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TOTAL ABSOLUTE CURVATURE OF IMMERSED SUBMANIFOLDS OF SPHERES

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Let M be an m-dimensional compact differentiable manifold and f be an immersion of M into S^{m+k}. Let $p\in S^{m+k}$ and π_p be the stereographic projection from p onto the tangent plane of S^{m+k} at $-p$. If $p\notin f(M)$, $\pi_p\cdot f$ is an immersion of M into the $(m+k)$-dimensional euclidean space. Let i be the inclusion of S^{m+k} into E^{m+k+1}. In general, $\tau(\pi_p\cdot f)$, the total absolute curvature of $\pi_p\cdot f$, is not necessarily equal to $\tau(i\cdot f)$, the total absolute curvature of $i\cdot f$. If $m=1$, however, T. Banchoff showed in [1] that the average value of $\tau(\pi_p\cdot f)$ over all possible base points is equal to $\tau(i\cdot f)$. In this paper we generalize this theorem to immersions of m-dimensional compact differentiable manifolds into S^{m+k}.

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1. Preliminaries.

Let M be an m-dimensional compact differentiable manifold and F be an immersion of M into E^{m+k}. Let ν_F^1 be the unit normal vector bundle of F with the total space N_F^1 and the projection $\pi: N_F^1\to M$. For any $e\in N_F^1$, we denote by $S(e)$ the second fundamental tensor of F with respect to e. $S(e)$ is an endomorphism of $T_{\pi(e)}M$. Let $G(e)=\det S(e)$ and we call $G(e)$ the *Lipschitz-Killing curvature* of F with respect to e. N_F^1 is an $(m+k-1)$-dimensional orientable Riemannian manifold with the metric naturally induced from E^{m+k}. Let $d\mu$ be the volume element of N_F^1.

Set

$$\tau(F)=(c_{m+k-1})^{-1}\int_{N_F^1}|G(e)|d\mu(e)$$

where c_{m+k-1} is the volume of S^{m+k-1}.
We call $\tau(F)$ the *total absolute curvature* of F.

Let $N_F^1(x)$ be the fibre of N_F^1 at $x\in M$. $N_F^1(x)$ has the natural metric of the sphere S^{k-1} and we denote its volume element by $d\sigma_x^{k-1}$.

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Set

$$\tau(F,\ x)=(c_{m+k-1})^{-1}\int_{N_F^1(x)}|G(e)|d\sigma_x^{k-1}(e)$$

We call $\tau(F, x)$ the *absolute curvature* of F at x.

We now assume that M is orientable and let dV be the volume element of M with respect to the metric induced from E^{m+k} by F. Then there is a differential form ω of degree $k-1$ on N_F^1 such that its restriction to $N_F^1(x)$ is equal to $d\sigma_x^{k-1}$ and $d\mu=\omega\wedge dV$. Thus it holds that

$$\tau(F)=\int_M \tau(F,\ x)\,dV(x)$$

When M is non-orientable, there is a double covering of M, $\bar{\pi}: \bar{M}\rightarrow M$ such that M is orientable. Then we have $\tau(F\cdot\bar{\pi})=2\tau(F)$.

We now consider a stereographic projection on S^n. Let $p\in S^n$ and π_p : $S^n-\{p\}\rightarrow T_{-p}S^n$ be the stereographic projection from p. Then for any q in $S^n-\{p\}$, $\pi_p(q)=p+k(q-p)$ where $k=\frac{2}{1-\langle q,\ p\rangle}$. Let $(\pi_p)_*: T_qS^n\rightarrow E^n$ be the differential of π_p at q. Then we have the following:

LEMMA 1. *Let $q\in S^n-\{p\}$ and $X\in T_qS^n$. Then we have*

$$(\pi_p)_*(X)=kX+\frac{k^2}{2}\langle X,\ p\rangle(q-p)$$

and

$$\|(\pi_p)_*(X)\|=k\|X\|$$

that is, π_p is a conformal mapping with the scale function k.

Proof. Let $\sigma(t)$ be a C^1-curve in S^n with $\sigma(0)=q$ and $\sigma'(0)=X$. Let $k(t):=k(\sigma(t))=\frac{2}{1-\langle\sigma(t),\ p\rangle}$. Since $(\pi_p)_*(X)=(\pi_p\cdot\sigma)'(0)$,

$$\begin{aligned}(\pi_p)_*(X)&=\frac{d}{dt}(p+k(t)(\sigma(t)-p))|_{t=0}\\ &=k\sigma'(0)+k'(0)(q-p)\\ &=kX+\frac{k^2}{2}\langle X,\ p\rangle(q-p)\end{aligned}$$

Since $\langle q-p,\ q-p\rangle=2(1-\langle q,\ p\rangle)=\frac{4}{k}$ and $\langle q,\ X\rangle=0$,

$$\begin{aligned}\langle(\pi_p)_*(X),\ (\pi_p)_*(X)\rangle&=k^2\langle X,\ X\rangle-k^3\langle X,\ p\rangle^2+\frac{1}{4}k^4\langle X,\ p\rangle^2\langle q-p,\ q-p\rangle\\ &=k^2\langle X,\ X\rangle.\end{aligned}$$

q. e. d.

2. Main theorem.

We now state the main theorem of this paper.

THEOREM. *Let M be an m-dimensional compact differentiable manifold and f be an immersion of M into S^{m+k}. Let $\imath$ be the inclusion of S^{m+k} into E^{m+k+1}. Then*

$$\tau(\imath\cdot f)=(c_{m+k})^{-1}\int_{S^{m+k}}\tau(\pi_p\cdot f)\,d\sigma^{m+k}(p)$$

where $d\sigma^{m+k}$ is the standard volume element of S^{m+k}.

Note: The function $p\mapsto\tau(\pi_p\cdot f)$ is defined almost everywhere in S^{m+k} (in fact it is defined in $S^{m+k}-f(M)$) and continuous there.

We first consider the absolute curvature of $\imath\cdot f$. Let $N_f^1(x)$ be the unit normal vector space of f at $x\in M$. Let $e\in N_f^1(x)$ and $a_\imath(e)$ $(1\leqq \imath\leqq m)$ be a system of principal curvatures of f with respect to e. Let $X_\imath(e)$ be (unit) principal vectors of f with respect to e with the principal curvatures $a_i(e)$ respectively. For convenience, we use the same notation x for the image of x by f, $f(x)$. If $\bar{e}$ is a unit normal vector of $\imath\cdot f$ at x, $\bar{e}=x\sin\rho+e\cos\rho$ for some $e\in N_f^1(x)$ and $\rho\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

LEMMA 2. *$X_\imath(e)$ are the principal vectors of $i\cdot f$ with respect to $\bar{e}$ with the principal curvatures $\sin\rho+a_i(e)\cos\rho$ respectively.*

Proof. Let S_e be the second fundamental tensor of f with respect to e and $\bar{S}_{\bar{e}}$ the one of $\imath\cdot f$ with respect to $\bar{e}$. Since $\bar{S}_e(X_i(e))=S_e(X_i(e))=a_\imath(e)X_i(e)$,

$$\bar{S}_{\bar{e}}(X_\imath(e))=\sin\rho\,\bar{S}_x(X_i(e))+\cos\rho\,\bar{S}_e(X_\imath(e))$$
$$=(\sin\rho+a_i(e)\cos\rho)X_\imath(e)$$

q. e. d.

Let $\bar{G}(\bar{e})$ be the Lipschitz-Killing curvature of $i\cdot f$ with respect to $\bar{e}$. Then we have $\bar{G}(\bar{e})=\prod_{\imath=1}^{m}(\sin\rho+a_i(e)\cos\rho)$. Hence, denoting by $d\sigma^l$ the standard volume element of S^l,

$$\tau(\imath\cdot f,\ x)=(c_{m+k})^{-1}\int_{N^1_{\imath\cdot f}(x)}|G(e)|\,d\sigma^k(\bar{e})$$

$$=(c_{m+k})^{-1}\int_{-\pi/2}^{\pi/2}\int_{N_f^1(x)}\prod_{\imath=1}^{m}|\sin\rho+a_\imath(e)\cos\rho|\cos^{k-1}\rho\,d\rho\wedge d\sigma^{k-1}(e) \quad \cdots\cdots(1)$$

Now we consider the absolute curvature of $\pi_p \cdot f$. For $p \in S^{m+k} - f(M)$, define a mapping $\lambda_p : N_f^1(x) \to N_{\pi_p \cdot f}^1(x)$ by $\lambda_p(e) := 1/k(\pi_p)_*(e)$ $(e \in N_f^1(x))$. Let $\tilde{e} := \lambda_p(e)$ and let $\tilde{S}_p(\tilde{e})$ be the second fundamental tensor of $\pi_p \cdot f$ with respect to $\tilde{e}$. Then we have the following lemma:

LEMMA 3. *$1/k(X_i(e))$ are the principal vectors of $\pi_p \cdot f$ with respect to $\tilde{e} = \lambda_p(e)$ with the principal curvatures $1/k(a_i(e) + k/2\langle e, p\rangle)$ respectively.*

Proof. In this proof we identify M with its image by f. Let $\alpha(t)$ be a C^1-curve in M with $\alpha(0) = x$ and $\alpha'(0) = X_i(e)$. Take $e(t) = e(\alpha(t))$, a normal vector field of f along $\alpha(t)$, such that $e(0) = e$. Let $(\quad)^T : T_{\pi_p \cdot f(x)} E^{m+k} \to T_x M$ be the tangential projection of $\pi_p \cdot f$. Since

$$\tilde{S}_p(\tilde{e})\Big(\frac{1}{k} X_i(e)\Big) = \frac{1}{k}\Big(\frac{d}{dt} \tilde{e}(t)|_{t=0}\Big)^T$$

where $\tilde{e}(t) = \lambda_{\alpha(t)} e(t)$, denoting

$$k(\alpha(t)) = \frac{2}{1 - \langle \alpha(t), p\rangle} \qquad \text{by} \quad k(t),$$

$$\begin{aligned}\tilde{S}_p(\tilde{e})\Big(\frac{1}{k} X_i(e)\Big) &= \frac{1}{k}\Big(\frac{d}{dt}\Big(e(t) + \frac{k(t)}{2}\langle e(t), p\rangle(\alpha(t) - p)\Big)\Big|_{t=0}\Big)^T \\ &= \frac{1}{k}\Big(e'(0) + \frac{k'(0)}{2}\langle e, p\rangle(x-p) + \frac{k}{2}\langle e'(0), p\rangle(x-p) \\ &\qquad + \frac{k}{2}\langle e, p\rangle X_i(e)\Big)^T\end{aligned}$$

Since $((\pi_p)_*(e'(0)))^T = S_e(X_i(e)) = a_i(e) X_i(e)$,

$$\begin{aligned}\tilde{S}_p(\tilde{e})\Big(\frac{1}{k} X_i(e)\Big) &= \frac{1}{k}\Big(\frac{1}{k}(\pi_p)_*(e'(0)) \\ &\qquad + \frac{k}{2}\langle e, p\rangle(\pi_p)_*\Big(\frac{1}{k} X_i(e)\Big)\Big)^T \\ &= \frac{1}{k}\Big(a_i(e) + \frac{k}{2}\langle e, p\rangle\Big)\Big(\frac{1}{k} X_i(e)\Big)\end{aligned}$$

q. e. d.

Let $\tilde{G}_p(\tilde{e})$ be the Lipschitz-Killing curvature of $\pi_p \cdot f$ with respect to $\tilde{e} = \lambda_p(e) \in N_{\pi_p \cdot f}^1(x)$. Since $\tilde{G}_p(\tilde{e}) = \prod_{i=1}^m (a_i(e) + k/2\langle e, p\rangle)$ and λ_p is an isometry of $N_f^1(x)$ into $N_{\pi_p \cdot f}^1(x)$,

$$\tau(\pi_p\cdot f,\ x)=(c_{m+k-1})^{-1}\int_{N^1_{\pi_p\cdot f}(x)}|\tilde{G}_p(\tilde{e})|d\sigma^{k-1}(\tilde{e})$$

$$=(c_{m+k-1})^{-1}\int_{N^1_f(x)}\frac{1}{k^m}\prod_{i=1}^{m}\left|a_i(e)+\frac{k}{2}\langle e,\ p\rangle\right|d\sigma^{k-1}(e) \qquad \cdots\cdots(2)$$

Proof of Theorem.

We now assume that M is orientable. When M is non-orientable, the proof of the theorem can be reduced to the orientable case through the double covering of M.

Let dV be the volume element of M with respect to the metric induced from E^{m+k+1} by $\iota\cdot f$ and dV_p be the one with respect to the metric induced from E^{m+k} by $\pi_p\cdot f$. Then we have $dV_p=k^m dV$. Since

$$\tau(\iota\cdot f)=\int_M\tau(\iota\cdot f,\ x)\,dV(x) \quad \text{and} \quad \tau(\pi_p\cdot f)=\int_M\tau(\pi_p\cdot f,\ x)\,dV_p(x),$$

by (1) and (2), it is sufficient for the proof of the theorem to show the following equality (3) for any $x\in M$ and $e\in N^1_f(x)$:

$$\int_{-\pi/2}^{\pi/2}\prod_{i=1}^{m}|\sin\rho+a_i(e)\cos\rho|\cos^{k-1}\rho\,d\rho$$

$$=(c_{m+k-1})^{-1}\int_{S^{m+k}}\prod_{i=1}^{m}\left|a_i(e)+\frac{k}{2}\langle e,\ p\rangle\right|d\sigma^{m+k}(p) \qquad \cdots\cdots(3)$$

We parameterize S^{m+k} by $p=(\sin^2\theta_1-\cos^2\theta_1\sin\theta_2,\ \sin\theta_1\cos\theta_1(1+\sin\theta_2),\ \cos\theta_1\cos\theta_2\sin\theta_3,\ \cos\theta_1\cos\theta_2\cos\theta_3\sin\theta_4,\ \cdots\cdots,\ \cos\theta_1\cos\theta_2\cdots\cdots\cos\theta_{m+k-1}\sin\theta_{m+k},\ \cos\theta_1\cos\theta_2\cdots\cdots\cos\theta_{m+k-1}\cos_{m+k})$ $(-\pi/2\leqq\theta_1,\ \cdots\cdots,\ \theta_{m+k-1}\leqq\pi/2,\ -\pi\leqq\theta_{m+k}\leqq\pi)$.

The volume element of S^{m+k} in terms of this coordinate system is

$$d\sigma^{m+k}(p)=\cos^{m+k-1}\theta_1(1+\sin\theta_2)\cos^{m+k-2}\theta_2\cos^{m+k-3}\theta_3\cdots\cdots$$
$$\cdots\cdots\cos\theta_{m+k-1}\,d\theta_1\wedge d\theta_2\cdots\cdots\wedge d\theta_{m+k-1}\wedge d\theta_{m+k}$$

We may assume $x=(1,\ 0,\ \cdots\cdots,\ 0)$ and $e=(0,\ 1,\ 0,\ \cdots\cdots,\ 0)$. Then $k/2\langle e,\ p\rangle=\tan\theta_1$. Hence

$$\int_{S^{m+k}}\prod_{i=1}^{m}\left|a_i(e)+\frac{k}{2}\langle e,\ p\rangle\right|d\sigma^{m+k}(p)$$

$$=\int_{-\pi/2}^{\pi/2}\cdots\cdots\int_{-\pi/2}^{\pi/2}\int_{-\pi}^{\pi}\prod_{i=1}^{m}|a_i(e)+\tan\theta_1|\cos^{m+k-1}\theta_1(1+\sin\theta_2)$$
$$\times\cos^{m+k-2}\theta_2\cos^{m+k-3}\theta_3\cdots\cdots\cos\theta_{m+k-1}\,d\theta_1\,d\theta_2\cdots\cdots d\theta_{m+k}$$

$$= c_{m+k-1}\int_{-\pi/2}^{\pi/2}\prod_{i=1}^{m}|a_i(e)+\tan\theta_1|\cos^{m+k-1}\theta_1\,d\theta_1$$

$$= c_{m+k-1}\int_{-\pi/2}^{\pi/2}\prod_{i=1}^{m}|\sin\theta_1+a_i(e)\cos\theta_1|\cos^{k-1}\theta_1\,d\theta_1$$

Thus the equality (3) is shown and the proof of the theorem is completed.

REFERENCES

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