m+\delta)}<\infty .$$

Then, for every sequence $\{a_k\}$ and for every integer n, we have

(17) $$E(\sum_{i=b+1}^{b+n}a_i\xi_i)^m\leqq c'_m A_{b,n}^m \qquad (\text{all } b\geqq 0,\ n\geqq 1),$$

where c'_m is an absolute constant depending only on m. Hence, the analogous inequalities to (15) and (16) hold.

The first part of Theorem 3 is analogously proved to the proof of Theorem 1, using Lemma B instead of Lemma A and so is omitted.

5. Functionals of mixing sequences. For a strictly stationary mixing process $\{\xi_j\}$, let H_a^b be a Hilbert space of random variables, measurable with respect to M_a^b, and U an isometric operator on $H_{-\infty}^{\infty}$. Let $Y\in H_{-\infty}^{\infty}$ be a random element such that $EY=0$ and $E|Y|^{2+\delta}<\infty$ for some $\delta\geqq 0$. Define

(18) $$Y_j=U^j Y \qquad (j=0, \pm 1, \pm 2, \cdots)$$

and put

(19) $$\psi(k)=E|Y-E(Y|M_{-k}^k)|^{2+\delta} \qquad (k=1, 2, \cdots).$$

THEOREM 4. *Let $\{\xi_j\}$ be a strictly stationary, ϕ-mixing sequence. Let $\{Y_j\}$ be the strictly stationary sequence defined by* (18) *with $\delta=0$. If $\sum_{k=1}^{\infty}\phi^{1/2}(k)<\infty$ and $\sum_{k=1}^{\infty}\psi^{1/2}(k)<\infty$, then for every sequence $\{a_k\}$ and for every $n(\geqq 1)$*

(20) $$\operatorname{Var}(\sum_{i=b+1}^{b+n}a_iY_i)\leqq KM_0A_{b,n}^2 \qquad (\text{all } b\geqq 0).$$

Hence, for every $n\geqq 1$

(21) $$E(\max_{1\leqq j\leqq n}(\sum_{i=b+1}^{b+j}a_iY_i)^2)\leqq KM_0A_{b,n}^2(\log 2n)^2 \qquad (\text{all } b\geqq 0)$$

Here, $M_0=\max\{EY^2, \{EY^2\}^{1/2}\}$.

Proof. Without loss of generality, we may assume that $b=0$. From the proof of (18.6.4) in [3]

$$|E(a_iY_i)(a_jY_j)|=|a_ia_j||EY_0Y_{j-i}|$$
$$\leqq M_0(a_i^2+a_j^2)\Big\{\phi^{1/2}\Big(\Big[\frac{j-i}{3}\Big]\Big)+\psi^{1/2}\Big(\Big[\frac{j-i}{3}\Big]\Big)\Big\}$$

where $j>i$ and $[s]$ denotes the largest integer p such that $p\leqq s$. Thus, (20) follows, since

$$\operatorname{Var}\Big(\sum_{i=1}^n a_iY_i\Big)\leqq M_0\Big[\sum_{i=1}^n a_i^2+2\sum_{i<j}(a_i^2+a_j^2)\Big\{\phi^{1/2}\Big(\Big[\frac{j-i}{3}\Big]\Big)+\psi^{1/2}\Big(\Big[\frac{j-i}{3}\Big]\Big)\Big\}\Big]$$
$$\leqq KM_0\Big[\sum_{i=1}^n a_i^2+\sum_{i=1}^n a_i^2\sum_{p=1}^{n-i}\{\phi^{1/2}(p)+\psi^{1/2}(p)\}$$
$$+\sum_{j=2}^n a_j^2\sum_{q=1}^{j-1}\{\phi^{1/2}(q)+\psi^{1/2}(q)\}\Big]$$
$$\leqq KM_0A_{0,n}^2\,.$$

(21) follows easily from (20).

Analogously, using inequalities in the proof of Theorem 18.6.2 in [3] we have the following

THEOREM 5. *Let the strictly stationary sequence* $\{\xi_i\}$ *be s.m. and consider the strictly stationary sequence* $\{Y_j\}$ *defined by* (18) *with some* $\delta>0$. *If*

$$\sum_{k=1}^{\infty}\{\alpha(k)\}^{\delta/(2+\delta)}<\infty \quad \textit{and} \quad \sum_{k=1}^{\infty}\{\psi(k)\}^{\delta/(2+\delta)}<\infty\,,$$

then for any $n(\geqq 1)$

$$\operatorname{Var}\Big(\sum_{i=b+1}^{b+n}a_iY_i\Big)\leqq KM_1A_{b,n}^2 \qquad (\text{all } b\geqq 0) \tag{22}$$

and so

$$E\Big(\max_{1\leqq j\leqq n}\sum_{i=b+1}^{b+n}a_iY_i\Big)^2\leqq KM_1A_{b,n}^2(\log 2n)^2 \qquad (\text{all } b\geqq 0)\,. \tag{23}$$

Here, $M_1=\max(E|Y|^{2+\delta}, \{E|Y|^{2+\delta}\}^{2/2+\delta})$.

6. Some applications.

(I) Almost sure convergence of series $\sum_{i=1}^{\infty}a_i\xi_i$.

THEOREM 6. *Let* $\{\xi_i\}$ *be a s.m. mixing sequence of random variables with* $E\xi_i=0$. *Then, the series* $\sum_{i=1}^{\infty}a_i\xi_i$ *is convergent almost surely, if* $\sum_{i=1}^{\infty}a_i^2\log^2 i$ *and for* *some* $\delta>0$ *the following conditions are satisfied:*

(i) $$E|\xi_i|^{2+\delta}\leqq K(i=1, 2, \cdots), \quad \textit{and}$$

(ii) $$\sum_{n=1}^{\infty} \{\alpha(n)\}^{\delta/(2+\delta)}<\infty .$$

Proof. Let $N=N(n)$ be an arbitrary function of n such that $N>n$. If (i) holds, then from Theorem 1

$$E(\sum_{i=n}^{N} a_i\xi_i)^2\leqq K\sum_{i=n}^{N} a_i^2E\xi_i^2\leqq Kd\log^{-2}n$$

where $d=\sum_{i=1}^{\infty} a_i^2\log^2 i$, and so

$$\sum_{n=1}^{\infty} E(\sum_{i=2^n}^{N} a_i\xi_i)^2\leqq K\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty .$$

Hence, by the Beppo-Levi theorem

$$\sum_{i=2^n}^{\infty}\xi_i \longrightarrow 0 \qquad \text{a. s.}$$

The rest of the proof is obtained by the method of the proof of Theorem 3.2.1 in [8], using Theorem 2 instead of Theorem 3.1.1 in [8] and so is omitted.

If (ii) holds, from Theorem 3 we have the desired conclusion analogously.

For functionals of mixing processes the following theorem holds:

THEOREM 7. *For a strictly stationary mixing process* $\{\xi_j\}$, *let* $\{Y_j\}$ *be the process defined in Section* 5. *Then the series* $\sum_{i=1}^{\infty} a_i\xi_i$ *is convergent almost surely if* $\sum_{i=1}^{\infty} a_i^2\log^2 i<\infty$ *and one of the following conditions holds:*

(i) $$\{\xi_i\} \textit{ is } \phi\textit{-mixing with } \sum_{i=1}^{\infty}\phi^{1/2}(i)<\infty \quad \textit{and} \quad \sum_{i=1}^{\infty}\psi^{1/2}(i)<\infty ,$$

or

(ii) $$\{\xi_i\} \textit{ is s. m. with } \sum_{n=1}^{\infty}\{\alpha(n)\}^{\delta/(2+\delta)}<\infty , \; E|Y|^{2+\delta}<\infty \quad \textit{and}$$

$$\sum_{n=1}^{\infty}\{\psi(n)\}^{\delta/(2+\delta)}<\infty \qquad (\delta>0).$$

Remark. It is obvious from the proof of Theorem 6 that the conclusions of Theorems 6 and 7 remain true, if we replace the condition $\sum_{i=1}^{\infty} a_i^2\log^2 i<\infty$ by the condition

$$\sum_{i=1}^{\infty} a_i^2(\log i)(\log\log i)(\log\log\log i)^{1+\varepsilon}<\infty$$

for some $\varepsilon>0$.

(II) The rate of the convergence in the strong law of large numbers.

THEOREM 8. *Let $m\geqq 4$ be an even integer. If the conditions of Theorem* 2 *or* 3 *are satisfied, then the followings hold:*
(i) *if $A_n\to\infty$, then for each $\varepsilon>0$ and $\delta>0$*

$$P\Big(\sum_{i=1}^{n} a_i\xi_i=o\{A_n(\log A_n)^{1/m}(\log\log A_n)^{(1+\delta)/m}\}\Big)=1 \tag{24}$$

and

$$\sum_n P\Big(\frac{a_n^{2/m}}{A_n^{(m+2)/m}(\log A_n)^{(1+\delta)/m}}\max_{1\leqq j\leqq n}\Big|\sum_{i=1}^{j} a_i\xi_i\Big|\geqq\varepsilon\Big)<\infty \tag{25}$$

(ii) *if $A_n\to\infty$ and $a_n^2\leqq cA_n^2(n\geqq n_0,\ 0<c<1)$, then for each $\varepsilon>0$ and $\delta>0$*

$$\sum_n \frac{a_n^2A_n^{m-2}}{(\log A_n)^{1+\delta}}P\Big(\sup_{k\geqq n}\frac{1}{A_k^2}\Big|\sum_{i=1}^{k} a_i\xi_i\Big|\geqq\varepsilon\Big)<\infty \tag{26}$$

and

$$\sum_n \frac{a_n^2}{A_n^2(\log A_n)^{1-b_1}}P\Big(\sup_{k\geqq n}\frac{1}{A_k^2(\log A_k)^{b_2}}\Big|\sum_{i=1}^{k} a_i\xi_i\Big|\geqq 1\Big) \tag{27}$$

$$(0\leqq b_1<b_2m-1).\quad \textit{Here, } A_n^2=A_{0,n}^2=\sum_{i=1}^{n} a_i^2\,.$$

This theorem follows from Theorems 5-8 in [4] and Theorems 2 and 3.

(III) The functional central limit theorem for (not necessarily strictly stationary) mixing sequences. In what follows, we assume that $\{\xi_i\}$ is a sequence of random variables centered at expectations with variances $E\xi_n^2$ uniformly bounded by 1. Put

$$S_n=\sum_{i\leqq n}\xi_i,\ s_n^2=E(S_n^2),\qquad \sigma_N^*=\max_{1\leqq n\leqq N}E\xi_n^2 \tag{28}$$

We shall assume that $s_n^2\to\infty$.

Consider the point $s_k^2/s_n^2(1\leqq k\leqq n)$ on the real line. Order them linearly and discard those bigger than 1. Set

$$X_n(s_k^2/s_n^2)=s_n^{-1}S_k$$

and define a random function $X_N(t)$ in $C[0, 1]$ by

$$X_n(t)=s_n^{-1}s_k^2 \tag{30}$$

and linear between those points. Similarly define a random function $Y_n(t)$ in $D[0, 1]$ by setting $Y_n(t)=s_n^{-1}S_k$ if $t=s_n^{-2}s_k^2$. Throughout the interior of the partition intervals (t_{i-1}, t_i) we define $Y_n(t)$ to be constant equaling any value between $Y_n(t_{i-1})$ and $Y_n(t_i)$.

We shall suppose that one of the following conditions holds.

(a) $\{\xi_n\}$ satisfies Condition (I) with $\sum_{n=1}^{\infty} n\{\phi(n)\}^{1/4}<\infty$, and $E\xi_i^4\leqq K(i=1, 2, \cdots)$, and

(b) $\{\xi_n\}$ satisfies Condition (II) with $\sum_{n=1}^{\infty} n\alpha^{\delta/4+\delta}(n)<\infty$, and $E|\xi_i|^{4+\delta}\leqq K(i=1, 2, \cdots)$

for some $\delta>0$.

Now, we write

$$S_n=\sum_{i=1}^{n}\xi_i=\sum_{j=1}^{l}y_j+\sum_{j=1}^{l+1}z_j \tag{31}$$

where we set

$$y_1=\xi_1+\cdots+\xi_{h_1},\qquad z_1=\xi_{h_1+1}+\cdots+\xi_{h_1+k},\ \cdots,$$

$$y_l=\xi_{\rho_l+1}+\cdots+\xi_{\rho_l+h_l},\qquad z_l=\xi_{\rho_l+h_l+1}+\cdots+\xi_{\rho_l+1}$$

$$z_{l+1}=\xi_{\rho_l+1}+\cdots+\xi_n\,.$$

Here, we put

$$\rho_i=\sum_{\nu<i}(h_\nu+k)$$

the integers h and k being at our disposal.

A double sequence of real numbers is called an admissible pair for $\{\xi_n\}$ if

$$\kappa_n\longrightarrow 0,\ \frac{\kappa_n B_n}{\sigma_n^{*2}}\longrightarrow 0,\ \frac{s_n^2}{B_n}\longrightarrow\infty$$

$$\phi\Big(\frac{\kappa_n B_n}{\sigma_n^*}\Big)\frac{s_n^2}{B_n}\longrightarrow 0\quad\text{or}\quad \alpha\Big(\frac{\kappa_n B_n}{\sigma_n^*}\Big)\frac{s_n^2}{B_n}\longrightarrow 0 \tag{32}$$

according to whether Condition (I) or (II) is assumed to hold.

LEMMA. *Suppose that (a) or (b) holds. Let* (κ_n, s_n) *be any admissible pair for* $\{\xi_n\}$. *Then we can represent* S_n *in the form* (31) *subject to the following conditions.*

$$E(y_j^2)=B_n(1+o(1)),\qquad E(z_j^2)\leqq K\kappa_n B_n$$

$$E(z_{l+1}^2)\leqq B_n(1+o(1)) \tag{33}$$

uniformly in $1\leqq j\leqq l$. *Moreover*

$$E(\sum_{j\leqq 1}z_j)^2\leqq K\kappa_n s_n^2\,,\qquad E(\sum_{j\leqq 1}y_j)^2=s_n^2(1+o(1))\,. \tag{34}$$

The proof of this lemma is easily obtained by the method of the proof of Lemma 4 in [7], using Theorems 1 and 3, and so is omitted.

By Lemma we have the following theorem which is a generalization of Theorem 1 in [8].

THEOREM 10. *Suppose that* $\{\xi_n\}$ *satisfies either (a) or (b). Let* (κ_n, B_N) *be any admissible pair and let* $y_j=y_{nj}$*(with df* F_{nj}*) be the sequence of random variables associated with it according to Lemma. Then*

$$X_n\xrightarrow{D}W\quad\text{and}\quad Y_n\xrightarrow{D}W \tag{35}$$

where W is standard Brownian motion if and only if, for any $\varepsilon>0$

(36) $$s_n^{-2}\sum_{j\le l}\int_{|y|\ge\varepsilon s_n} y^2 dF_{nj} \longrightarrow 0 \qquad (n\longrightarrow\infty)$$

Proof. The proof is carried out by the same method of the proof of Theorem 1 in [8], using Theorems 1 and 3 instead of condition a) in [8] and so is omitted.

(IV) The rate of convergence to normality. Let $\{\xi_i\}$ be a strictly stationary, s. m. sequence of random variables with $E\xi_i=0$. Put $S_0=0$ and $S_n=\sum_{j=1}^{n}\xi_j$, and assume that

(37) $$\sigma^2=E\xi_0^2+2\sum_{j=1}^{\infty}E\xi_0\xi_j>0$$

if the series is convergent. It is known that if $E|\xi_i|^{2+\delta}<\infty$ and $\sum_{j=1}^{\infty}\{\alpha(j)\}^{\delta/2+\delta}<\infty$ for some $\delta>0$, then the series in (37) is absolutely convergent. (cf. [3], Theorem 18.5.3)

THEOREM 11. *Let $\{\xi_i\}$ be a strictly stationary, s. m. sequence of random variables with $E\xi_i=0$ and $E|\xi_i|^{4+\delta}<\infty$ for some $\delta>0$. If $\alpha(n)=O(e^{-\gamma n})$ for some $\gamma>0$, then*

(38) $$\Delta_n=\sup_x|P(S_n<x\sigma\sqrt{n})-\Phi(x)|=O(n^{-1/7})$$

where

(39) $$\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du\,.$$

Proof. Let n be any positive integer fixed. Let

$$p=[n^{4/7}]\,,\qquad q=[c\log n]\qquad (c\gamma>2)\,,$$

$$k=[n(p+q)^{-1}]\,.$$

Define

$$\eta_i=\sum_{j=1}^{p}\xi_{i(p+q)+j}\qquad (i=1,\cdots,k)$$

and

$$\zeta_i=\sum_{j=1}^{p}\xi_{i(p+q)+p+j}\qquad (i=1,\cdots,k)\,.$$

Put

$$\eta_i^*=(\mathrm{var}\,\eta_1)^{-1/2}\eta_i\qquad (i=1,\cdots,k)\,.$$

Then

$$\Delta_n \leqq \sup_x |P(\sum_{i=1}^{k} \eta_i^* \leqq x\sqrt{k}) - \Phi(x)|$$

$$+\sup_x |\Phi(x-2\varepsilon_n)-\Phi(x)| + \sup_x \left|\Phi\left(\frac{\sqrt{k \operatorname{Var} \eta_1}}{\sqrt{n}\ \sigma} x\right) - \Phi(x)\right|$$

$$+P(|\sum_{i=1}^{k} \zeta_i| \geqq \varepsilon_n n^{1/2}) + P(|\sum_{i=k(p+q)+1}^{n} \xi_i| \geqq \varepsilon_n n^{1/2})$$

where $\varepsilon_n = n^{-1/7}$.

Now, by the method used in the proof of Theorem 2 in [5], we shall show

$$\Delta_n' = \sup_x |P(\sum_{i=1}^{k} \eta_i^* \leqq x\sqrt{k}) - \Phi(x)| = O(n^{-1/7}). \tag{40}$$

Let $Y_1, \cdots, Y_k$ be independently and identically distributed random variables each having the same df as that of η_1^*. Thus, $EY_i=0$, $\operatorname{Var} Y_i = 1$ and from Theorem 1

$$E|Y_i|^3 = (\operatorname{var} \eta_1)^{-3/2} E|\eta_1|^3 \leqq (\operatorname{Var} \eta_1)^{-3/2} (E|\eta_1|^4)^{3/4}$$

$$\leqq K_0 p^{-3/2} p^{3/2} = K_0 .$$

Applying Lemma 1 in [6, p. 109] to the sum $k^{-1/2} \sum_{j=1}^{k} Y_j$, we obtain

$$\left| \frac{\prod_{j=1}^{k} Ee^{itk^{-1/2}Y_j} - e^{-t^2/2}}{t} \right| \leqq Kk^{-1/2} \{\operatorname{Var} Y_1\}^{-3/2} E|Y_1|^3 t^2 e^{-t^2/4}$$

$$\leqq Kk^{-1/2} t^2 e^{-t^2/4}$$

for all t such that $|t| \leqq K_1 \sqrt{k}$.

On the other hand, as η_i^*'s are s. m., so for all n sufficiently large and for all t

$$|Ee^{itk^{-1/2}\sum_{j=1}^{k}\eta_j^*} - \prod_{j=1}^{k} Ee^{itk^{-1/2}Y_j}| \leqq K\alpha(q) = O(n^{-1/7})$$

and from Theorem 1

$$|Ee^{itk^{-1/2}\sum_{j=1}^{k}\eta_j^*} - \prod_{j=1}^{k} Ee^{itk^{-1/2}Y_j}|$$

$$\leqq \frac{t^2}{2k} \{E|\sum_{j=1}^{k} \eta_j^*|^2 + kEY_1^2\} \leqq Kt^2$$

for all $|t|$ sufficiently small.

Hence, from Theorem 3 in [6, p. 111] it follows that for some $a>0$

$$\Delta_n' \leqq K_3 \int_{-ak^{1/2}}^{ak^{1/2}} \left| \frac{E\{\exp(itk^{-1/2}\sum_{j=1}^{k}\eta_j^*)\} - e^{-t^2/2}}{t} \right| dt + K_4 k^{-1/2}$$

$$=K_3\left[\left\{\int_{|t|\le n^{-1}}+\int_{n^{-1}\le|t|\le ak^{1/2}}\right\}\left|\frac{E\{\exp(itk^{-1/2}\sum_{j=1}^{k}\eta_j^*)\}-\prod_{j=1}^{k}E\exp(itk^{-1/2}Y_j)}{t}\right|dt\right.$$

$$+\int_{|t|\le ak^{1/2}}\left|\frac{\prod_{j=1}^{k}E\exp(itk^{-1/2}Y_j)-e^{-t^2/2}}{t}\right|dt+K_4k^{-1/2}$$

$$=K_3\{O(n^{-2})+k\alpha(q)\int_{n^{-1}\le|t|\le ak^{1/2}}|t|^{-1}dt\}+K_4k^{-1/2}$$

$$=K_3\{O(n^{-2})+O(n^{-1})\}+K_4k^{-1/2}=O(n^{-1/7}).$$

Thus, we have (40).

From inequalities (3.3) and (3.4) in [6, p. 114] and Theorem 1 we have the following inequalities:

(41) $$\sup_x|\Phi(x-2\varepsilon_n)-\Phi(x)|\leqq 2\varepsilon_n=2n^{-1/7}$$

(42) $$\sup_x\left|\Phi\left(\frac{\sqrt{k\operatorname{Var}\eta_1}}{\sqrt{n}\,\sigma}x\right)-\Phi(x)\right|\leqq K\left|\frac{\sqrt{k\operatorname{Var}\eta_1}}{n}-1\right|$$
$$=K\left|\frac{k\operatorname{Var}\eta_1-n\sigma^2}{n(\sqrt{k\operatorname{Var}\eta_1}+\sqrt{n})\sigma}\right|\leqq K\frac{p}{n}=Kn^{-3/7}$$

(43) $$P(|\sum_{i=1}^{k}\zeta_i|\geqq\varepsilon_n n^{1/2})\leqq n^{-5/7}E|\sum_{i=1}^{k}\zeta_i|^2$$
$$\leqq Kn^{-5/7}k\{E|\zeta_1|^{2+\delta}\}^{2/2+\delta}=O(n^{-1/7})$$

(44) $$P(|\sum_{i=k(p+q)+1}^{n}\xi_i|\geqq\varepsilon_n n^{1/2})\leqq Kn^{-5/7}\{n-k(p+q)\}$$
$$=O(n^{-1/7}).$$

Hence, by (40)-(44), we have (38) and the proof is completed.

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