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KODAI MATH. J.
1 (1978), 85—88

ON A SINGULAR PERTURBATION PROBLEM FOR LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS, I

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1. In a paper [1], we had to treat the following singular perturbation problem.

Let us consider two linear systems of ordinary differential equations containing a small positive parameter ε:

$$\varepsilon \frac{d\boldsymbol{u}}{dt} = (A_0 - \varepsilon A_1)\boldsymbol{u} + \boldsymbol{\delta}_1(\varepsilon)\,, \tag{1}$$

$$\varepsilon \frac{d\boldsymbol{u}}{dt} = (A_0 + \varepsilon A_1)\boldsymbol{u} + \boldsymbol{\delta}_2(\varepsilon)\,, \tag{2}$$

where

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & -\beta & \gamma \\ -\alpha & -\gamma & \beta \end{pmatrix}, \quad A_1 = \begin{pmatrix} \alpha & \alpha & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and α, β, γ are positive constants such that $\gamma < \beta$.

Further,

$$\boldsymbol{\delta}_1(\varepsilon) = \begin{pmatrix} -\varepsilon\delta_1(\varepsilon) \\ \delta_2(\varepsilon) \\ \delta_3(\varepsilon) \end{pmatrix}, \quad \boldsymbol{\delta}_2(\varepsilon) = \begin{pmatrix} \varepsilon\delta_1(\varepsilon) \\ \delta_2(\varepsilon) \\ \delta_3(\varepsilon) \end{pmatrix},$$

and $\delta_j(\varepsilon) \to 0$ $(j=1, 2, 3)$ for $\varepsilon \to +0$.

Given an interval $t_1 \leqq t \leqq t_2$ and a point t_0 such that $t_1 \leqq t_0 \leqq t_2$, we need a set of continuous functions $u_1(t\,;\varepsilon)$, $u_2(t\,;\varepsilon)$, $u_3(t\,;\varepsilon)$ with the following properties.

(I) The conditions

$$u_1(t_0\,;\varepsilon) = P(\varepsilon)\,, \quad u_2(t_1\,;\varepsilon) = Q(\varepsilon)\,, \quad u_3(t_2\,;\varepsilon) = R(\varepsilon)$$

are fulfilled, where $P(\varepsilon)$, $Q(\varepsilon)$ and $R(\varepsilon)$ are suitable positive quantities tending to zero with ε, such that $P_0(\varepsilon) \leqq P(\varepsilon)$, $Q_0(\varepsilon) \leqq Q(\varepsilon)$, $R_0(\varepsilon) \leqq R(\varepsilon)$ for given positive quantities $P_0(\varepsilon)$, $Q_0(\varepsilon)$, $R_0(\varepsilon)$ tending to zero with ε.

(II) $u_1(t\,;\varepsilon)$, $u_2(t\,;\varepsilon)$, $u_3(t\,;\varepsilon)$ are positive on $t_1 \leqq t \leqq t_2$.

Received August 29, 1976

(III) $u = u(t; \varepsilon)$ satisfies the system (1) for $t_1 \leqq t \leqq t_0$ and satisfies the system (2) for $t_0 \leqq t \leqq t_2$.

In the paper [1], we gave a solution of the above problem, but the proof was not complete.

In this paper, we will give a new proof of the existence of such a solution rather in a special case. That is, we take the case $t_0 = t_1$ and hence we omit the system (1).

2. Put

$$A = \frac{A_0}{\varepsilon} + A_1 = \begin{pmatrix} \alpha & \alpha & \alpha \\ \frac{\alpha}{\varepsilon} & -\frac{\beta}{\varepsilon} & \frac{\gamma}{\varepsilon} \\ -\frac{\alpha}{\varepsilon} & -\frac{\gamma}{\varepsilon} & \frac{\beta}{\varepsilon} \end{pmatrix}, \tag{3}$$

then the characteristic equation of A is

$$\varepsilon^2\lambda^3 - \varepsilon^2\alpha\lambda^2 - (\beta^2 - \gamma^2)\lambda + \alpha\beta^2 - \alpha\gamma^2 + 2\alpha^2\beta + 2\alpha^2\gamma = 0. \tag{4}$$

Since the roots ρ_1, ρ_2, ρ_3 of this equation can be regarded as algebraic functions of ε, we put

$$\lambda = a_0 + a_1\varepsilon + \cdots,$$

or

$$\lambda = \frac{b_{-1}}{\varepsilon} + b_0 + b_1\varepsilon + \cdots,$$

to the end of finding these roots.

Substituting these series into (4) and determining the coefficients $a_0, a_1, \cdots$, or $b_{-1}, b_0, \cdots$, as the characteristic roots ρ_1, ρ_2, ρ_3 of A, we get

$$\rho_1 = \frac{2\alpha^2 + \alpha(\beta - \gamma)}{\beta - \gamma} + O(\varepsilon),$$

$$\rho_2 = -\frac{\mu}{\varepsilon} + O(1), \qquad \rho_3 = \frac{\mu}{\varepsilon} + O(1),$$

where $\mu = \sqrt{\beta^2 - \gamma^2}$.

Furthermore the canonical form $\hat{A}$ of A is

$$\hat{A} = \begin{pmatrix} \rho_1 & & 0 \\ & \rho_2 & \\ 0 & & \rho_3 \end{pmatrix} \tag{5}$$

and the transformation matrix $S(\varepsilon)$ that transforms A into $\hat{A}$ is

$$(6)\qquad S(\varepsilon)=(s_{ij}(\varepsilon))$$
$$=\begin{pmatrix} \beta-\gamma+O(\varepsilon) & O(\varepsilon) & O(\varepsilon) \\ \alpha+O(\varepsilon) & \gamma+O(\varepsilon) & \gamma+O(\varepsilon) \\ \alpha+O(\varepsilon) & \beta-\sqrt{\beta^2-\gamma^2}+O(\varepsilon) & \beta+\sqrt{\beta^2-\gamma^2}+O(\varepsilon) \end{pmatrix}.$$

That is, $S(\varepsilon)^{-1}AS(\varepsilon)=\hat{A}$.

By the transformation of unknowns $\boldsymbol{u}=S(\varepsilon)\boldsymbol{v}$, the system (2) is changed into

$$(7)\qquad \frac{d\boldsymbol{v}}{dt}=\hat{A}\boldsymbol{v}+\frac{1}{\varepsilon}\hat{\boldsymbol{\delta}}_2(\varepsilon)\,,\quad \hat{\boldsymbol{\delta}}_2(\varepsilon)=S(\varepsilon)^{-1}\boldsymbol{\delta}_2(\varepsilon)\,.$$

Now, let

$$\boldsymbol{\omega}(\varepsilon)=\begin{pmatrix} \omega_1(\varepsilon) \\ \omega_2(\varepsilon) \\ \omega_3(\varepsilon) \end{pmatrix}$$

be a solution of a linear equation

$$(A_0+\varepsilon A_1)\boldsymbol{\omega}+\boldsymbol{\delta}_2(\varepsilon)=\boldsymbol{0}\,.$$

Then, clearly $\omega_j(\varepsilon)\to 0$ $(j=1, 2, 3)$ for $\varepsilon\to+0$.

We will seek for a desired solution in the following form:

$$\boldsymbol{u}(t\,;\varepsilon)=\begin{pmatrix} u_1(t\,;\varepsilon) \\ u_2(t\,;\varepsilon) \\ u_3(t\,;\varepsilon) \end{pmatrix}=S(\varepsilon)\begin{pmatrix} C_1(\varepsilon)e^{\rho_1(t-t_1)} \\ C_2(\varepsilon)e^{\rho_2(t-t_1)} \\ C_3(\varepsilon)e^{\rho_3(t-t_2)} \end{pmatrix}+\begin{pmatrix} \omega_1(\varepsilon) \\ \omega_2(\varepsilon) \\ \omega_3(\varepsilon) \end{pmatrix}.$$

3. It is sufficient to determine the positive quantities $P(\varepsilon)$, $Q(\varepsilon)$, $R(\varepsilon)$ and the coefficients $C_j(\varepsilon)(j=1, 2, 3)$ so that

$$(8)\qquad \begin{cases} C_1(\varepsilon)s_{11}(\varepsilon)+C_2(\varepsilon)s_{12}(\varepsilon)+C_3(\varepsilon)s_{13}(\varepsilon)e^{\rho_3(t_1-t_2)}=\hat{P}(\varepsilon)(=P(\varepsilon)-\omega_1(\varepsilon))\,, \\ C_1(\varepsilon)s_{21}(\varepsilon)+C_2(\varepsilon)s_{22}(\varepsilon)+C_3(\varepsilon)s_{23}(\varepsilon)e^{\rho_3(t_1-t_2)}=\hat{Q}(\varepsilon)(=Q(\varepsilon)-\omega_2(\varepsilon))\,, \\ C_1(\varepsilon)s_{31}(\varepsilon)e^{\rho_1(t_2-t_1)}+C_2(\varepsilon)s_{32}(\varepsilon)e^{\rho_2(t_2-t_1)}+C_3(\varepsilon)s_{33}(\varepsilon)=\hat{R}(\varepsilon)(=R(\varepsilon)-\omega_3(\varepsilon))\,, \end{cases}$$

and $u_1(t\,;\varepsilon)$, $u_2(t\,;\varepsilon)$, $u_3(t\,;\varepsilon)$ are positive on the interval $t_1\leqq t\leqq t_2$.

By virtue of $\rho_2<0$, $\rho_3>0$, we see easily

$$\Delta(\varepsilon)=\begin{vmatrix} s_{11}(\varepsilon) & s_{12}(\varepsilon) & s_{13}(\varepsilon)e^{\rho_3(t_1-t_2)} \\ s_{21}(\varepsilon) & s_{22}(\varepsilon) & s_{23}(\varepsilon)e^{\rho_3(t_1-t_2)} \\ s_{31}(\varepsilon)e^{\rho_1(t_2-t_1)} & s_{32}(\varepsilon)e^{\rho_2(t_2-t_1)} & s_{33}(\varepsilon) \end{vmatrix}$$
$$=(\beta-\gamma)\gamma(\beta+\sqrt{\beta^2-\gamma^2})+O(\varepsilon)\,,$$

$$\Delta_1(\varepsilon)=\begin{vmatrix} \hat{P}(\varepsilon) & s_{12}(\varepsilon) & s_{13}(\varepsilon)e^{\rho_3(t_1-t_2)} \\ \hat{Q}(\varepsilon) & s_{22}(\varepsilon) & s_{23}(\varepsilon)e^{\rho_3(t_1-t_2)} \\ \hat{R}(\varepsilon) & s_{32}(\varepsilon)e^{\rho_2(t_2-t_1)} & s_{33}(\varepsilon) \end{vmatrix}$$

$$=\gamma(\beta+\sqrt{\beta^2-\gamma^2})\hat{P}(\varepsilon)+O(\varepsilon),$$

$$\Delta_2(\varepsilon)=\begin{vmatrix} s_{11}(\varepsilon) & \hat{P}(\varepsilon) & s_{13}(\varepsilon)e^{\rho_3(t_1-t_2)} \\ s_{21}(\varepsilon) & \hat{Q}(\varepsilon) & s_{23}(\varepsilon)e^{\rho_3(t_1-t_2)} \\ s_{31}(\varepsilon)e^{\rho_1(t_2-t_1)} & \hat{R}(\varepsilon) & s_{33}(\varepsilon) \end{vmatrix}$$

$$=(\beta+\sqrt{\beta^2-\gamma^2})\{-\alpha\hat{P}(\varepsilon)+(\beta-\gamma)\hat{Q}(\varepsilon)\}+O(\varepsilon),$$

$$\Delta_3(\varepsilon)=\begin{vmatrix} s_{11}(\varepsilon) & s_{12}(\varepsilon) & \hat{P}(\varepsilon) \\ s_{21}(\varepsilon) & s_{22}(\varepsilon) & \hat{Q}(\varepsilon) \\ s_{31}(\varepsilon)e^{\rho_1(t_2-t_1)} & s_{32}(\varepsilon)e^{\rho_2(t_2-t_1)} & \hat{R}(\varepsilon) \end{vmatrix}$$

$$=\gamma\{-\alpha e^{\rho_1(t_2-t_1)}\hat{P}(\varepsilon)+(\beta-\gamma)\hat{R}(\varepsilon)\}+O(\varepsilon).$$

Hence we have

$$C_1(\varepsilon)=\frac{1}{\beta-\gamma}\hat{P}(\varepsilon)+O(\varepsilon),$$

$$C_2(\varepsilon)=\frac{1}{\gamma(\beta-\gamma)}\{-\alpha\hat{P}(\varepsilon)+(\beta-\gamma)\hat{Q}(\varepsilon)\}+O(\varepsilon),$$

$$C_3(\varepsilon)=\frac{1}{(\beta-\gamma)(\beta+\sqrt{\beta^2-\gamma^2})}\{-\alpha e^{\rho_1(t_2-t_1)}\hat{P}(\varepsilon)+(\beta-\gamma)\hat{R}(\varepsilon)\}+O(\varepsilon).$$

This shows that we can take

$$P(\varepsilon)=(\hat{P}(\varepsilon)+\omega_1(\varepsilon)),$$

$$Q(\varepsilon)=(\hat{Q}(\varepsilon)+\omega_2(\varepsilon)),$$

$$R(\varepsilon)=(\hat{R}(\varepsilon)+\omega_3(\varepsilon)),$$

such that $C_j(\varepsilon)>0$ $(j=1,2,3)$, and $u_j(t;\varepsilon)>0$ $(j=1,2,3)$ on the interval $t_1\leqq t\leqq t_2$.

REFERENCE

[1] Y. HIRASAWA, On singular perturbation problems of non-linear systems of differential equations, III, Comment. Math. Univ. Sancti. Pauli, 4 (1955), 93-104.

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