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ON THE LINEAR TRANSFORMATIONS OF AHLFORS FUNCTIONS

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1. Introduction.

Let R be a finite open Riemann surface of genus p with n boundary components Γ_j $(j=0, \cdots, n-1)$. For two distinct fixed points a, b in R, it is known that there exist the extremal functions which maximize $|f(b)|$ in the family

$$\{f;\ f\in A(R),\ f(a)=0,\ |f|\leqq 1 \text{ on } R\}$$

and they are unique up to a constant multiple of absolute value 1. The same assertion holds when for a fixed local coordinate we take $|f'(a)|$ instead of $|f(b)|$. They are called Ahlfors functions at a and we denote any of them by f_{ab}, f_a respectively. In his lecture note, J. D. Fay [3] has given an explicit representation of the Ahlfors functions of a planar domain by means of theta function. In the present paper, using Fay's result, we obtain a necessary and sufficient condition for the linear transformation of f_{ab}

$$T_\alpha(f_{ab})(z)=\frac{f_{ab}(z)-f_{ab}(\alpha)}{1-\overline{f_{ab}(\alpha)}\cdot f_{ab}(z)}, \qquad \alpha\in R$$

to be $f_{\alpha\beta}$ $(\beta\in R)$ when R is planar. Let $Q(R)$ denote the set of points a in R such that $T_\alpha(f_a)=f_\alpha$ for some $\alpha\in R$ distinct from a. As a corollary, if $Q(R)$ is not empty, then the double $\hat{R}$ of R is hyperelliptic and $Q(R)$ is the union of n mutually disjoint analytic simple curves in R. However in the non-planar case the situation changes. In section 4 we shall show an example of the bordered surface whose $Q(R)$ has non-empty interior. Prof. Suita pointed out to the author that this fact enables us to give a negative answer for the real analyticity of the analytic capacities of non-planar surfaces. But they are real analytic for every plane region $\notin O_{AB}$ [4]. These phenomena show interesting contrasts between non-planar and planar cases. The author wishes to express his sincere thanks to Prof. Suita for his valuable advice and remarks.

2. Notation

Fay's notation and definitions [3] will be used in this paper. Let ϕ be the

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canonical anti-conformal involution of $\hat{R}$, and fix a symmetric canonical homology basis on $\hat{R}$:

$$A_1, B_1, \cdots, A_\rho, B_\rho, A_{\rho+1}, B_{\rho+1}, \cdots, A_{\rho+n-1}, B_{\rho+n-1}, A_{1'}, B_{1'}, \cdots, A_{\rho'}, B_{\rho'},$$

such that $A_{\rho+k}=\Gamma_k$ for $k=1, \cdots, n-1$, and $A_1, B_1, \cdots, A_\rho, B_\rho$ (resp. $A_{1'}, B_{1'}, \cdots, A_{\rho'}, B_{\rho'}$) are cycles in R (resp. $\phi(R)$) satisfying the relations in $H_1(\hat{R}, \boldsymbol{Z})$:

$$\phi(A_i)=A_{i'}, \quad \phi(B_i)=-B_{i'}, \quad 1\leqq i\leqq \rho$$

$$\phi(A_i)=A_i, \quad \phi(B_i)=-B_i, \quad \rho+1\leqq i\leqq \rho+n-1.$$

Let $u_1, \cdots, u_g$ $(g=2\rho+n-1)$ be a basis of the holomorphic differentials on $\hat{R}$ normalized so that the period matrix with respect to the symmetric canonical homology basis has the form $(2\pi iI, \tau)$ where $I=$the identity matrix and τ is a symmetric matrix with $\operatorname{Re}\tau\leqq 0$. Let ω_{b-a} be the normalized differential of the third kind on $\hat{R}$ with poles at b of residue 1 and at a of residue -1, $E(x, y)$ the prime form, $\theta(z)=\sum_{m\in \boldsymbol{Z}^g}\exp\left(\frac{1}{2}{}^t m\tau m+{}^t m\cdot z\right)$ Riemann's theta function. Then the following symmetries hold [3, Chap. 6]: for $x, y\in\hat{R}$, $z\in\boldsymbol{C}^g$

$$\omega_{b-a}(x)=\overline{\omega_{b-a}(\bar{x})}, \qquad \theta(z)=\overline{\theta(\phi(z))}, \qquad E(x, y)=\overline{E(\bar{x}, \bar{y})}, \tag{1}$$

where $\bar{x}=\phi(x)$ the conjugate point of x, and for $z_1, \cdots, z_{\rho+n-1}, z_{1'}, \cdots, z_{\rho'}\in\boldsymbol{C}$

$$\phi(z_1, \cdots, z_\rho, z_{\rho+1}, \cdots, z_{\rho+n-1}, z_{1'}, \cdots, z_{\rho'})$$
$$=-(\bar{z}_{1'}, \cdots, \bar{z}_{\rho'}, \bar{z}_{\rho+1}, \cdots, \bar{z}_{\rho+n-1}, \bar{z}_1, \cdots, \bar{z}_\rho).$$

We denote by $D(f)$ the divisor of a meromorphic function f.

3. Linear transformations of Ahlfors functions.

Let us write $x-y=\int_y^x u=\left(\int_y^x u_j\right)_{j=1,\cdots,g}$ for the sake of simplicity. When R is planar $(\rho=0)$, the Ahlfors function can be represented by the theta function and the prime form:

$$f_{ab}(x)=\varepsilon\cdot\frac{\theta(x-\bar{a}-s)E(x, a)}{\theta(x-a-s)E(x, \bar{a})}, \quad |\varepsilon|=1, \quad s=\frac{1}{2}\int_{a+\bar{a}}^{b+\bar{b}} u\in\boldsymbol{C}^g, \tag{2}$$

where the paths of integration of s must be chosen to be symmetric with respect to ∂R so that s becomes a purely imaginary vector [3, Prop. 6.17].

THEOREM 1. *Let R be a plane regular region of finite connectivity. Then the following conditions are equivalent.*

(i) $T_\alpha(f_{ab})=f_{\alpha\beta}$,

(ii) $\frac{1}{2}(a+\bar{a}+b+\bar{b})=\frac{1}{2}(\alpha+\bar{\alpha}+\beta+\bar{\beta})$ *in* $J_0=\boldsymbol{C}^g/(2\pi iI, \tau)$,

(iii) *there exists a meromorphic function f on R satisfying $f(x)>0$ on ∂R and $D(f)=\alpha+\beta-a-b$.*

Proof. Clearly we may assume $\varepsilon=1$ in (2) to calculate $T_\alpha(f_{ab})$. From the symmetries (1),

$$T_\alpha(f_{ab})(x)=\frac{\dfrac{\theta(x-\bar a-s)E(x,a)}{\theta(x-a-s)E(x,\bar a)}-\dfrac{\theta(\alpha-\bar a-s)E(\alpha,a)}{\theta(\alpha-a-s)E(\alpha,\bar a)}}{1-\dfrac{\overline{\theta(\alpha-\bar a-s)E(\alpha,a)}}{\overline{\theta(\alpha-a-s)E(\alpha,\bar a)}}\,\dfrac{\theta(x-\bar a-s)E(x,a)}{\theta(x-a-s)E(x,\bar a)}}$$

$$=\eta\cdot\frac{\theta(x-\bar a-s)\theta(\alpha-a-s)E(x,a)E(\alpha,\bar a)-\theta(x-a-s)\theta(\alpha-\bar a-s)E(x,\bar a)E(\alpha,a)}{\theta(x-a-s)\theta(\bar\alpha-\bar a-s)E(x,\bar a)E(\bar\alpha,a)-\theta(x-\bar a-s)\theta(\bar\alpha-a-s)E(x,a)E(\bar\alpha,\bar a)}$$

where

$$\eta=\frac{\overline{\theta(\alpha-a-s)E(\alpha,\bar a)}}{\theta(\alpha-a-s)E(\alpha,\bar a)},\qquad |\eta|=1.$$

This expression can be simplified by the addition-formula of the theta function [3]: for $x, y, a, b\in\hat R$ and $e\in \boldsymbol{C}^g$,

$$\theta(x-b-e)\theta(y-a-e)E(x,a)E(y,b)-\theta(x-a-e)\theta(y-b-e)E(x,b)E(y,a)$$

$$=\theta(e)\theta(x+y-a-b-e)E(x,y)E(a,b).$$

Disregarding a constant multiple of absolute value 1, we now have

$$T_\alpha(f_{ab})=\frac{\theta(x+\alpha-a-\bar a-s)E(x,\alpha)}{\theta(x+\bar\alpha-a-\bar a-s)E(x,\bar\alpha)}.$$

It is obvious that $T_\alpha(f_{ab})=f_{\alpha\beta}$ if and only if they have the same divisor on $\hat R$. By using Riemann's vanishing theorem [3, Th. 1.1] this condition can be expressed as

$$s+a+\bar a-\alpha-\bar\alpha=\frac{1}{2}\int_{\alpha+\bar\alpha}^{\beta+\bar\beta}u \qquad \text{in } J_0$$

or

$$\frac{1}{2}(a+\bar a+b+\bar b)=\frac{1}{2}(\alpha+\bar\alpha+\beta+\bar\beta) \qquad \text{in } J_0.$$

Thus (i) and (ii) are equivalent. On the other hand the equivalence of (ii) and (iii) is a consequence of the next lemma which is a variant of Abel's theorem.

LEMMA 1. *Let R be planar. Then*

$$\frac{1}{2}(a+\bar a+b+\bar b)=\frac{1}{2}(\alpha+\bar\alpha+\beta+\bar\beta) \qquad \textit{in } J_0$$

if and only if there exists a meromorphic function f on R with $D(f)=a+b-\alpha-\beta$ satisfying $f>0$ on ∂R.

Proof. Let us introduce the notation $\left\{\begin{matrix}\delta\\ \varepsilon\end{matrix}\right\}_\tau=2\pi i\varepsilon+\tau\delta$ for $\varepsilon, \delta\in \boldsymbol{R}^g$. Note that for any $x, y\in\hat{R}$, $x+\bar{x}-y-\bar{y}$ has the form $\left\{\begin{matrix}0\\ \varepsilon\end{matrix}\right\}_\tau (\varepsilon\in\boldsymbol{R}^g)$ if we take symmetric paths of integration. Thus the condition $1/2(a+\bar{a}+b+\bar{b})=1/2(\alpha+\bar{\alpha}+\beta+\bar{\beta})$ in J_0 is equivalent to

$$a+\bar{a}+b+\bar{b}=\alpha+\bar{\alpha}+\beta+\bar{\beta}+2\left\{\begin{matrix}0 & \cdots & 0\\ \nu_1 & \cdots & \nu_g\end{matrix}\right\}_\tau \quad \text{in } \boldsymbol{C}^g,\ \nu_j\in\boldsymbol{Z}\ (j=1,\cdots,g).$$

Assume that this is true, then by Abel's theorem there exists a meromorphic function f on $\hat{R}$ with divisor $D=a+\bar{a}+b+\bar{b}-\alpha-\bar{\alpha}-\beta-\bar{\beta}$. In fact for any fixed $p_0\in\Gamma_0$ the function

$$f(p)=\exp\int_{p_0}^{p}\omega_D\,,\qquad p\in\hat{R}$$

has the divisor D where $\omega_D=\omega_{a-\alpha}+\omega_{\bar{a}-\bar{\alpha}}+\omega_{b-\beta}+\omega_{\bar{b}-\bar{\beta}}$. We must examine the argument of f along ∂R. If $p_j\in\Gamma_j$ $(j=1,\cdots,n-1)$, using the symmetries (1) and Riemann's bilinear relation, we have

$$\arg f(p_j)=\operatorname{Im}\int_{p_0}^{p_j}\omega_D=\frac{1}{2i}\left(\int_{-C_j}\omega_D-\int_{-C_j}\bar{\omega}_D\right)=\frac{-1}{2i}\left(\int_{C_j}\omega_D-\int_{\phi(C_j)}\omega_D\right)$$

$$=-\frac{1}{2i}\int_{B_j}\omega_D=-\frac{1}{2i}\int_D u_j=-2\nu_j\pi\,,\quad \nu_j\in\boldsymbol{Z}\,,$$

where C_j is a smooth curve in R connecting $p_j\in\Gamma_j$ with $p_0\in\Gamma_0$. Thus $f(p_j)>0$. Similarly, $f(p)>0$ for $p\in\Gamma_0$, hence we conclude $f>0$ on ∂R.

Next assume that f is a meromorphic function on R with $D(f)=a+b-\alpha-\beta$ satisfying $f>0$ on ∂R. Then by reflection with respect to ∂R, f can be extended to $\tilde{f}$ which is meromorphic on $\hat{R}$ with $D(\tilde{f})=a+\bar{a}+b+\bar{b}-\alpha-\bar{\alpha}-\beta-\bar{\beta}$. If $B-A$ $(A, B>0)$ is the divisor of a meromorphic function F on $\hat{R}$, by Abel's theorem we have

$$\int_A^B u=\left\{\begin{matrix}m_1 & \cdots & m_g\\ n_1 & \cdots & n_g\end{matrix}\right\}_\tau$$

where $m_j=-\dfrac{1}{2\pi i}\displaystyle\int_{A_j}d\log F\in\boldsymbol{Z}$ and $n_j=\dfrac{1}{2\pi i}\displaystyle\int_{B_j}d\log F\in\boldsymbol{Z}$ $(j=1,\cdots,g)$. Therefore $m_j=0$ for $D(\tilde{f})$ since $\tilde{f}>0$ on ∂R. On the other hand we have

$$n_j=\frac{1}{2\pi i}\int_{B_j}d\log\tilde{f}=\frac{1}{2\pi i}\int_{C_j}d\log\tilde{f}+\frac{1}{2\pi i}\int_{-\phi(C_j)}d\log\tilde{f}$$

$$=\frac{1}{2\pi i}\int_{C_j}d\log\tilde{f}-\frac{1}{2\pi i}\int_{C_j}\overline{d\log\tilde{f}}=\frac{1}{\pi}\operatorname{Im}\int_{C_j}d\log\tilde{f}$$

$$=\frac{1}{\pi}\cdot\arg(f(p_0)/f(p_j))=2\nu_j\,,\qquad \nu_j\in\boldsymbol{Z}\qquad\text{q. e. d.}$$

For example if R is an annulus with center at the origin, the condition of the lemma reduces to the one that $\arg\alpha\beta=\arg ab$.

Letting $b\to a$, $\beta\to\alpha$ in Theorem 1, we easily obtain

COROLLARY. *$T_\alpha(f_a)=f_\alpha$ if and only if $a+\bar{a}=\alpha+\bar{\alpha}$ in J_0. Moreover assume that the genus of $R\geqq 2$. Then $T_\alpha(f_a)=f_\alpha$ if and only if $\hat{R}$ is hyperelliptic and $\phi(a)=J(a)$, $\phi(\alpha)=J(\alpha)$ where J is the hyperelliptic involution of $\hat{R}$.*

It is natural from this corollary that the next problem is to determine, when $\hat{R}$ is hyperelliptic, the locus of $\phi=J$, or as is the same, the set of fixed points of $\phi\circ J$. Let h be a non-trivial automorphism of $\hat{R}$ and H be the set of fixed points of h.

LEMMA 2. *If $h\circ\phi=\phi\circ h$, then either $H\subset\partial R$ or $H\cap\partial R=\phi$ (empty set). In the second case, for any $p\in H\cap R$*

$$H\cap R\subset\{x\in R\,;\,f_p(x)=0\}\,.$$

Proof. If $H\subset\partial R$, there is nothing to prove. Thus we may assume $H\backslash\partial R\neq\phi$. It is then easily verified from the hypothesis of the lemma that $h(R)=R$ and h is an automorphism of R. Fix $p\in H\cap R$ and consider the function $f_p\circ h$.

$$f_p\circ h(p)=0\,,\qquad |f_p\circ h|\leqq 1\qquad\text{on } R$$

For a suitable choice of the local coordinate z centered at p, h has the form $h(z)=\varepsilon z$ ($\varepsilon^N=1$, $\varepsilon\neq 1$) in a neighborhood of p. Taking the derivative of $f_p\circ h$ at p, we have $(f_p\circ h)'(p)=\varepsilon f_p'(p)$. Hence $f_p\circ h$ is an Ahlfors function at p and $f_p\circ h(x)=\varepsilon f_p(x)$ for all x in $R\cup\partial R$, since Ahlfors functions are unique. Suppose $x\in H\cap(R\cup\partial R)$, then $f_p(x)=\varepsilon f_p(x)$, so that $f_p(x)=0$. This means that $x\in R$ which gives $H\cap\partial R=\phi$. q. e. d.

Let $\hat{R}$ be hyperelliptic and $h=J$ its hyperelliptic involution. Since $J\circ\phi=\phi\circ J$, it follows from lemma 2 that $W\subset\partial R$ or $W\cap\partial R=\phi$, where W is the set of all Weierstrass points. More precisely we have

THEOREM 2. *Let the double $\hat{R}$ of R be hyperelliptic of genus g.*

(i) *Assume $\rho=0$: W is contained in ∂R and there are two Weierstrass points on each Γ_j ($j=0,\cdots,n-1$). The locus of $\phi=J$ is a union of n mutually disjoint analytic simple closed curves in R passing through the Weierstrass points on ∂R.*

(ii) *Assume $\rho>0$: The number of the boundary components of R is one or two, and $W\cap\partial R=\phi$. The Ahlfors functions at the $g+1$ Weierstrass points in R are identical and have $g+1$ zeros at these Weierstrass points. The locus of $\phi=J$ is empty.*

Proof. First we consider the case where $W\subset\partial R$. Since $\hat{R}$ is hyperelliptic,

$\hat{R}$ has $2g+2$ Weierstrass points. From $2g+2 \geqq 2n$, we see that one of the boundary components of R, say Γ_0, contains at least two Weierstrass points. If $W_1, W_2 \in \Gamma_0$ are those Weierstrass points, then from the hypothesis there exists a meromorphic function $w(x)$ on $\hat{R}$ with $D(w)=2W_1-2W_2$. Comparison of the divisors gives

$$w(\phi(x))=\lambda\cdot\overline{w(x)}, \qquad \lambda\in\boldsymbol{C},\ |\lambda|=1,\ x\in\hat{R}\,.$$

For a suitable $\varepsilon\in\boldsymbol{C}$, $w_0(x)=\varepsilon\cdot w(x)$ satisfies

$$w_0(\phi(x))=\overline{w_0(x)}\,.$$

Therefore $w_0(\Gamma_j)$ $(j=0, \cdots, n-1)$ is a continuum in $\boldsymbol{R}\cup\{\infty\}$ so that a closed interval in $\boldsymbol{R}\cup\{\infty\}$. It is easy to see that each Γ_j covers $w_0(\Gamma_j)$ exactly twice. Since w_0 is of order 2, we have

$$w_0(\Gamma_j)\cap w_0(\Gamma_k)=\phi \qquad (j\neq k)\,.$$

From the argument principle w_0 maps R conformally onto a slit region $w_0(R)$ $=\boldsymbol{C}\setminus\bigcup_{j=0}^{n-1} w_0(\Gamma_j)$. Consequently ρ must be zero and on each Γ_j there are two Weierstrass points which correspond to the end points of $w_0(\Gamma_j)$. To determine the locus of $\phi=J$ we claim that

$$\{x\in\hat{R}\,;\ \phi(x)=J(x)\}=W\cup\{x\in\hat{R}\,;\ w_0(x)\in\boldsymbol{R},\ x\notin\partial R\}\,.$$

This follows easily from the functional equation $\overline{w_0(x)}=w_0(\phi\circ J(x))$. Now it is clear that the locus consists of n disjoint simple closed analytic curves passing through the Weierstrass points. We will then be finished with the proof of (i) if we can show $\rho>0$ whenever $W\cap\partial R=\phi$.

Suppose that $W\cap\partial R=\phi$. Let Q be a Weierstrass point in R. Then there exists, as in the proof of the first part, a mermorphic function w_1 on $\hat{R}$ with $D(w_1)=2Q-2\phi(Q)$ satisfying

$$w_1(\phi(x))\overline{w_1(x)}=1\,.$$

The restriction of w_1 to R is a unitary function on R having only a double zero at $Q\in R$. But on bordered surfaces every non-constant unitary holomorphic function has at least n zeros, where n is the number of the boundary components. Thus n must be one or two. The hyperellipticity of $\hat{R}$ implies that $g=2\rho+n-1$ $\geqq 2$. From $n\leqq 2$ we conclude that $\rho>0$. This completes the proof of (i) and the first part of (ii).

Next we proceed to prove the rest of (ii). Let $p\in W\cap R$. Then by Lemma 2 we have

$$W\cap R\subset\{x\in R\,;\ f_p(x)=0\}\,.$$

But it is known that the Ahlfors function has at most $g+1$ zeros [1]. Noting that $W \cap R$ consists of $g+1$ points, we have indeed

$$W \cap R = \{x \in R\,;\ f_p(x)=0\}\,.$$

Thus the Ahlfors functions at the Weierstrass points in R are identical with each other, since they have the same divisor. To complete the proof we have to show that the locus of $\phi=J$ is empty. Assume that $\phi(x)=J(x)$ for $x\in\hat{R}$. Then we obtain

$$1=w_1(\phi(x))\overline{w_1(x)}=w_1(J(x))\overline{w_1(x)}=|w_1(x)|^2\,.$$

Hence $x\in\partial R$ and so $x=J(x)$ or $x\in W$, contradicting the fact that $W\cap\partial R=\phi$.

q. e. d.

The proof of Theorem 2 contains the following: every regular plane region whose double is hyperelliptic is conformally equivalent to a region slit along a finite number of segments on a line. The converse is clearly also true.

4. Example.

Let D_1 and D_2 be two copies of a closed unit disk with slits along the segments $[s, t]$, $[-t, -s]$ $(0<s<t<1)$. We construct the desired finite bordered surface S by joining D_2 to D_1 along their common distinguished slits in the standard manner (i.e. the upper edge of such a slit of a given copy being joined to the lower edge of the corresponding slit of the other copy). The double $\hat{S}$ of S can be expressed explicitly by the algebraic equation

$$y^2=(x^2-s^2)(x^2-t^2)(x^{-2}-s^2)(x^{-2}-t^2)\,. \tag{3}$$

Thus $\hat{S}$ is hyperellitic of genus 3. It follows that $\phi(x, y)=((1/\bar{x}), \bar{y})$ is the canonical anti-conformal involution of $\hat{S}$ and that $S=\{(x, y)\in\hat{S}\,;\ |x|\leqq 1\}$. Denote by O_1, O_2 (resp. ∞_1, ∞_2) the points of $\hat{S}$ lying over the origin (resp. the point at infinity) in such a way that we have

$$x^2y=st+O(x^2),\quad x^2y=-st+O(x^2),\quad x^{-2}y=st+O(x^{-2}),\quad x^{-2}y=-st+O(x^{-2}) \tag{4}$$

near these points, respectively. In this section we shall show that the Ahlfors function for S at p in a sufficiently small neighborhood of O_1, O_2 has the form

$$\varepsilon\cdot\frac{x-p}{1-\bar{p}x}\,,\qquad |\varepsilon|=1\,.$$

Consequently, S is an example of the bordered surface whose $Q(S)$ has non-empty interior.

To show this we begin by studying the meromorphic differential ψ on S with $D(\psi)\geqq O_2-O_1+\infty_2-\infty_1$. Such differentials, by the Riemann-Roch theorem, form a 2-dimentional vector space. Indeed, easy calculation shows that its basis is given by

$$\frac{dx}{xy}, \quad \frac{\{y+st(x-x^{-1})^2\}\,dx}{xy}.$$

The former is holomorphic and the latter is meromorphic with simple poles at O_1, ∞_1.

LEMMA 3. *Let* $\phi=\dfrac{\{y+st(x-x^{-1})^2+\lambda\}\,dx}{ixy}$, $\lambda\in C$. *Then* ϕ *is a positive differential if and only if*

$$-(1-s^2)(1-t^2)\leqq\lambda\leqq(1-s^2)(1-t^2). \tag{5}$$

Proof. By definition the positiveness of ϕ means that

$$\frac{y+st(x-x^{-1})^2+\lambda}{y}\geqq 0 \qquad \text{for all } |x|=1.$$

Since y is real for $|x|=1$, λ must also be real. Since $\pm y$ correspond to a fixed x, we have

$$\pm\frac{st(x-x^{-1})^2+\lambda}{y}\geqq -1 \qquad \text{for all } |x|=1,$$

or

$$\{st(x-x^{-1})^2+\lambda\}^2\leqq y^2 \qquad \text{for all } |x|=1.$$

Set $r=x^2+x^{-2}$ for convenience, then from (3) we can replace the above inequality by

$$\{st(r-2)+\lambda\}^2\leqq(1+s^4-s^2r)(1+t^4-t^2r), \qquad -2\leqq r\leqq 2.$$

This inequality is linear with respect to r, and so setting $r=\pm 2$, we obtain

$$\lambda^2\leqq(1-s^2)^2(1-t^2)^2, \qquad (\lambda-4st)^2\leqq(1+s^2)^2(1+t^2)^2.$$

These inequalities yield (5). q. e. d.

Next let W be the family of all differentials ω holomorphic on $S\cup\partial S$ except at $a\in S$ where ω has a principal part

$$\frac{dz(x)}{(z(x)-z(a))^2}+\frac{\eta\cdot dz(x)}{z(x)-z(a)}, \qquad \eta\in \boldsymbol{C}$$

for a fixed local coordinate z around $a\in S$. Following Ahlfors [1], for any $a\in S$ there exists an extremal differential minimizing the expression $\int_{\partial S}|\omega|$ in W. It is in general not unique. But if we denote any of it by ω_a and set $\phi_a=f_a\omega_a$, then it is known to be characterized by $\arg\phi_a=$const. on ∂S. This characterization and the fact that $|f_a|=1$ on ∂S imply that f_a, ω_a, ϕ_a are extended meromorphically across ∂S to $\hat{S}$. Setting $D(f_a)=a+A_a$ on S we obtain

$$D(f_a)=a+A_a-\bar{a}-\bar{A}_a$$

$$D(\omega_a)=B_a-2a+2\bar{A}_a+\bar{B}_a \qquad \text{on } \hat{S} \tag{6}$$

$$D(\phi_a)=A_a+B_a-a+\bar{A}_a+\bar{B}_a-\bar{a}$$

where B_a is a positive divisor satisfying $a\notin\{B_a\}\subset S\cup\partial S$. Conversely, if f is a unitary function on $\hat{S}$ and ϕ a positive differential on $\hat{S}$ with $D(f)$ and $D(\phi)$ having the form (6) for some divisors A and B, then f is an Ahlfors function at a. Since the degree of a meromorphic differential is $2g-2$, we have

$$\deg A_a+\deg B_a=g\,. \tag{7}$$

Set $N(a)=1+\deg A_a$ (the number of the zeros of f_a) and call any extremal differential ω_a a *conjugate differential* of f_a.

LEMMA 4. *If there exists a conjugate differential ω_a having no zeros on ∂S, then $N(p)=N(a)$ for p in a neighborhood of a.*

Proof. It is easily verified from Hurwitz's theorem and a normal family argument that $N(p)$ is lower semi-continuous, i.e.,

$$N(a)\leqq N(p)\,, \qquad p\in U_1 \tag{8}$$

in some neighborhood U_1 of a. We shall show under the hypothesis of Lemma 4 that $N(p)$ is also upper semi-continuous. To this end we proceed as follows.

Since ω_a is a Schottky differential with only a double pole at a, the residue of ω_a at a vanishes. Therefore if we set

$$\omega_p(x)=\pi L(x, p)+\Omega_p(x)$$

for $x, p\in\hat{S}$ where $L(x, p)$ is the adjoint Bergman kernel [2] and $\Omega_p(x)$ is a holomorphic differential on $\hat{S}$, then we obtain

$$\frac{1}{2\pi}\int_{t\in\partial S}|\Omega_p(t)|\leqq\frac{1}{2\pi}\int_{t\in\partial S}|\omega_p(t)|+\frac{1}{2}\int_{t\in\partial S}|L(t, p)|$$

$$=|f_p'(p)|+\frac{1}{2}\int_{t\in\partial S}|L(t, p)|.$$

Note that the last equality holds by the duality relation between f_p and ω_p [1]. It is clear that the last expression is bounded for $p\in U_1$, so that

$$\frac{1}{2\pi}\int_{t\in\partial S}|\Omega_p(t)|\leqq K\,, \qquad \text{for } p\in U_1$$

with finite K. Let U_2 be a sufficiently small neighborhood of a satisfying $\bar{U}_2\subset U_1\backslash\{B_a\}$. This is possible because $a\notin\{B_a\}$. Fix a nowhere-vanishing holomorphic differential $\xi(x)$ on $S\cup\partial S$. Then clearly $\dfrac{L(x, p)}{\xi(x)}$ is locally bounded in $S\backslash\bar{U}_2$

for $p\in U_2$. On the other hand, we obtain from Green's formula

$$\left|\frac{\Omega_p(x)}{\xi(x)}\right|=\frac{1}{2\pi}\left|\int_{t\in\partial S}\frac{\Omega_p(t)}{\xi(t)}dG_x(t)\right|\leqq\frac{1}{2\pi}\operatorname*{Max}_{t\in\partial S}\left|\frac{dG_x(t)}{\xi(t)}\right|\cdot\int_{t\in\partial S}|\Omega_p(t)|$$

$$\leqq K\cdot\operatorname*{Max}_{t\in\partial S}\left|\frac{dG_x(t)}{\xi(t)}\right|,\qquad p\in U_2,$$

where $dG_x(p)=dg(p, x)+i^*dg(p, x)$, g being the Green's function of S. Thus the family $\{\omega_p(x)/\xi(x)\}_{p\in U_2}$ is locally bounded in $S\backslash\bar{U}_2$, so that it is normal there. Since $\{B_a\}\subset S\backslash\bar{U}_2$, a similar reasoning as in the case of Ahlfors functions gives that $\deg B_a$ is also lower semi-continuous at a, i.e., there is a neighborhood U $(\subset U_2)$ of a such that

(9) $$\deg B_a\leqq\deg B_p,\qquad p\in U.$$

From (7), (8), (9) one obtains the desired result. q.e.d.

Now it is almost clear how to prove the before mentioned assertion concerning our example. By Lemma 3 and the remark stated below (6), x is an Ahlfors function at O_1 since $D(x)=O_1+O_2-\infty_1-\infty_2$ in $\hat{S}$. Thus $N(O_1)=2$. But the proof of Lemma 3 shows that $\phi>0$ on ∂S if $-(1-s^2)(1-t^2)<\lambda<(1-s^2)(1-t^2)$. Hence by Lemma 4 $N(p)=2$ in some neighborhood U of O_1, so that f_p $(p\in U)$ is a fractional linear transformation of x. Since $|f_p|=1$ on ∂S, f_p must have the form

$$\varepsilon\cdot\frac{x-p}{1-\bar{p}x},\qquad |\varepsilon|=1,\ p\in U.$$

By symmetry the same assertion holds for O_2.

It is of some interest to see that at $\pm s$, $\pm t$, the Weierstrass points in S, the Ahlfors functions are given by

$$\varepsilon\cdot\frac{y}{x^2(x^{-2}-s^2)(x^{-2}-t^2)}=\varepsilon\cdot\sqrt{\frac{(x^2-s^2)(x^2-t^2)}{(1-s^2x^2)(1-t^2x^2)}},\qquad |\varepsilon|=1.$$

This is a direct consequence of part (ii) of Theorem 2. Thus $N(p)=4$ in some neighborhoods of $\pm s$, $\pm t$. Furthermore, a closer examination using the method of this paper will show that $N(p)=4$ also in a neighborhood of ∂S.

Finally we remark that the analytic capacity $C_B(z)$ is not real analytic on S. The Gaussian curvature of C_B is given by

$$-\frac{1}{C_B{}^2}\cdot\frac{\partial^2\log C_B(z)}{\partial z\partial\bar{z}}$$

and is constant -1 in some neighborhood of $O_1\in S$ since there we have

$$C_B(z)=\frac{|f_{o_1}{}'(z)|\,|dz|}{1-|f_{o_1}(z)|^2}.$$

If C_B is real analytic on S, then the curvature is -1 everywhere on S by analytic continuation. However, when compaired with the Poincaré metric by using Ahlfors' version of Schwarz's lemma, this gives rise to a contradiction.

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