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MINIMAL HYPERSURFACES WITH THREE PRINCIPAL CURVATURE FIELDS IN S^{n+1}

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As is well known, there are many works on minimal hypersurfaces with two regular principal curvature fields in the space forms, especially the spheres. On the contrary, it seems we have very few works on minimal hypersurfaces with more than two principal curvature fields in such spaces. In the present paper, we shall study minimal hypersurfaces in S^{n+1} which have three regular and nonsimple principal curvature fields, say μ_1, μ_2 and μ_3.

In § 1, we shall state some fundamental theorems in the following argument. In § 2 and § 3, we shall develope a general theory on such hypersurfaces and find that the three tangent vector fields $H(\mu_i)$ (defined by (3.13)) corresponding to μ_i, $i=1, 2, 3$, play important role in our investigation. In § 4 and § 5, we shall treat the case in which one of $H(\mu_i)$ vanishes identically and give an example of such hypersurfaces. Finally, we shall investigate the case in which $H(\mu_i) \not\equiv 0$, $i=1, 2, 3$, and show that each μ_i can not be constant (Theorem 5), which tells us that in order to construct examples of such hypersurfaces each μ_i must be considered as a nonconstant function.

§ 1. Preliminaries

Let $M=M^n$ be a hypersurface in an $(n+1)$-dimensional Riemannian manifold $\bar{M}=\bar{M}^{n+1}$ of constant curvature $\bar{c}$. Let $\bar{\omega}_A$, $\bar{\omega}_{AB}=-\bar{\omega}_{BA}$, $A, B=1, 2, \cdots, n+1$, be the basic and connection forms of $\bar{M}$ on the orthonormal frame bundle $F(\bar{M})$ over $\bar{M}$, which satisfy the structure equations

$$d\bar{\omega}_A=\sum_B \bar{\omega}_{AB}\wedge\bar{\omega}_B , \qquad d\bar{\omega}_{AB}=\sum_C \bar{\omega}_{AC}\wedge\bar{\omega}_{CB}-\bar{c}\bar{\omega}_A\wedge\bar{\omega}_B . \tag{1.1}$$

Let B be the submanifold of $F(\bar{M})$ over M composed of $b=(x, e_1, \cdots, e_{n+1})$ such that $(x, e_1, \cdots, e_n)\in F(M)$, where $F(M)$ is the orthonormal frame bundle of M with the induced Riemannian metric from $\bar{M}$. Then, deleting the bars of $\bar{\omega}_A$, $\bar{\omega}_{AB}$ on B, we have

$$\omega_{n+1}=0 , \qquad \omega_{i(n+1)}=\sum_j A_{ij}\omega_j , \qquad A_{ij}=A_{ji} , \tag{1.2}$$
$$i, j=1, 2, \cdots, n .$$

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Let $\Phi(\omega):=\sum_{i,j}A_{ij}\omega_i\omega_j$ and $\Psi(\omega):=\sum_{i,j,k}B_{ijk}\omega_i\omega_j\omega_k$ be the 2nd and 3rd fundamental forms of M respectively, where B_{ijk} are defined by

$$DA_{ij}:=dA_{ij}-\sum_k\omega_{ik}A_{kj}-\sum_k\omega_{jk}A_{ik}=\sum_k B_{ijk}\omega_k \tag{1.3}$$

and

$$B_{ijk}=B_{jik}=B_{ikj}\,. \tag{1.4}$$

Now, let k be a principal curvature of M at $x\in M$, an eigen value of Φ at x, and denote the tangent subspace of all principal tangent vectors for k and the zero vector at x by $E(k, x)$. $\dim E(k, x)$ is equal to the multiplicity of k.

Let μ be a smooth principal curvature field of M. If $E(\mu, x)=E(\mu(x), x)$ is of constant dimension, then $E(\mu, x)$, $x\in M$, make a smooth distribution of M, which we denote by $E(\mu)$. We call such field μ regular. We have the following theorem.

THEOREM A. *Let M be a hypersurface immersed in an $(n+1)$-dimensional Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$ and suppose that M has a regular principal curvature field μ, then the distribution $E(\mu)$ is completely integrable.*

Proof. Let m be the dimension of the distribution of $E(\mu)$ and take only the frames b such that $e_\alpha\in E(\mu)$, $\alpha=1, 2, \cdots, m$. Then, we have

$$\omega_{\alpha(n+1)}=\mu\omega_\alpha\,,\qquad \omega_{r(n+1)}=\sum_t A_{rt}\omega_t\,. \tag{1.5}$$

In the proof, we suppose the ranges of indexes as follows:

$$\alpha, \beta, \cdots=1, 2, \cdots, m\,;\qquad r, t, \cdots=m+1, \cdots, n\,.$$

From (1.1) and (1.5), we get easily

$$d\omega_{\alpha(n+1)}=d\mu\wedge\omega_\alpha+\mu(\sum_\beta\omega_\beta\wedge\omega_{\beta\alpha}+\sum_r\omega_r\wedge\omega_{r\alpha})\,,$$

$$d\omega_{\alpha(n+1)}=\sum_\beta\omega_{\alpha\beta}\wedge\omega_{\beta(n+1)}+\sum_r\omega_{\alpha r}\wedge\omega_{r(n+1)}$$

$$=\mu\sum_\beta\omega_\beta\wedge\omega_{\beta\alpha}-\sum_{r,t}A_{rt}\omega_t\wedge\omega_{\alpha r}$$

and hence

$$d\mu\wedge\omega_\alpha-\sum_r\omega_{\alpha r}\wedge\sum_t(A_{rt}-\mu\delta_{rt})\omega_t=0\,.$$

Setting $d\mu=\sum_i\mu_i\omega_i$, we can put

$$\mu_t\omega_\alpha+\sum_r\omega_{\alpha r}(A_{rt}-\mu\delta_{rt})=\sum_s K_{\alpha ts}\omega_s\,,\qquad K_{\alpha ts}=K_{\alpha st}\,,$$

by E. Cartan's lemma. Thus we have

$$\sum_r(A_{rt}-\mu\delta_{rt})\sum_\alpha\omega_\alpha\wedge\omega_{\alpha r}=\sum_{\alpha,s}K_{\alpha ts}\omega_\alpha\wedge\omega_s\,,$$

which implies

(1.6) $$\sum_\alpha \omega_\alpha \wedge \omega_{\alpha r}=0 \qquad (\mathrm{mod}\ \omega_{m+1}, \cdots, \omega_n),$$

because

$$\det(A_{rt}-\mu\delta_{rt})\neq 0$$

by the regularity of μ. On the other hand, we have also

$$d\omega_t=\sum_\alpha \omega_\alpha\wedge\omega_{\alpha t}+\sum_r \omega_r\wedge\omega_{rt}$$

$$\equiv 0 \qquad (\mathrm{mod}\ \omega_{m+1}, \cdots, \omega_n)$$

by (1.6). This shows that the system of Pfaff equations:

$$\omega_{m+1}=\cdots=\omega_n=0,$$

i. e. the distribution $E(\mu)$ is completely integrable. Q. E. D.

Remark. We have proved Theorem A under the condition of regularity of the other principal curvatures (Theorem 2 in [3]).

We prove the following

THEOREM B. *Let M be a hypersurface immersed in an $(n+1)$-dimensional Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$. If the 3rd fundamental form of M vanishes, then M is totally geodesic or umbilic, otherwise M has just two principal curvatures λ, μ such that $\lambda\mu=-\bar{c}$.*

Proof. Since we have $A_{ij,k}=0$, we can choose special frames such that $A_{ii}=\lambda_i$, $A_{ij}=0 (i\neq j)$ and $\lambda_1, \cdots, \lambda_n$ are all constant. If $\lambda_1=\lambda_2=\cdots=\lambda_n$, then M is totally geodesic or umbilic. Otherwise, we have $\omega_{ij}=0$ for $\lambda_i\neq\lambda_j$. In fact, for such i and j, we have from (1.3)

$$0=dA_{ij}-\sum_k \omega_{ik}A_{kj}-\sum_k \omega_{jk}A_{ik}=(\lambda_i-\lambda_j)\omega_{ij},$$

hence $\omega_{ij}=0$. Furthermore, from this relation we get

$$0=d\omega_{ij}=\sum_k \omega_{ik}\wedge\omega_{kj}+\omega_{i(n+1)}\wedge\omega_{(n+1)j}-\bar{c}\omega_i\wedge\omega_j$$

$$=-(\lambda_i\lambda_j+\bar{c})\omega_i\wedge\omega_j,$$

i. e. $$\lambda_i\lambda_j+\bar{c}=0, \qquad \text{for} \quad \lambda_i\neq\lambda_j.$$

This fact shows that the number of different principal curvatures is at most two. Q. E. D.

Theorem B shows that if a hypersurface M in $\bar{M}$ have more than two principal curvatures, its 3rd fundamental form Ψ does not vanish.

In the following, we consider also Ψ as a symmetric tensor field of M. We denote the tangent space M at $x\in M$ by M_x and the inner product of $X, Y\in M_x$ by $\langle X, Y\rangle$.

§ 2. On the distributions for two regular principal curvature fields

In this section, we shall consider M in $\bar{M}$ as in Theorem A, whose 3rd fundamental form Ψ does not vanish.

LEMMA 1. *Let μ be a regular principal curvature field of M. For any tangent vectors $X, Y \in E(\mu, x)$, such that $\langle X, Y\rangle = 0$ and any $Z \in M_x$, we have*

$$\Psi(X, Y, Z) = 0 .$$

Proof. Let us put $\dim E(\mu, x) = m$. We restrict $\omega_i, \omega_{ij}, A_{ij}$ on the submanifold of the frames $b = (x, e_1, \cdots, e_n, e_{n+1})$ such that $e_\alpha \in E(\mu, x)$, $\alpha = 1, 2, \cdots, m$. Then we have

$$A_{\alpha i} = \mu\delta_{\alpha i}, \ \alpha = 1, 2, \cdots, m ; \ i = 1, 2, \cdots, n .$$

Hence, by (1.3) we get

$$\text{(2.1)} \qquad \sum_{k=1}^{n} B_{\alpha ik}\omega_k = \delta_{\alpha i}d\mu - \sum_{k=1}^{n} \omega_{\alpha k}A_{ki} - \sum_{k=1}^{n} \omega_{ik}A_{\alpha k}$$

$$= \delta_{\alpha i}d\mu - \mu\sum_{\gamma=1}^{m} \delta_{\gamma i}\omega_{\alpha\gamma} - \sum_{t=m+1}^{n} \omega_{\alpha t}A_{ti} + \mu\omega_{\alpha i} .$$

Especially, we have

$$\text{(2.2)} \qquad \sum_{k=1}^{n} B_{\alpha\beta k}\omega_k = \delta_{\alpha\beta}d\mu , \qquad \alpha, \beta = 1, 2, \cdots, m ,$$

which imply easily this lemma. Q. E. D.

LEMMA 2. *If* $\dim E(\mu) = \dim E(\mu, x) \geqq 2$, *then*

$$\Psi(X, Y, Z) = 0 , \qquad X, Y, Z \in E(\mu, x) .$$

Proof. On the submanifold of B used in the proof of Lemma 1, we have from (2.2)

$$d\mu = \sum_{k=1}^{n} B_{\alpha\alpha k}\omega_k = B_{\alpha\alpha\alpha}\omega_\alpha + \sum_{r=m+1}^{n} B_{\alpha\alpha r}\omega_r$$

for a fixed $\alpha \leqq m$. Hence, putting $d\mu = \sum_{k=1}^{n} \mu_{,k}\omega_k$, we have

$$\text{(2.3)} \qquad \mu_{,k} = B_{\alpha\alpha k} , \qquad \alpha = 1, 2, \cdots, m ; \ k = 1, 2, \cdots, n$$

and then, using the assumption $m \geqq 2$, we obtain

$$\text{(2.4)} \qquad \mu_{,1} = \cdots = \mu_{,m} = 0 .$$

Hence we have $B_{\alpha\beta\gamma} = 0$, which imply the lemma. Q .E. D.

Since any integral submanifold of the distribution $E(\mu)$ is a solution of the system of Pfaff equations

$$\omega_{m+1}=\cdots=\omega_n=0\,,$$

we have easily the following

COROLLARY. *If* $\dim E(\mu)\geqq 2$, μ *is constant on any integral submanifold of the distribution* $E(\mu)$.

LEMMA 3. *Let* μ_1 *and* μ_2 *be two regular principal curvature fields of* M. *Let* $E(\mu_1)+E(\mu_2)$ *be the distribution of the tangent subspace* $E(\mu_1, x)+E(\mu_2, x)$, $x\in M$. *Then,* $E(\mu_1)+E(\mu_2)$ *is completely integrable, if and only if for any* $X\in E(\mu_1, x)$, $Y\in E(\mu_2, x)$ *and any* $Z\perp E(\mu_1, x)+E(\mu_2, x)$ *we have* $\Psi(X, Y, Z)=0$.

Proof. Let us put $\dim E(\mu_1, x)=m_1$ and $\dim E(\mu_2, x)=m_2$. We restrict $\omega_1, \omega_{ij}, A_{ij}$ on the submanifold of the frames $b=(x, e_1, \cdots, e_n, e_{n+1})$ such that $e_{\alpha_1}\in E(\mu_1, x)$, $\alpha_1=1, \cdots, m_1$, and $e_{\alpha_2}\in E(\mu_2, x)$, $\alpha_2=m_1+1, \cdots, m_1+m_2$. Then, we have

$$A_{\alpha_1 i}=\mu_1\delta_{\alpha_1 i} \quad\text{and}\quad A_{\alpha_2 i}=\mu_2\delta_{\alpha_2 i}$$

$$\alpha_1=1, \cdots, m_1\,;\ \alpha_2=m_1+1, \cdots, m_1+m_2\,,\ i=1, 2, \cdots, n\,.$$

From (2.1) and (2.3), we have for $s>m_1+m_2=m$

$$\mu_{1,s}\omega_{\alpha_1}+\sum_{\alpha_2}B_{\alpha_1\alpha_2 s}\omega_{\alpha_2}+\sum_{t>m}B_{\alpha_1 ts}\omega_t$$
$$=(\mu_1-A_{ss})\omega_{\alpha_1 s}-\sum_{t>m,\,t\neq s}\omega_{\alpha_1 t}A_{ts}\,,$$

hence

$$\sum_{\alpha_1,\alpha_2}B_{\alpha_1\alpha_2 s}\omega_{\alpha_1}\wedge\omega_{\alpha_2}+\sum_{\alpha_1}\sum_{t>m}B_{\alpha_1 ts}\omega_{\alpha_1}\wedge\omega_t$$
$$=(\mu_1-A_{ss})\sum_{\alpha_1}\omega_{\alpha_1}\wedge\omega_{\alpha_1 s}-\sum_{t>m,\,t\neq s}A_{ts}\omega_{\alpha_1}\wedge\omega_{\alpha_1 t}\,.$$

Analogously, we get

$$\sum_{\alpha_1,\alpha_2}B_{\alpha_1\alpha_2 s}\omega_{\alpha_2}\wedge\omega_{\alpha_1}+\sum_{\alpha_2}\sum_{t>m}B_{\alpha_2 ts}\omega_{\alpha_2}\wedge\omega_t$$
$$=(\mu_2-A_{ss})\sum_{\alpha_2}\omega_{\alpha_2}\wedge\omega_{\alpha_2 s}-\sum_{t>m,\,t\neq s}A_{ts}\omega_{\alpha_2}\wedge\omega_{\alpha_2 t}\,.$$

Here, we restrict locally the submanifold of the above frames to the one such that $A_{ts}=\delta_{ts}\lambda_t$, $t, s>m$. Then, from the regularity of μ_1 and μ_2, we have $\mu_1\neq\lambda_s$, $\mu_2\neq\lambda_s$, $s>m$. Hence the above equalities turn out

$$\sum_{\alpha_1}\omega_{\alpha_1}\wedge\omega_{\alpha_1 s}=\frac{1}{\mu_1-\lambda_s}\{\sum_{\alpha_1,\alpha_2}B_{\alpha_1\alpha_2 s}\omega_{\alpha_1}\wedge\omega_{\alpha_2}+\sum_{\alpha_1}\sum_{t>m}B_{\alpha_1 ts}\omega_{\alpha_1}\wedge\omega_t\}\,,$$

$$\sum_{\alpha_2}\omega_{\alpha_2}\wedge\omega_{\alpha_2 s}=\frac{1}{\mu_2-\lambda_s}\{-\sum_{\alpha_1,\alpha_2}B_{\alpha_1\alpha_2 s}\omega_{\alpha_1}\wedge\omega_{\alpha_2}+\sum_{\alpha_2}\sum_{t>m}B_{\alpha_2 ts}\omega_{\alpha_2}\wedge\omega_t\}\,.$$

Making use of these equalities, we get

$$d\omega_s=\sum_{\alpha_1}\omega_{\alpha_1}\wedge\omega_{\alpha_1 s}+\sum_{\alpha_2}\omega_{\alpha_2}\wedge\omega_{\alpha_2 s}+\sum_{t>m}\omega_t\wedge\omega_{ts}$$

$$
\begin{aligned}
(2.5) \qquad & =-\frac{\mu_1-\mu_2}{(\mu_1-\lambda_s)(\mu_2-\lambda_s)}\sum_{\alpha_1,\alpha_2} B_{\alpha_1\alpha_2 s}\omega_{\alpha 1}\wedge\omega_{\alpha 2} \\
& +\frac{1}{\mu_1-\lambda_s}\sum_{\alpha_1}\sum_{t>m} B_{\alpha_1 ts}\omega_{\alpha_1}\wedge\omega_t+\frac{1}{\mu_2-\lambda_s}\sum_{\alpha_2}\sum_{t>m} B_{\alpha_2 ts}\omega_{\alpha_2}\wedge\omega_t \\
& +\sum_{t>m}\omega_t\wedge\omega_{ts}\,.
\end{aligned}
$$

Since $E(\mu_1)+E(\mu_2)$ is given by $\omega_s=0$, $s>m$, it is completely integrable, if and only if

$$
(2.6) \qquad B_{\alpha_1\alpha_2 s}=0\,,
$$

$$
\alpha_1=1,\cdots,m_1\,;\ \alpha_2=m_1+1,\cdots,m_1+m_2\,;\ s=m_1+m_2+1,\cdots,n\,,
$$

by means of (2.5). (2.6) is equivalent to the condition in the statement of this lemma. Q.E.D.

From Lemma 3, we obtain easily the following

THEOREM 1. *Let M be a hypersurface immersed in a Riemannian manifold $\bar{M}$ of constant curvature and Ψ its 3rd fundamental form. If M has just three regular principal curvature fields μ_1, μ_2, μ_3 such that $E(\mu_1)+E(\mu_2)+E(\mu_3)=TM$, then the distributions $E(\mu_2)+E(\mu_3)$, $E(\mu_3)+E(\mu_1)$ and $E(\mu_1)+E(\mu_2)$ are simultaneously completely integrable if and only if for any $X_i\in E(\mu_i, x)$, $x\in M$, $i=1, 2, 3$,*

$$
\Psi(X_1, X_2, X_3)=0\,.
$$

In the case of integrable in Theorem 1, by (2.2) and (2.3) the equation (2.5) becomes

$$
\begin{aligned}
& d\omega_{\alpha_3}=-\frac{\mu_1-\mu_2}{(\mu_1-\mu_3)(\mu_2-\mu_3)}\sum_{\alpha_1,\alpha_2} B_{\alpha_1\alpha_2\alpha_3}\omega_{\alpha_1}\wedge\omega_{\alpha_2} \\
(2.7) \qquad & +\frac{1}{\mu_1-\mu_3}\sum_{\alpha_1}\mu_{3,\alpha_1}\omega_{\alpha_1}\wedge\omega_{\alpha_3}+\frac{1}{\mu_2-\mu_3}\sum_{\alpha_2}\mu_{3,\alpha_2}\omega_{\alpha_2}\wedge\omega_{\alpha_3} \\
& +\sum_{\beta_3}\omega_{\beta_3}\wedge\omega_{\beta_3\alpha_3}\,,
\end{aligned}
$$

where $e_{\alpha_1}\in E(\mu_1, x)$, $\alpha_1=1,\cdots,m_1$; $e_{\alpha_2}\in E(\mu_2, x)$, $\alpha_2=m_1+1,\cdots,m_1+m_2$; $e_{\alpha_3}\in E(\mu_3,x)$, $\alpha_3=m_1+m_2+1,\cdots,n$.

§ 3. Minimal hypersurfaces with three non-simple regular principal curvature fields

Using the notation in § 2, we shall investigate a minimal hypersurface M in $\bar{M}$ as in Theorem 1 with

$$
(3.1) \qquad \Psi(X_1, X_2, X_3)=0 \quad \text{for} \quad X_i\in E(\mu_i, x),\ x\in M,\ i=1, 2, 3
$$

and

$$\dim E(\mu_i, x)=m_i\geqq 2\,,\ i=1, 2, 3\,, \tag{3.2}$$

in this section.

LEMMA 4. *There exist a local coordinate system* $u_1, \cdots, u_n$ *and functions* $f_1=f_1(u_{\alpha_1})$, $f_2=f_2(u_{\alpha_2})$, $f_3=f_3(u_{\alpha_3})$ *such that*

$$\mu_1=\frac{1}{m_1}(f_2-f_3)\,,\qquad \mu_2=\frac{1}{m_2}(f_3-f_1)\,,\qquad \mu_3=\frac{1}{m_3}(f_1-f_2)\,.$$

Proof. Since M is minimal in $\bar{M}$, we have

$$m_1\mu_1+m_2\mu_2+m_3\mu_3=0\,. \tag{3.3}$$

By (3.1), we have

$$B_{\alpha_1\alpha_2\alpha_3}=0\,,\qquad \alpha_i\in I(\mu_i)\,, \tag{3.4}$$

where

$$I(\mu_1)=\{1, 2, \cdots, m_1\},\quad I(\mu_2)=\{m_1+1, \cdots, m_1+m_2\},\quad I(\mu_3)=\{m_1+m_2+1, \cdots, n\}.$$

By (3.2), (2.4), (2.7) and (3.4), there exist a local coordinate system $u_1, \cdots, u_n$ such that u_{α_i}=constant, $\alpha_i\in I(\mu_i)$, give an integral submanifold of the distribution $E(\mu_j)+E(\mu_k)$, $\{i, j, k\}=\{1, 2, 3\}$, and $\mu_i=\mu_i(u_{\alpha_j}\,;\, u_{\alpha_k})$. Taking a fixed point of M with local coordinates $(c_1, \cdots, c_n)$, from (3.3) we get easily the following equalities:

$$\begin{aligned} &m_2\mu_2(u_{\alpha_1}\,;\, u_{\alpha_3})+m_3\mu_3(u_{\alpha_1}\,;\, u_{\alpha_2})=m_2\mu_2(c_{\alpha_1}\,;\, u_{\alpha_3})+m_3\mu_3(c_{\alpha_1}\,;\, u_{\alpha_2})\,,\\ &m_3\mu_3(u_{\alpha_1}\,;\, u_{\alpha_2})+m_1\mu_1(u_{\alpha_2}\,;\, u_{\alpha_3})=m_3\mu_3(u_{\alpha_1}\,;\, c_{\alpha_2})+m_1\mu_1(c_{\alpha_2}\,;\, u_{\alpha_3})\,,\\ &m_1\mu_1(u_{\alpha_2}\,;\, u_{\alpha_3})+m_2\mu_2(u_{\alpha_1}\,;\, u_{\alpha_3})=m_1\mu_1(u_{\alpha_2}\,;\, c_{\alpha_3})+m_2\mu_2(u_{\alpha_1}\,;\, c_{\alpha_3})\,, \end{aligned} \tag{3.5}$$

from which we obtain

$$\begin{aligned} &\{m_2\mu_2(u_{\alpha_1}\,;\, c_{\alpha_3})+m_3\mu_3(u_{\alpha_1}\,;\, c_{\alpha_2})\}+\{m_3\mu_3(c_{\alpha_1}\,;\, u_{\alpha_2})+m_1\mu_1(u_{\alpha_2}\,;\, c_{\alpha_3})\}\\ &\qquad+\{m_1\mu_1(c_{\alpha_2}\,;\, u_{\alpha_3})+m_2\mu_2(c_{\alpha_1}\,;\, u_{\alpha_3})\}=0\,. \end{aligned}$$

Hence there exist constants a_1, a_2, a_3 such that

$$\begin{aligned} &m_2\mu_2(u_{\alpha_1}\,;\, c_{\alpha_3})+m_3\mu_3(u_{\alpha_1}\,;\, c_{\alpha_2})=a_1\,,\\ &m_3\mu_3(c_{\alpha_1}\,;\, u_{\alpha_2})+m_1\mu_1(u_{\alpha_2}\,;\, c_{\alpha_3})=a_2\,,\\ &m_1\mu_1(c_{\alpha_2}\,;\, u_{\alpha_3})+m_2\mu_2(c_{\alpha_1}\,;\, u_{\alpha_3})=a_3 \end{aligned} \tag{3.6}$$

and

$$a_1+a_2+a_3=0\,.$$

Now, setting

$$\begin{aligned} &f_1(u_{\alpha_1})=m_3\mu_3(u_{\alpha_1}\,;\, c_{\alpha_2})+a_3,\ f_2(u_{\alpha_2})=-m_3\mu_3(c_{\alpha_1}\,;\, u_{\alpha_2})\,,\\ &f_3(u_{\alpha_3})=m_2\mu_2(c_{\alpha_1}\,;\, u_{\alpha_3})\,, \end{aligned}$$

we have easily from (3.5) and (3.6)

$$m_1\mu_1(u_{\alpha_2};\ u_{\alpha_3})=f_2(u_{\alpha_2})-f_3(u_{\alpha_3}),$$

$$\begin{aligned} m_2\mu_2(u_{\alpha_1};\ u_{\alpha_3}) &= -m_3\mu_3(u_{\alpha_1};\ c_{\alpha_2})-m_1\mu_1(c_{\alpha_2};\ u_{\alpha_3}) \\ &= -f_1(u_{\alpha_1})+a_3-m_1\mu_1(c_{\alpha_2};\ u_{\alpha_3}) \\ &= -f_1(u_{\alpha_1})+m_2\mu_2(c_{\alpha_1};\ u_{\alpha_3})=f_3(u_{\alpha_3})-f_1(u_{\alpha_1}) \end{aligned}$$

and

$$m_3\mu_3(u_{\alpha_1};\ u_{\alpha_2})=f_1(u_{\alpha_1})-f_2(u_{\alpha_2}). \qquad \text{Q. E. D.}$$

By Lemma 4, in the present case we obtain easily

$$\mu_2-\mu_3=\frac{F-nf_1}{m_2m_3}, \qquad \mu_3-\mu_1=\frac{F-nf_2}{m_3m_1}, \qquad \mu_1-\mu_2=\frac{F-nf_3}{m_1m_2}, \tag{3.7}$$

$$F=m_1f_1+m_2f_2+m_3f_3$$

and from (2.7)

$$\begin{cases} d\omega_{\alpha_1}=-\left(\dfrac{m_2df_2}{F-nf_3}+\dfrac{m_3df_3}{F-nf_2}\right)\wedge\omega_{\alpha_1}+\sum\limits_{\beta_1}\omega_{\beta_1}\wedge\omega_{\beta_1\alpha_1}, \\ d\omega_{\alpha_2}=-\left(\dfrac{m_3df_3}{F-nf_1}+\dfrac{m_1df_1}{F-nf_3}\right)\wedge\omega_{\alpha_2}+\sum\limits_{\beta_2}\omega_{\beta_2}\wedge\omega_{\beta_2\alpha_2}, \\ d\omega_{\alpha_3}=-\left(\dfrac{m_1df_1}{F-nf_2}+\dfrac{m_2df_2}{F-nf_1}\right)\wedge\omega_{\alpha_3}+\sum\limits_{\beta_3}\omega_{\beta_3}\wedge\omega_{\beta_3\alpha_3}. \end{cases} \tag{3.8}$$

Next, from (2.1) and (2.3) we obtain

$$\begin{aligned} (\mu_2-\mu_3)\omega_{\alpha_2\alpha_3} &= \sum_{k=1}^{n}B_{\alpha_2\alpha_3k}\omega_k=B_{\alpha_2\alpha_3\alpha_2}\omega_{\alpha_2}+B_{\alpha_2\alpha_3\alpha_3}\omega_{\alpha_3} \\ &= \mu_{2,\alpha_3}\omega_{\alpha_2}+\mu_{3,\alpha_2}\omega_{\alpha_3}, \end{aligned}$$

hence

$$\begin{cases} \omega_{\alpha_2\alpha_3}=\dfrac{1}{F-nf_1}(m_3f_{3,\alpha_3}\omega_{\alpha_2}-m_2f_{2,\alpha_2}\omega_{\alpha_3}), \\ \omega_{\alpha_3\alpha_1}=\dfrac{1}{F-nf_2}(m_1f_{1,\alpha_1}\omega_{\alpha_3}-m_3f_{3,\alpha_3}\omega_{\alpha_1}), \\ \omega_{\alpha_1\alpha_2}=\dfrac{1}{F-nf_3}(m_2f_{2,\alpha_2}\omega_{\alpha_1}-m_1f_{1,\alpha_1}\omega_{\alpha_2}). \end{cases} \tag{3.9}$$

Using these, we get

$$\begin{aligned} d\omega_{\alpha_1\beta_1} &= \sum_{k=1}^{n}\omega_{\alpha_1k}\wedge\omega_{k\beta_1}-(\mu_1^2+\bar{c})\omega_{\alpha_1}\wedge\omega_{\beta_1} \\ &= \sum_{\gamma_1}\omega_{\alpha_1\gamma_1}\wedge\omega_{\gamma_1\beta_1}-(\mu_1^2+\bar{c})\omega_{\alpha_1}\wedge\omega_{\beta_1} \\ &\quad -\frac{1}{(F-nf_3)^2}\sum_{\gamma_2}(m_2f_{2,\gamma_2}\omega_{\alpha_1}-m_1f_{1,\alpha_1}\omega_{\gamma_2})\wedge(m_2f_{2,\gamma_2}\omega_{\beta_1}-m_1f_{1,\beta_1}\omega_{\gamma_2}) \end{aligned}$$

$$-\frac{1}{(F-nf_2)^2}\sum_{\gamma_3}(m_1 f_{1,\alpha_1}\omega_{\gamma_3}-m_3 f_{3,\gamma_3}\omega_{\alpha_1})\wedge(m_1 f_{1,\beta_1}\omega_{\gamma_3}-m_3 f_{3,\gamma_3}\omega_{\beta_1})$$

$$=\sum_{\gamma_1}\omega_{\alpha_1\gamma_1}\wedge\omega_{\gamma_1\beta_1}-\left\{\frac{m_2^2}{(F-nf_3)^2}|\nabla f_2|^2+\frac{m_3^2}{(F-nf_2)^2}|\nabla f_3|^2+\mu_1^2+\bar{c}\right\}\omega_{\alpha_1}\wedge\omega_{\beta_1}$$

$$+\frac{m_1 m_2}{(F-nf_3)^2}(\omega_{\alpha_1}\wedge f_{1,\beta_1}df_2+f_{1,\alpha_1}df_2\wedge\omega_{\beta_1})$$

$$+\frac{m_1 m_3}{(F-nf_2)^2}(\omega_{\alpha_1}\wedge f_{1,\beta_1}df_3+f_{1,\alpha_1}df_3\wedge\omega_{\beta_1}),$$

i. e.

$$d\omega_{\alpha_1\beta_1}-\sum_{\gamma_1}\omega_{\alpha_1\gamma_1}\wedge\omega_{\gamma_1\beta_1}$$

$$=-\left[\frac{1}{(F-nf_2)^2(F-nf_3)^2}\{m_2^2(F-nf_2)^2|\nabla f_2|^2+m_3^2(F-nf_3)^2|\nabla f_3|^2\}\right.$$

(3.10)

$$\left.+\mu_1^2+\bar{c}\right]\omega_{\alpha_1}\wedge\omega_{\beta_1}+\frac{m_1}{(F-nf_2)^2(F-nf_3)^2}(f_{1,\beta_1}\omega_{\alpha_1}$$

$$-f_{1,\alpha_1}\omega_{\beta_1})\wedge\{m_2(F-nf_2)^2df_2+m_3(F-nf_3)^2df_3\}.$$

Analogously, we get the formulas for $d\omega_{\alpha_2\beta_2}$ and $d\omega_{\alpha_3\beta_3}$ by cyclic changes of the suffixes in (3.10). Making use of the above formulas we get the following theorem.

THEOREM 2. *Let M be a minimal hypersurface immersed in a Riemannian manifold $\bar{M}$ of constant curvature with three non-simple regular principal curvature fields μ_1, μ_2 and μ_3 such that $E(\mu_1)+E(\mu_2)+E(\mu_3)=T(M)$ and $\Psi(X_1, X_2, X_3)=0$ for $X_i\in E(\mu_i, x)$, $i=1, 2, 3$, $x\in M$. Then the integral submanifolds of $E(\mu_i)$ are totally umbilic in $\bar{M}$.*

Proof. The integral submanifolds of the distribution $E(\mu_1)$ are the solutions of the Pfaff equations:

$$\omega_{\alpha_2}=\omega_{\alpha_3}=0,\ \alpha_2\in I(\mu_2),\ \alpha_3\in I(\mu_3). \tag{3.11}$$

Hence, along any integral submanifold K^{m_1} of $E(\mu_1)$ we have from (3.9), (1.2) and Lemma 4 the following:

(3.12)

$$\omega_{\alpha_1\beta_2}=\frac{m_2}{F-nf_3}f_{2,\beta_2}\omega_{\alpha_1},\qquad \beta_2\in I(\mu_2),$$

$$\omega_{\alpha_1\beta_3}=\frac{m}{F-nf_2}f_{3,\beta_3}\omega_{\alpha_1},\qquad \beta_3\in I(\mu_3),$$

$$\omega_{\alpha_1(n+1)}=\frac{1}{m_1}(f_2-f_3)\omega_{\alpha_1}.$$

These equalities show that K^{m_1} is totally umbilic in $\bar{M}$. Q. E. D.

By means of (3.12) the mean curvature vector field $H(\mu_1)$ of K^{m_1} is given by

$$H(\mu_1)=\frac{m_2}{F-nf_3}\nabla f_2+\frac{m_3}{F-nf_2}\nabla f_3+\frac{1}{m_1}(f_2-f_3)e_{n+1} \tag{3.13}$$
$$=\frac{1}{(F-nf_2)(F-nf_3)}\{m_2(F-nf_2)\nabla f_2+m_3(F-nf_3)\nabla f_3\}$$
$$+\frac{1}{m_1}(f_2-f_3)e_{n+1}\,.$$

We have also for K^{m_1} the following formulas:

$$\begin{cases} dx=\sum_{\alpha_1}\omega_{\alpha_1}e_{\alpha_1} \\ De_{\alpha_1}=\sum_{\beta_1}\omega_{\alpha_1\beta_1}e_{\beta_1}+H(\mu_1)\omega_{\alpha_1}\,, \\ De_{\alpha_2}=-\langle H(\mu_1),\, e_{\alpha_2}\rangle dx+\sum_{\beta_2}\omega_{\alpha_2\beta_2}e_{\beta_2}\,, \\ De_{\alpha_3}=-\langle H(\mu_1),\, e_{\alpha_3}\rangle dx+\sum_{\beta_3}\omega_{\alpha_3\beta_3}e_{\beta_3}\,, \\ De_{n+1}=-\langle H(\mu_1),\, e_{n+1}\rangle dx\,, \end{cases} \tag{3.14}$$

where D denotes the covariant differential operator of $\bar{M}$.

COROLLARY. *K^{m_1} is a submanifold with m-index 0 in $\bar{M}$.*

Proof. For any normal vector ξ of K^{m_1}

$$\xi=\sum_{\alpha_2}\xi_{\alpha_2}e_{\alpha_2}+\sum_{\alpha_3}\xi_{\alpha_3}e_{\alpha_3}+\xi_{n+1}e_{n+1}\,,$$

the corresponding 2nd fundamental form A_ξ is given by

$$A_\xi=\left\{\frac{m_2}{F-nf_3}\sum_{\beta_2}f_{2,\beta_2}\xi_{\beta_2}+\frac{m_3}{F-nf_2}\sum_{\beta_3}f_{3,\beta_3}\xi_{\beta_3}+\frac{1}{m_1}(f_2-f_3)\xi_{n+1}\right\}I\,.$$

Hence, Trace $A_\xi=0$ if and only if $A_\xi=0$. Therefore, the minimal index of K^{m_1} ($=\dim\{A_\xi\,|\,\text{Trace}\,A_\xi=0\}$) must be zero (see [1]). Q. E. D.

§ 4. Certain minimal hypersurfaces in S^{n+1}

In this section, we shall investigate minimal hypersurfaces as in Theorem 2 in the case of $\bar{M}^{n+1}=S^{n+1}$(unit $(n+1)$-sphere).

For any integrald submanifold K^{m_1} of $E(\mu_1)$, (3.14) becomes

$$(4.1)\qquad \begin{cases} dx=\sum_{\alpha_1}\omega_{\alpha_1}e_{\alpha_1}, \\ de_{\alpha_1}=\sum_{\beta_1}\omega_{\alpha_1\beta_1}e_{\beta_1}+(H(\mu_1)-x)\omega_{\alpha_1}, \\ de_{\alpha_2}=-\langle H(\mu_1), e_{\alpha_2}\rangle dx+\sum_{\beta_2}\omega_{\alpha_2\beta_2}e_{\beta_2}, \\ de_{\alpha_3}=-\langle H(\mu_1), e_{\alpha_3}\rangle dx+\sum_{\beta_3}\omega_{\alpha_3\beta_3}e_{\beta_3}, \\ de_{n+1}=-\langle H(\mu_1), e_{n+1}\rangle dx, \end{cases}$$

where d is the ordinary differential operator in $R^{n+2}(\supset S^{n+1})$.

LEMMA 5. *K^{m_1} is an m_1-dimensional small sphere or an open subset of this sphere.*

Proof. From (4.1), we get easily

$$\begin{aligned} 0=d^2e_{\alpha_1}=&\sum_{\beta_1}d\omega_{\alpha_1\beta_1}e_{\beta_1}-\sum_{\beta_1}\omega_{\alpha_1\beta_1}\wedge\{\sum_{\gamma_1}\omega_{\beta_1\gamma_1}e_{\gamma_1}+(H(\mu_1)-x)\omega_{\beta_1}\} \\ &+(H(\mu_1)-x)d\omega_{\alpha_1}+d(H(\mu_1)-x)\wedge\omega_{\alpha_1} \\ =&\sum_{\beta_1}\{d\omega_{\alpha_1\beta_1}-\sum_{\gamma_1}\omega_{\alpha_1\gamma_1}\wedge\omega_{\gamma_1\beta_1}\}e_{\beta_1} \\ &+\{d\omega_{\alpha_1}-\sum_{\beta_1}\omega_{\alpha_1\beta_1}\wedge\omega_{\beta_1}\}(H(\mu_1)-x)+d(H(\mu_1)-x)\wedge\omega_{\alpha_1}. \end{aligned}$$

Hence, for any vector Y orthogonal to $E(\mu_1, x)$ and $H(\mu_1)-x$, we have

$$d\langle H(\mu_1)-x, Y\rangle\wedge\omega_{\alpha_1}=0, \qquad \alpha_1=1, 2, \cdots, m_1.$$

Since $m_1=\dim E(\mu_1, x)\geqq 2$, this equalities imply

$$(4.2)\qquad \langle d(H(\mu_1)-x), Y\rangle=0, \qquad \text{for} \quad Y\perp E(\mu_1, x) \quad \text{and} \quad H(\mu_1)-x.$$

By means of (4.1) and (4.2), we can easily see that the Euclidean (m_1+1)-vector $e_1\wedge\cdots\wedge e_{m_1}\wedge(H(\mu_1)-x)$ is parallel to a fixed one along K^{m_1}. Hence, there exists an m_1+1 dimensional Euclidean plane $E^{m_1+1}\supset K^{m_1}$. Q.E.D.

LEMMA 6. *Let M^n be as in Theorem 2 and $\bar{M}^{n+1}=S^{n+1}$. Then, at least two of the vector fields $H(\mu_1)$, $H(\mu_2)$ and $H(\mu_3)$ do not vanish identically.*

Proof. Assume that $H(\mu_1)\equiv H(\mu_2)\equiv 0$, Then, from (3.13) we obtain

$$\nabla f_1=\nabla f_2=\nabla f_3=0, \qquad f_1=f_2=f_3.$$

Hence, f_1, f_2 and f_3 are the same constant, and so $\mu_1=\mu_2=\mu_3=0$ by Lemma 4. This is a contradiction. Q.E.D.

In the following of this section, we shall consider the case:

$$(4.3)\qquad H(\mu_1)\equiv 0 \quad \text{on} \quad M.$$

From (3.13) we may put

$$f_2=f_3=a \quad \text{(constant)}. \tag{4.4}$$

By virtue of the proof of Lemma 5, we see that the (m_1+1)-plane E^{m_1+1} passes through the origin of R^{n+2} in the present case, that is, the sphere$\supset K^{m_1}$ is an m_1-dimensional great sphere. We have from Lemma 4, (3.9), (3.8) and (3.13) the following:

$$\mu_1=0, \quad \mu_2=-\frac{1}{m_2}(f_1-a), \quad \mu_3=\frac{1}{m_3}(f_1-a), \tag{4.5}$$

$$\begin{cases} \omega_{\alpha_2\alpha_3}=0, & \omega_{\alpha_3\alpha_1}=\dfrac{1}{f_1-a}f_{1,\alpha_1}\omega_{\alpha_3}, \quad \omega_{\alpha_1\alpha_2}=-\dfrac{1}{f_1-a}f_{1,\alpha_1}\omega_{\alpha_2} \\ \omega_{\alpha_1(n+1)}=0, & \omega_{\alpha_2(n+1)}=-\dfrac{f_1-a}{m_2}\omega_{\alpha_2}, \quad \omega_{\alpha_3(n+1)}=\dfrac{f_1-a}{m_3}\omega_{\alpha_3}, \end{cases} \tag{4.6}$$

$$\begin{cases} d\omega_{\alpha_1}=\sum_{\beta_1}\omega_{\alpha_1\beta_1}\wedge\omega_{\beta_1} \\ d\omega_{\alpha_2}=-\dfrac{1}{f_1-a}df_1\wedge\omega_{\alpha_2}+\sum_{\beta_2}\omega_{\alpha_2\beta_2}\wedge\omega_{\beta_2}, \\ d\omega_{\alpha_3}=-\dfrac{1}{f_1-a}df_1\wedge\omega_{\alpha_3}+\sum_{\beta_3}\omega_{\alpha_3\beta_3}\wedge\omega_{\beta_3} \end{cases} \tag{4.7}$$

and

$$H(\mu_2)=\frac{1}{f_1-a}\nabla f_1-\frac{f_1-a}{m_2}e_{n+1}, \quad H(\mu_3)=\frac{1}{f_1-a}\nabla f_1+\frac{f_1-a}{m_3}e_{n+1}. \tag{4.8}$$

LEMMA 7. *Let M^n be as in Theorem 2 and $\bar{M}^{n+1}=S^{n+1}$. If $H(\mu_1)\equiv 0$, then f_1 can not be constant.*

Proof. Let us suppose that $H(\mu_1)\equiv 0$ and f_1 is a constant, then from Theorem 1 and (2.3) we obtain $\Psi\equiv 0$, which is impossible in the present case by Theorem B. Q.E.D.

By this lemma, we see that f_1 is a non-constant function of $u_1, \cdots, u_{m_1}$. Now, we consider an integral submanifold $K^{m_2+m_3}$ of the distribution $E(\mu_2)+E(\mu_3)$. Along $K^{m_2+m_3}$, by (4.5) and (4.6) we have

$$\begin{cases} dx=\sum_{\alpha_2}\omega_{\alpha_2}e_{\alpha_2}+\sum_{\alpha_3}\omega_{\alpha_3}e_{\alpha_3}, \\ de_{\alpha_2}=\sum_{\beta_2}\omega_{\alpha_2\beta_2}e_{\beta_2}+(H(\mu_2)-x)\omega_{\alpha_2}, \\ de_{\alpha_3}=\sum_{\beta_3}\omega_{\alpha_3\beta_3}e_{\beta_3}+(H(\mu_3)-x)\omega_{\alpha_3}, \\ de_{\alpha_1}=-\dfrac{1}{f_1-a}f_{1,\alpha_1}dx+\sum_{\beta_1}\omega_{\alpha_1\beta_1}e_{\beta_1}, \\ de_{n+1}=(f_1-a)\left\{\dfrac{1}{m_2}\sum_{\alpha_2}\omega_{\alpha_2}e_{\alpha_2}-\dfrac{1}{m_3}\sum_{\alpha_3}\omega_{\alpha_3}e_{\alpha_3}\right\}. \end{cases} \tag{4.9}$$

LEMMA 8. *Let M^n be as in Theorem 2 and $\bar{M}^{n+1}=S^{n+1}$. If $H(\mu_1)\equiv 0$, then any integral submanifold $K^{m_2+m_3}$ of $E(\mu_2)+E(\mu_3)$ is locally contained in an (m_2+m_3+2)-dimensional Euclidean plane in R^{n+2} not containing the origin.*

Proof. From (4.9), we get easily

$$0=d^2e_{\alpha_2}=\sum_{\beta_2}d\omega_{\alpha_2\beta_2}e_{\beta_2}-\sum_{\beta_2}\omega_{\alpha_2\beta_2}\wedge\{\sum_{\gamma_2}\omega_{\beta_2\gamma_2}e_{\gamma_2}+(H(\mu_2)-x)\omega_{\beta_2}\}$$
$$+(H(\mu_2)-x)d\omega_{\alpha_2}+d(H(\mu_2)-x)\wedge\omega_{\alpha_2},$$

i. e.

$$\sum_{\beta_2}\{d\omega_{\alpha_2\beta_2}-\sum_{\gamma_2}\omega_{\alpha_2\gamma_2}\wedge\omega_{\gamma_2\beta_2}\}e_{\beta_2}$$
$$+\{d\omega_{\alpha_2}-\sum_{\beta_2}\omega_{\alpha_2\beta_2}\wedge\omega_{\beta_2}\}\left\{\left(\frac{1}{f_1-a}\nabla f_1-x\right)-\frac{f_1-a}{m_2}e_{n+1}\right\}$$
$$+\left\{d\left(\frac{1}{f_1-a}\nabla f_1-x\right)-\frac{f_1-a}{m_2}de_{n+1}\right\}\wedge\omega_{\alpha_2}=0,$$

and analogously

$$\sum_{\beta_3}\{d\omega_{\alpha_3\beta_3}-\sum_{\gamma_3}\omega_{\alpha_3\gamma_3}\wedge\omega_{\gamma_3\beta_3}\}e_{\beta_3}$$
$$+\{d\omega_{\alpha_3}-\sum_{\beta_3}\omega_{\alpha_3\beta_3}\wedge\omega_{\beta_3}\}\left\{\left(\frac{1}{f_1-a}\nabla f_1-x\right)+\frac{f_1-a}{m_3}e_{n+1}\right\}$$
$$+\left\{d\left(\frac{1}{f_1-a}\nabla f_1-x\right)+\frac{f_1-a}{m_3}de_{n+1}\right\}\wedge\omega_{\alpha_3}=0.$$

Hence, for any vector Y orthogonal to $E(\mu_2, x)+E(\mu_3, x)$, e_{n+1} and $\frac{1}{f_1-a}\nabla f_1-x$, we have

$$d\left\langle\frac{1}{f_1-a}\nabla f_1-x, Y\right\rangle\wedge\omega_{\alpha_2}=0, \qquad d\left\langle\frac{1}{f_1-a}\nabla f_1-x, Y\right\rangle\wedge\omega_{\alpha_3}=0,$$

which imply

$$\left\langle d\left(\frac{1}{f_1-a}\nabla f_1-x\right), Y\right\rangle=0 \tag{4.10}$$
$$\text{for } Y\perp E(\mu_2, x),\ E(\mu_3, x),\ e_{n+1},\ \frac{1}{f_1-a}\nabla f_1-x.$$

Now, by means of (4.9) and (4.10), we obtain easily the equality

$$d\left\{\left(\frac{1}{f_1-a}\nabla f_1-x\right)\wedge e_{m_1+1}\wedge\cdots\wedge e_n\wedge e_{n+1}\right\}$$
$$=d\left(\frac{1}{f_1-a}\nabla f_1-x\right)\wedge e_{m_1+1}\wedge\cdots\wedge e_n\wedge e_{n+1}$$

$$=\frac{1}{\left|\frac{1}{f_1-a}\nabla f_1\right|^2+1}\left\langle d\left(\frac{1}{f_1-a}\nabla f_1-x\right), \frac{1}{f_1-a}\nabla f_1-x\right\rangle$$

$$\times\left(\frac{1}{f_1-a}\nabla f_1-x\right)\wedge e_{m_1+1}\wedge \cdots \wedge e_n\wedge e_{n+1}$$

along $K^{m_2+m_3}$, which shows that the simple (m_2+m_3+2)-vector field

$$\left(\frac{1}{f_1-a}\nabla f_1-x\right)\wedge e_{m_1+1}\wedge \cdots \wedge e_n\wedge e_{n+1}$$

is parallel to a fixed (m_2+m_3+2)-dimensional direction in R^{n+2}. Therefore, there exists an (m_2+m_3+2)-dimensional Euclidean plane $\tilde{E}^{m_2+m_3+2}\supset K^{m_2+m_3}$. Since $\nabla f_1 \neq 0$, $\tilde{E}^{m_2+m_3+2}$ can not contain the origin of R^{n+2}. Q. E. D.

LEMMA 9. *Let M^n be as in Lemma 8 with $H(\mu_1)\equiv 0$. Then $K^{m_2+m_3}$ is a Riemannian product of an m_2-dimensional sphere and an m_3-dimensional sphere.*

Proof. By Lemma 8, $K^{m_2+m_3}$ is contained in the (m_2+m_3+1)-dimensional sphere $\tilde{S}^{m_2+m_3+1}:=S^{n+1}\cap\tilde{E}^{m_2+m_3+2}$ as a hypersurface with normal unit vector field e_{n+1}. In $\tilde{E}^{m_2+m_3+2}$ with the origin $\frac{1}{f_1-a}\nabla f_1$, the position vector of x is given by

$$y:=x-\frac{1}{f_1-a}\nabla f_1.$$

Hence, in $\tilde{E}^{m_2+m_3+2}$, (4.9) can be written as

$$dy=\sum_{\alpha_2}\omega_{\alpha_2}e_{\alpha_2}+\sum_{\alpha_3}\omega_{\alpha_3}e_{\alpha_3}$$

$$de_{\alpha_2}=\sum_{\beta_2}\omega_{\alpha_2\beta_2}e_{\beta_2}-\frac{f_1-a}{m_2}\omega_{\alpha_2}e_{n+1}-\omega_{\alpha_2}y,$$

$$\text{(4.11)}\qquad de_{\alpha_3}=\sum_{\beta_3}\omega_{\alpha_3\beta_3}e_{\beta_3}+\frac{f_1-a}{m_3}\omega_{\alpha_3}e_{n+1}-\omega_{\alpha_3}y,$$

$$de_{n+1}=(f_1-a)\left\{\frac{1}{m_2}\sum_{\alpha_2}\omega_{\alpha_2}e_{\alpha_2}-\frac{1}{m_3}\sum_{\alpha_3}\omega_{\alpha_3}e_{\alpha_3}\right\}.$$

These equalities show that $K^{m_2+m_3}$ is a minimal hypersurface in $\tilde{S}^{m_2+m_3+1}$ with two principal curvatures $-\frac{f_1-a}{m_2}$ and $\frac{f_1-a}{m_3}$ of multiplicities m_2 and m_3 respectively. Hence, by Theorem 3 in [3], $K^{m_2+m_3}$ is locally a Riemannian product of two spheres of dimension m_2 and m_3 respectively. Q. E. D.

Remark 1. The two spheres in Lemma 9 can be considered as integral submanifolds of the distribution $E(\mu_2)$ and $E(\mu_3)$ respectively. On the other hand, by means of the equalities in the proof Lemma 8 we obtain

$$d\omega_{\alpha_2\beta_2}-\sum_{\gamma_2}\omega_{\alpha_2\gamma_2}\wedge\omega_{\gamma_2\beta_2}=\Big\{\langle e_{\beta_2}, dy\rangle+\frac{f_1-a}{m_2}\langle e_{\beta_2}, de_{n+1}\rangle\Big\}\wedge\omega_{\alpha_2}$$

$$=-\Big\{1+\Big(\frac{f_1-a}{m_2}\Big)^2\Big\}\omega_{\alpha_2}\wedge\omega_{\beta_2},$$

$$d\omega_{\alpha_3\beta_3}-\sum_{\gamma_3}\omega_{\alpha_3\gamma_3}\wedge\omega_{\gamma_3\beta_3}=\Big\{\langle e_{\beta_3}, dy\rangle-\frac{f_1-a}{m_3}\langle e_{\beta_3}, de_{n+1}\rangle\Big\}\wedge\omega_{\alpha_3}$$

$$=-\Big\{1+\Big(\frac{f_1-a}{m_3}\Big)^2\Big\}\omega_{\alpha_3}\wedge\omega_{\beta_3}.$$

Hence, in $\tilde{E}^{m_2+m_3+2}$ we have locally the product:

$$K^{m_2+m_3}\cong\tilde{S}^{m_2}\Big(\Big\{1+\Big(\frac{f_1-a}{m_2}\Big)^2\Big\}^{-1/2}\Big)\times\tilde{S}^{m_3}\Big(\Big\{1+\Big(\frac{f_1-a}{m_3}\Big)^2\Big\}^{-1/2}\Big), \tag{4.12}$$

where $\tilde{S}^m(r)$ denotes the m-dimensional sphere of radius r.

LEMMA 10. *Let M^n be as in Lemma* 8 *with $H(\mu_1)\equiv 0$. Then, f_1 satisfies*:

$$|\nabla f_1|^2=\frac{1}{m_2m_3}(f_1-a)^4-(f_1-a)^2. \tag{4.13}$$

Proof. By means of (4.6) we have

$$0=d\omega_{\alpha_2\alpha_3}=\sum_{j=1}^{n}\omega_{\alpha_2 j}\wedge\omega_{j\alpha_3}+\omega_{\alpha_2(n+1)}\wedge\omega_{(n+1)\alpha_3}-\omega_{\alpha_2}\wedge\omega_{\alpha_3}$$

$$=\sum_{\beta_1}\omega_{\alpha_2\beta_1}\wedge\omega_{\beta_1\alpha_3}-(\mu_2\mu_3+1)\omega_{\alpha_2}\wedge\omega_{\alpha_3}$$

$$=-\Big\{\frac{1}{(f_1-a)^2}\sum_{\beta_1}(f_{1,\beta_1})^2-\frac{(f_1-a)^2}{m_2m_3}+1\Big\}\omega_{\alpha_2}\wedge\omega_{\alpha_3},$$

from which we get immediately (4.13). Q.E.D.

Remark 2. Setting $\rho=\dfrac{1}{f_1-a}$, (4.13) becomes

$$|\nabla\rho|^2+\rho^2=\frac{1}{m_2m_3}\quad\text{on } M^n. \tag{4.14}$$

Let C be an orthogonal trajectory of the function f_1 parameterized with arclength s, then we get from (4.14)

$$(d\rho/ds)^2+\rho^2=\frac{1}{m_2m_3},$$

hence by integrating along C we may put

$$\rho=\frac{1}{\sqrt{m_2m_3}}\sin(s+c_0), \tag{4.15}$$

where c_0 is a constant and $\sin(s+c_0)\neq 0$. This curve C lies in an integral submanifold of $E(\mu_1)$, which is a great m_1-sphere of S^{n+1} in the present case.

LEMMA 11. *Let M be an n-dimensional Riemannian manifold and u a non-constant function such that $|\nabla u|^2$ is a non-zero function of u only. Then, the integral curves of ∇u are geodesics of M with certain parameters.*

Proof. We choose local coordinates $x^1, \cdots, x^{n-1}, x^n$ such that $u=x^n$ and the metric of M takes the form:

$$ds^2=\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(x)dx^\alpha dx^\beta+g_{nn}(x)dx^n dx^n .$$

Then, we have $|\nabla u|^2=g^{nn}(x)$. From the assumption, we get

$$\partial g^{nn}/\partial x^\alpha=-2g^{nn}\Gamma^n_{n\alpha}=0 ,$$

which imply $\partial g_{nn}/\partial x^\alpha=0$, $\alpha=1, 2, \cdots, n-1$, where Γ^i_{jk} are the Christoffel symbols of the Riemannian connection of M.

On the other hand, the equations of a geodesic with respect to any parameter t are

$$\frac{\dfrac{d^2x^1}{dt^2}+\sum_{j,k}\Gamma^1_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt}}{\dfrac{dx^1}{dt}}=\cdots\cdots=\frac{\dfrac{d^2x^n}{dt^2}+\sum_{j,k}\Gamma^n_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt}}{\dfrac{dx^n}{dt}}.$$

Now, for any curve $x^\alpha=$constant, $\alpha=1, 2, \cdots, n-1$, and $x^n=t$, we have

$$\frac{d^2x^\alpha}{dt^2}+\sum_{j,k}\Gamma^\alpha_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=\Gamma^\alpha_{nn}=-\frac{1}{2}\sum_{\beta=1}^{n-1}g^{\alpha\beta}\frac{\partial g_{nn}}{\partial x^\beta}=0$$

and

$$\frac{d^2x^n}{dt^2}+\sum_{j,k}\Gamma^n_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=\Gamma^n_{nn}=\frac{1}{2}g^{nn}\frac{\partial g_{nn}}{\partial x^n}.$$

These equalities show that this curve is a geodesic of M. Q.E.D.

Remark 3. In Lemma 11, if we suppose thal $|\nabla u|^2=F(u)$, then we obtain from the above computation $g_{nn}(x^n)=1/F(x^n)$, Therefore, the arclength s of the curve between $u=a$ and $u=b$ $(a<b)$ is given by

$$s=\int_a^b\frac{du}{\sqrt{F(u)}}.$$

Remark 4. By means of Lemma 11, Remark 2, Remark 3 and (4.14) the orthogonal trajectories of the function f_1 are all great circles of S^{n+1}. From these lemmas and remarks, we get the following theorem.

THEOREM 3. *Let M^n be as in Theorem 2 and $\bar{M}^{n+1}=S^{n+1}$ and $H(\mu_1)\equiv 0$. Then, we have the following*:

(i) *There exist a constant a and a non-trivial function f such that*

$$\mu_1=0, \qquad \mu_2=-\frac{1}{m_2}(f-a), \qquad \mu_3=\frac{1}{m}(f-a).$$

(ii) *Any integral submanifold of the distribution $E(\mu_1)$ is an m_1-dimensional great sphere of S^{n+1} and any integral submanifolds of the distributions $E(\mu_2)$ and $E(\mu_3)$ are m_2 and m_3-dimensional small spheres of S^{n+1} respectively.*

(iii) *Any integral submanifold of the distribution $E(\mu_2)+E(\mu_3)$ is a Riemannian product of an m_2-dimensional sphere and an m_2-dimensional sphere.*

(iv) *Any orthogonal trajectory C of the level hypersurfaces of the function f is an arc of a great circle of S^{n+1}.*

(v) *As function of the arclength s of C, f can be written as*

$$f=a+\frac{\sqrt{m_2 m_3}}{\sin(s+c_0)},$$

where c_0 is a constant.

(v) of Theorem 3 implies immediately the following fact.

COROLLARY. *Any n-dimensional complete Riemannian manifold can not be isometrically immersed in S^{n+1} as in Theorem 3.*

§ 5. An example of minimal hypersurfaces in S^{n+1} for Theorem 3

In this section, we shall give an example of minimal hypersurfaces in S^{n+1} with three non-simple regular principal curvatures μ_1, μ_2 and μ_3 such that $H(\mu_1)\equiv 0$, by making use of the facts obtained in § 4.

Let m_1, m_2 and m_3 be any integers greater than 1 and put $n=m_1+m_2+m_3$. Let us consider as

$$R^{n+2}=R^{m_1}\times R^{m_2+1}\times R^{m_3+1}.$$

First of all, we take two hyperspheres in R^{m_2+1} and R^{m_3+1} as follows:

$$\tilde{S}_0^{m_2}(r_2)\subset R^{m_2+1}, \qquad r_2=\left\{1+\left(\frac{b-a}{m_2}\right)^2\right\}^{-1/2},$$

$$\tilde{S}_0^{m_3}(r_3)\subset R^{m_3+1}, \qquad r_3=\left\{1+\left(\frac{b-a}{m_3}\right)^2\right\}^{-1/2}$$

and put

$$K_0^{m_2+m_3}=\tilde{S}_0^{m_2}(r_2)\times\tilde{S}_0^{m_3}(r_3), \tag{5.1}$$

which is contained in the (m_2+m_3+1)-dimensional sphere $\tilde{S}_0^{m_2+m_3+1}(r_0)$ in $R^{m_2+1}\times R^{m_3+1}$, where

$$r_0^2=r_2^2+r_3^2=\left\{2+\left(\frac{1}{m_2^2}+\frac{1}{m_3^2}\right)(b-a)^2\right\}\Big/\left\{1+\left(\frac{b-a}{m_2}\right)^2\right\}\left\{1+\left(\frac{b-a}{m_3}\right)^2\right\}. \tag{5.2}$$

From (5.2), we have

$$1-r_0^2=\left\{\frac{(b-a)^4}{(m_2m_3)^2}-1\right\}\Big/\left\{1+\left(\frac{b-a}{m_2}\right)^2\right\}\left\{1+\left(\frac{b-a}{m_3}\right)^2\right\}:=r_1^2 \tag{5.3}$$

and so we suppose here the inequality:

$$(b-a)^2>m_2m_3\,. \tag{5.4}$$

Let η be a normal unit vector field of $K_0^{m_2+m_3}$ in $\tilde{S}_0^{m_2+m_3+1}(r_0)$.

Next, we take the hypersphere $\tilde{S}_0^{m_1-1}(r_1)\subset R^{m_1}$, then we have

$$\tilde{S}_0^{m_1-1}(r_1)\times K_0^{m_2+m_3}\subset S^{n+1}\,.$$

Let $x_0=(y_0, z_0)$ be any point of $\tilde{S}_0^{m_1-1}(r_1)\times K_0^{m_2+m_3}$ and $E_{x_0}^{m_1}$ be the m_1-dimensional linear subspace of R^{n+2} which is tangent to S^{n+1} at x_0, orthogonal to $O\times T_{z_0}(K_0^{m_2+m_3})$ and $O\times\eta(z_0)$. Let $S_{x_0}^{m_1}$ be the great m_1-sphere of S^{n+1} which is the intersection of S^{n+1} and (m_1+1)-dimensional linear subspace including $E_{x_0}^{m_1}$ and the origin of R^{n+2}. Then, we define a hypersurface of S^{n+1} by

$$M^n:=\cup\{S_{x_0}^{m_1}\,|\,x_0\in\tilde{S}_0^{m_1-1}(r_1)\times K_0^{m_2+m_3}\}\,. \tag{5.5}$$

In the rest of this section, we shall prove that this M^n is a minimal hypersurface of S^{n+1} as in Theorem 3 under an additional condition.

First, we define a tangent unit vector field ξ of M^n by the following way. At the point x_0 above, $\xi(x_0)$ be one of the tangent unit vectors to $S_{x_0}^{m_1}$ orthogonal to $T_{y_0}(S_0^{m_1-1}(r_1))\times O$, since this makes sense by means of $\tilde{S}_0^{m_1-1}(r_1)\times z_0\subset S_{x_0}^{m_1}$. Then, we extend the domain of definition of ξ along the great circle C_{x_0} of S^{n+1} which passes through x_0 and has $\xi(x_0)$ as the tangent vector. Let C_{x_0} be parameterized with arclength s such that

$$C_{x_0}:\ x=\gamma_{x_0}(s) \qquad \text{with} \quad x_0=\gamma_{x_0}(0)\,. \tag{5.6}$$

Then, considering (v) in Theorem 3, we put

$$c_0=\mathrm{Sin}^{-1}\frac{\sqrt{m_2m_3}}{b-a} \tag{5.7}$$

and define a function f on M^n by

$$f(x)=a+\frac{\sqrt{m_2m_3}}{\sin(s+c_0)}\,, \tag{5.8}$$

where $x=\gamma_{x_0}(s)$ and $0<s+c_0<\pi$.

Now, we compute the second fundamental form of M^n in S^{n+1}. Setting

$$x=(y,z)\,,\qquad z=(u,v)\,,\qquad y\in R^{m_1}\,,\qquad u\in R^{m_2+1}\,,\qquad v\in R^{m_3+1}$$

and $z_0=(u_0, v_0)$, we have $|y_0|=r_1$, $|u_0|=r_2$, $|v_0|=r_3$. The normal unit vector $\eta(z_0)$ at z_0 of $K_0^{m_2+m_3}$ in $S_0^{m_2+m_3+1}(r_0)$ is given by

$$\eta(z_0)=\left(\frac{r_3}{r_0r_2}u_0,\ -\frac{r_2}{r_0r_3}v_0\right). \tag{5.9}$$

Analogously, we can obtain

$$\xi(x_0)=\left(-\frac{r_0}{r_1}y_0,\ \frac{r_1}{r_0}u_0,\ \frac{r_1}{r_0}v_0\right). \tag{5.10}$$

Hence, we get

$$\begin{aligned} x&=x_0\cos\theta+\xi(x_0)\sin\theta\\ &=\left(\left(\cos\theta-\frac{r_0}{r_1}\sin\theta\right)y_0,\ \left(\cos\theta+\frac{r_1}{r_0}\sin\theta\right)u_0,\ \left(\cos\theta+\frac{r_1}{r_0}\sin\theta\right)v_0\right). \end{aligned} \tag{5.11}$$

Next, we compute the normal unit vector $N(x_0)$ of M^n in S^{n+1} at x_0 by means of (5.10). We get easily

$$N(x_0)=\left(0,\ \frac{r_3}{r_0r_2}u_0,\ -\frac{r_2}{r_0r_3}v_0\right)=(0,\ \eta(z_0)). \tag{5.12}$$

From (5.11), we have at x_0

$$d^2x:\begin{cases} d^2y=d^2y_0-2\dfrac{r_0}{r_1}d\theta\, dy_0-\left((d\theta)^2+\dfrac{r_0}{r_1}d^2\theta\right)y_0\,,\\[2ex] d^2u=d^2u_0+2\dfrac{r_1}{r_0}d\theta\, du_0-\left((d\theta)^2-\dfrac{r_1}{r_0}d^2\theta\right)u_0\,,\\[2ex] d^2v=d^2v_0+2\dfrac{r_1}{r_0}d\theta\, dv_0-\left((d\theta)^2-\dfrac{r_1}{r_0}d^2\theta\right)v_0 \end{cases}$$

and hence we get

$$\begin{aligned} \langle d^2x, N(x_0)\rangle&=\langle d^2y, 0\rangle+\frac{r_3}{r_0r_2}\langle d^2u, u_0\rangle-\frac{r_2}{r_0r_3}\langle d^2v, v_0\rangle\\ &=\frac{r_3}{r_0r_2}\left\{\langle d^2u_0, u_0\rangle-r_2^2\left((d\theta)^2-\frac{r_1}{r_0}d^2\theta\right)\right\}\\ &\quad-\frac{r_2}{r_0r_3}\left\{\langle d^2v_0, v_0\rangle-r_3^2\left((d\theta)^2-\frac{r_1}{r_0}d^2\theta\right)\right\} \end{aligned}$$

i. e.

$$\langle d^2x, N(x_0)\rangle=-\frac{r_3}{r_0r_2}\langle du_0, du_0\rangle+\frac{r_2}{r_0r_3}\langle dv_0, dv_0\rangle\,. \tag{5.13}$$

On the other hand, we have also at x_0

$$dx:\begin{cases} dy=dy_0-\dfrac{r_0}{r_1}d\theta\, y_0\,,\\[2ex] du=du_0+\dfrac{r_1}{r_0}d\theta\, u_0\,,\\[2ex] dv=dv_0+\dfrac{r_1}{r_0}d\theta\, v_0\,, \end{cases}$$

and hance we get

$$ds^2=\langle dy_0, dy_0\rangle+d\theta^2+\langle du_0, du_0\rangle+\langle dv_0, dv_0\rangle\,. \tag{5.14}$$

From (5.13) and (5.14), we see that M^n have three principal curvatures $\mu_1=0$, $\mu_2=-\frac{r_3}{r_0r_2}$, $\mu_3=\frac{r_2}{r_0r_3}$ of multiplicities m_1, m_2, m_3 respectively at x_0. From the argument above, especially the formula (5.11), we see that μ_1, μ_2, μ_3 can be considered as regular fields and $\mu_1\equiv 0$, by replacing r_2 and r_3 with

$$\bar{r}_2=\left(\cos\theta+\frac{r_1}{r_0}\sin\theta\right)r_2\,,\qquad \bar{r}_3=\left(\cos\theta+\frac{r_1}{r_0}\sin\theta\right)r_3$$

respectively. Then, we get easily

$$H(\mu_1)\equiv 0\,. \tag{5.15}$$

Finally, we shall check the condition that M^n is minimal in S^{n+1}. We have

$$\begin{aligned} m_2\mu_2+m_3\mu_3&=-\frac{m_2\bar{r}_3}{\bar{r}_0\bar{r}_2}+\frac{m_3\bar{r}_2}{\bar{r}_0\bar{r}_3} \\ &=\frac{\bar{r}_2\bar{r}_3}{\bar{r}_0}\left[-\frac{m_2}{\bar{r}_2^2}+\frac{m_3}{\bar{r}_3^2}\right]=\frac{r_2r_3}{\bar{r}_0}\left[-\frac{m_2}{r_2^2}+\frac{m_3}{r_3^2}\right] \\ &=\frac{r_2r_3}{\bar{r}_0}\left[-(m_2-m_3)+(b-a)^2\left(\frac{1}{m_3}-\frac{1}{m_2}\right)\right], \end{aligned}$$

that is

$$m_2\mu_2+m_3\mu_3=\frac{(m_2-m_3)r_2r_3}{m_2m_3\bar{r}_0}\{(b-a)^2-m_2m_3\}\,, \tag{5.16}$$

where $\bar{r}_0=\sqrt{\bar{r}_2^2+\bar{r}_3^2}$. Accordingly, by (5,4), M^n is minimal if and only if

$$m_2=m_3\,. \tag{5.17}$$

Now, assuming (5.17), at a general point x we have

$$\mu_2=-\mu_3=-\frac{\bar{r}_3}{\bar{r}_0\bar{r}_2}=-\frac{1}{\sqrt{2}\,\bar{r}_2}=-\left\{\sqrt{2}\left(\cos\theta+\frac{r_1}{r_0}\sin\theta\right)r_2\right\}^{-1}.$$

On the other hand, from (5.3), (5.7) and (5.17) we have

$$\frac{r_1}{r_0}=\frac{\sqrt{1-2r_2^2}}{\sqrt{2}\,r_2}=\sqrt{\frac{1}{2}\left\{1+\left(\frac{b-a}{m_2}\right)^2\right\}-1}=\frac{1}{\sqrt{2}\,|\tan c_0|},$$

$$r_2^2=\left\{1+\left(\frac{b-a}{m_2}\right)^2\right\}^{-1}=\frac{\sin^2 c_0}{1+\sin^2 c_0},$$

and

$$2\left(\cos\theta+\frac{r_1}{r_0}\sin\theta\right)^2 r_2^2=2\left(\cos\theta+\frac{\sin\theta}{\sqrt{2}\,|\tan c_0|}\right)^2\times\frac{\sin^2 c_0}{1+\sin^2 c_0}$$

$$=\frac{\{\pm\cos\theta\cdot\sqrt{2}\sin c_0+\sin\theta\cdot\cos c_0\}^2}{1+\sin^2 c_0}=\sin^2(\theta+\theta_0),$$

where

$$\sin\theta_0=\pm\frac{\sqrt{2}\sin c_0}{\sqrt{1+\sin^2 c_0}},\qquad \cos\theta_0=\frac{\cos c_0}{\sqrt{1+\sin^2 c_0}},$$

Hence, we may put

$$\mu_2=-\mu_3=-\frac{1}{\sin(\theta+\theta_0)}. \tag{5.18}$$

Thus, we obtain a conclusion as follows:

THEOREM 4. *Let M^n be a hypersurface constructed by* (5.5) *in S^{n+1}. Then, it is an example of minimal hypersurfaces as in Theorem* 3, *if and only if $m_2=m_3$.*

Remark 5. Since we took $\tilde{S}_0^{m_1-1}\times K_0^{m_2+m_3}\subset S^{n+1}$ as a base hypersurface of M^n for our construction, the above argument does not entirely treat with this kind of minimal hypersurfaces in S^{n+1}.

§ 6. The case in which $H(\mu_i)\not\equiv 0$, i=1, 2, 3

In this section, we shall investigate minimal hypersurfaces as in Theorem 2 with $H(\mu_i)\not\equiv 0$, $i=1, 2, 3$, in the case of $\bar{M}^{n+1}=S^{n+1}$.

By (3.13), $H(\mu_1)\equiv 0$ is equivalent to the condition

$$f_2=f_3=a \text{ constant}.$$

LEMMA 12. *Let M^n be a minimal hypersurface in S^{n+1} as in Theorem* 2 *with $H(\mu_i)\not\equiv 0$, $i=1, 2, 3$, then f_i in Lemma* 4 *can not be constant for $i=1, 2, 3$.*

Proof. Let M^n be a hypersurface as in the statement. Let us suppose that $f_i=a_i$, $i=1, 2, 3$, where a_1, a_2 and a_3 are constants different from each others. Then, by (3.9) we have the equation

$$\omega_{\alpha_2\alpha_3}=\omega_{\alpha_3\alpha_1}=\omega_{\alpha_1\alpha_2}=0.$$

Hence we have from (1.3)

$$DA_{\alpha_i\alpha_j}=dA_{\alpha_i\alpha_j}-\sum_{k=1}^{n}\omega_{\alpha_i k}A_{k\alpha_j}-\sum_{k=1}^{n}\omega_{\alpha_j k}A_{\alpha_i k}$$

$$=-\mu_j\omega_{\alpha_i\alpha_j}-\mu_i\omega_{\alpha_j\alpha_i}=(\mu_i-\mu_j)\omega_{\alpha_i\alpha_j}=0,$$

for $i, j=1, 2, 3$, which imply $\Psi\equiv 0$. This contradicts to Theorem B. Q. E. D.

By means of Lemma 12, we may consider the case:

(6.1) $\quad f_1=f$ is a non-constant function, $f_2=a_2$, $f_3=a_3$, $a_2\neq a_3$, where a_2 and a_3 are constants, as the simplest one in the present situation. In the following, we shall investigate this case.

Under the condition (6.1), we have from Lemma 4 and (3.13)

$$\mu_1=\frac{a_2-a_3}{m_1}, \qquad \mu_2=\frac{a_3-f}{m_2}, \qquad \mu_3=\frac{f-a_2}{m_3}, \tag{6.2}$$

$$\begin{cases} H(\mu_1)=\mu_1 e_{n+1}, \\ H(\mu_2)=\dfrac{m_1}{F-na_3}\nabla f+\mu_2 e_{n+1}, \qquad H(\mu_3)=\dfrac{m_1}{F-na_2}\nabla f+\mu_3 e_{n+1} \end{cases} \tag{6.3}$$

where

$$F=m_1 f+m_2 a_2+m_3 a_3, \qquad n=m_1+m_2+m_3. \tag{6.4}$$

From (3.8) and (3.9) we have

$$\begin{cases} d\omega_{\alpha_1}=\sum_{\beta_1}\omega_{\beta_1}\wedge\omega_{\beta_1\alpha_1}, \\ d\omega_{\alpha_2}=-\dfrac{m_1}{F-na_3}df\wedge\omega_{\alpha_2}+\sum_{\beta_2}\omega_{\beta_2}\wedge\omega_{\beta_2\alpha_2}, \\ d\omega_{\alpha_3}=-\dfrac{m_1}{F-na_2}df\wedge\omega_{\alpha_3}+\sum_{\beta_3}\omega_{\beta_3}\wedge\omega_{\beta_3\alpha_3} \end{cases} \tag{6.5}$$

and

$$\omega_{\alpha_2\alpha_3}=0,\ \omega_{\alpha_3\alpha_1}=\frac{m_1}{F-na_2}f_{,\alpha_1}\omega_{\alpha_3}, \qquad \omega_{\alpha_1\alpha_2}=-\frac{m_1}{F-na_3}f_{,\alpha_1}\omega_{\alpha_2}. \tag{6.6}$$

Now, by means of (6.3) and (6.6), we have the following equalities along M^n:

$$\begin{cases} dx=\sum_{\alpha_1}\omega_{\alpha_1}e_{\alpha_1}+\sum_{\alpha_2}\omega_{\alpha_2}e_{\alpha_2}+\sum_{\alpha_3}\omega_{\alpha_3}e_{\alpha_3}, \\ de_{\alpha_1}=\sum_{\beta_1}\omega_{\alpha_1\beta_1}e_{\beta_1}-m_1 f_{,\alpha_1}\left\{\dfrac{1}{F-na_3}\sum_{\beta_2}\omega_{\beta_2}e_{\beta_2}+\dfrac{1}{F-na_2}\sum_{\beta_3}\omega_{\beta_3}e_{\beta_3}\right\} \\ \qquad +(\mu_1 e_{n+1}-x)\omega_{\alpha_1}, \\ de_{\alpha_2}=\sum_{\beta_2}\omega_{\alpha_2\beta_2}e_{\beta_2}+(H(\mu_2)-x)\omega_{\alpha_2}, \\ de_{\alpha_3}=\sum_{\beta_3}\omega_{\alpha_3\beta_3}e_{\beta_3}+(H(\mu_3)-x)\omega_{\alpha_3}, \\ de_{n+1}=-\mu_1\sum_{\alpha_1}\omega_{\alpha_1}e_{\alpha_1}-\mu_2\sum_{\alpha_2}\omega_{\alpha_2}e_{\alpha_2}-\mu_3\sum_{\alpha_3}\omega_{\alpha_3}e_{\alpha_3}. \end{cases} \tag{6.7}$$

From these equalities, we obtain

$$\begin{aligned} 0=d^2 e_{\alpha_2}=&\sum_{\beta_2}d\omega_{\alpha_2\beta_2}e_{\beta_2}-\sum_{\beta_2}\omega_{\alpha_2\beta_2}\wedge\{\sum_{\gamma_2}\omega_{\beta_2\gamma_2}e_{\gamma_2}+(H(\mu_2)-x)\omega_{\beta_2}\} \\ &+d(H(\mu_2)-x)\wedge\omega_{\alpha_2}+(H(\mu_2)-x)d\omega_{\alpha_2} \end{aligned}$$

i. e.

$$\sum_{\beta_2}(d\omega_{\alpha_2\beta_2}-\sum_{\gamma_2}\omega_{\alpha_2\gamma_2}\wedge\omega_{\gamma_2\beta_2})e_{\beta_2}+(d\omega_{\alpha_2}-\sum_{\beta_2}\omega_{\alpha_2\beta_2}\wedge\omega_{\beta_2})(H(\mu_2)-x) +d(H(\mu_2)-x)\wedge\omega_{\alpha_2}=0 \tag{6.8}$$

and analogously

$$\sum_{\beta_3}(d\omega_{\alpha_3\beta_3}-\sum_{\gamma_3}\omega_{\alpha_3\gamma_3}\wedge\omega_{\gamma_3\beta_3})e_{\beta_3}+(d\omega_{\alpha_3}-\sum_{\beta_3}\omega_{\alpha_3\beta_3}\wedge\omega_{\beta_3})(H(\mu_3)-x) +d(H(\mu_3)-x)\wedge\omega_{\alpha_3}=0\,. \tag{6.9}$$

Let Y_2 be any vector perpendicular to $E(\mu_2)$ and $H(\mu_2)-x$ at the point x, then we get from (6.8)

$$\langle Y_2,\, d(H(\mu_2)-x)\rangle\wedge\omega_{\alpha_2}=0\,.$$

Since $\dim E(\mu_2, x)\geqq 2$, this equality implies

$$\langle Y_2,\, d(H(\mu_2)-x)\rangle=0\,.$$

Hence, we have the following equality for the (m_2+1)-vector $(H(\mu_2)-x)\wedge e_{m_1+1}\wedge\cdots\wedge e_{m_1+m_2}$ in R^{n+2}

$$d\{(H(\mu_2)-x)\wedge e_{m_1+1}\wedge\cdots\wedge e_{m_1+m_2}\} =\frac{1}{|H(\mu_2)|^2+1}\langle d(H(\mu_2)-x),\, H(\mu_2)-x\rangle(H(\mu_2)-x)\wedge e_{m_1+1}\wedge\cdots\wedge e_{m_1+m_2} \tag{6.10}$$

and analogously

$$d\{(H(\mu_3)-x)\wedge e_{m_1+m_2+1}\wedge\cdots\wedge e_n\} =\frac{1}{|H(\mu_3)|^2+1}\langle d(H(\mu_3)-x),\, H(\mu_3)-x\rangle(H(\mu_3)-x)\wedge e_{m_1+m_2+1}\wedge\cdots\wedge e_n\,. \tag{6.11}$$

(6.10) and (6.11) imply that there exist two fixed (m_2+1)-plane $E_2^{m_2+1}$ and (m_3+1)-plane $E_3^{m_3+1}$ in R^{n+2} through the origin such that

$$(H(\mu_2)-x)\wedge e_{m_1+1}\wedge\cdots\wedge e_{m_1+m_2}//E_2^{m_2+1}\,,$$
$$(H(\mu_3)-x)\wedge e_{m_1+m_2+1}\wedge\cdots\wedge e_n//E_3^{m_3+1}\,.$$

Let $E_{\iota,x}^{m_\iota+1}$ be the $(m_\iota+1)$-plane through x parallel to $E_\iota^{m_\iota+1}$, $\iota=2, 3$. Then, we see easily that $E_{\iota,x}^{m_\iota+1}\cap S^{n+1}$ contains the integral submanifold of the distribution $E(\mu_\iota)$ through x. Since we have

$$x\wedge(H(\mu_2)-x)\wedge(H(\mu_3)-x)=x\wedge H(\mu_2)\wedge H(\mu_3)$$
$$=m_1\Big(\frac{f-a_2}{m_3(F-na_3)}+\frac{f-a_3}{m_2(F-na_2)}\Big)x\wedge\nabla f\wedge e_{n+1}\neq 0\,,$$

the (m_2+m_3+2)-plane containing $E_{2,x}^{m_2+1}$ and $E_{3,x}^{m_3+1}$, which we denote by $E_{2,3,x}^{m_2+m_3+2}$, does not contain the origin of R^{n+2}. Furthermore, the small (m_2+m_3+1)-sphere

$S^{m_2+m_3+1}_{2,3,x}=E^{m_2+m_3+2}_{2,3,x}\cap S^{n+1}$ contains the integral submanifold $K^{m_2+m_3}$ of the distribution $E(\mu_2)+E(\mu_3)$. Thus, we see that $K^{m_2+m_3}$ can be considered locally as a locus of moving $S^{m_2}_{2,x}=E^{m_2+1}_{2,x}\cap S^{n+1}$, where x is a moving point along $S^{m_3}_{3,x_0}=E^{m_3+1}_{3,x_0}\cap S^{n+1}$.

On the other hand, from (6.6) we have

$$\begin{aligned}0=d\omega_{\alpha_2\alpha_3}&=\sum_{j=1}^{n}\omega_{\alpha_2 j}\wedge\omega_{j\alpha_3}+\omega_{\alpha_2(n+1)}\wedge\omega_{(n+1)\alpha_3}-\omega_{\alpha_2}\wedge\omega_{\alpha_3}\\&=\sum_{\beta_1}\omega_{\alpha_2\beta_1}\wedge\omega_{\beta_1\alpha_3}-(\mu_2\mu_3+1)\omega_{\alpha_2}\wedge\omega_{\alpha_3}\\&=-\left\{\frac{m_1^2}{(F-na_2)(F-na_3)}\sum_{\beta_1}(f_{,\beta_1})^2-\frac{(f-a_2)(f-a_3)}{m_2m_3}+1\right\}\omega_{\alpha_2}\wedge\omega_{\alpha_3},\end{aligned}$$

from which we get the equality for f

$$\frac{m_1^2}{\{m_1f-(m_1+m_3)a_2+m_3a_3\}\{m_1f_1+m_2a_2-(m_1+m_2)a_3\}}|\nabla f|^2-\frac{(f-a_2)(f-a_3)}{m_2m_3}+1=0$$

i. e.

$$|\nabla f|^2=\left\{f-a_2-\frac{m_3}{m_1}(a_2-a_3)\right\}\left\{f-a_3-\frac{m_2}{m_1}(a_3-a_2)\right\}\times\left\{\frac{(f-a_2)(f-a_3)}{m_2m_3}-1\right\}. \tag{6.12}$$

Hence, we have

$$\langle H(\mu_2)-x,\ H(\mu_3)-x\rangle=\frac{m_1^2}{(F-na_2)(F-na_3)}|\nabla f|^2-\frac{(f-a_2)(f-a_3)}{m_2m_3}+1=0,$$

that is

$$(H(\mu_2)-x)\perp(H(\mu_3)-x), \tag{6.13}$$

and so

$$E_2^{m_2+1}\perp E_3^{m_3+1}. \tag{6.14}$$

We notice here that (6.12) will be reduced to (4.13), if we put $f=f_1$, $a_2=a_3=a$. And, by Lemma 11, any orthogonal trajectory C of the level hypersurfaces of the function f is an arc of a geodesic of M^n. C lies in an integral submanifold $K_1^{m_1}$ of the distribution $E(\mu_1)$.

Now, we prove the following

THEOREM 5. *Let M^n be a minimal hypersurface in S^{n+1} as in Theorem 2 with $H(\mu_i)\not\equiv 0$, $i=1, 2, 3$, then $d\mu_i\not\equiv 0$ for $i=1, 2, 3$.*

Proof. Let us suppose that $d\mu_1\equiv 0$, then f_2 and f_3 must be constants by Lemma 4. Setting $f_2=a_2$ and $f_3=a_3$, we have $a_2\neq a_3$ and $f_1=f$ is not a constant function by Lemma 12. Thus, this must be the case (6.1) and so we can use the argument above in this section. We may consider as

$$R^{n+2}=R_1^{m_1}\times R_2^{m_2+1}\times R_3^{m_3+1}$$

$$R_2^{m_2+1}=E_2^{m_2+1}, \qquad R_3^{m_3+1}=E_3^{m_3+1}, \qquad R_2\approx R_3\approx R=R_1$$

and we denote any point $x\in R^{n+2}$ as

$$x=(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v}), \qquad \boldsymbol{y}\in R_1^{m_1}, \qquad \boldsymbol{u}\in R_2^{m_2+1}, \qquad \boldsymbol{v}\in R_3^{m_3+1}.$$

Taking a fixed point $x_0=(\boldsymbol{y}_0, \boldsymbol{u}_0, \boldsymbol{v}_0)\in M^n$, we may assume that

$$R_2^{m_2+1}=R_2^{m_2}\times R_2, \qquad R_2^{m_2}\| E^{m_2}(\mu_2, x_0),$$

$$R_3^{m_3+1}=R_3^{m_3}\times R_3, \qquad R_3^{m_3}\| E^{m_3}(\mu_3, x_0).$$

Let $K_{x_0}^{m_1}$ be the integral submanifold of the distribution $E(\mu_1)$ through x_0 and $E_{1,x_0}^{m_1+1}$ the (m_1+1)-plane containing $K_{x_0}^{m_1}$. Since

$$e_1, \cdots, e_{m_1}, H(\mu_1)-x_0=\mu_1 e_{n+1}-x_0 \| E_{1,x_0}^{m_1+1}$$

we may consider as

$$E_{1,x_0}^{m_1+1}\| R_1^{m_1}\times R_2\times R_3$$

by (6.3) and (6.14). Therefore, $E_{1,x_0}^{m_1+1}$ is given by an equation such that

$$\sum_{\alpha_1=1}^{m_1} a_{\alpha_1}y_{\alpha_1}+c_2u_{m_2+1}+c_3v_{m_3+1}=c_0,$$

$$u_{\alpha_2}=u_{0\alpha_2}, \qquad v_{\alpha_3}=v_{0\alpha_3},$$

where a_{α_1}, c_2, c_3 and c_0 are real constants such that

$$\sum_{\alpha_1=1}^{m_1} a_{\alpha_1}^2+c_2^2+c_3^2=1, \qquad 0<c_0<1,\ c_2^2+c_3^2\neq 0$$

and

$$\boldsymbol{u}_0=(u_{01}, \cdots, u_{0m_2}, u_{0(m_2+1)}), \qquad \boldsymbol{v}_0=(v_{01}, \cdots, v_{0m_3}, v_{0(m_3+1)}).$$

(Since $K_{x_0}^{m_1}$ is a small m_1-sphere of S^{n+1}, c_0 must be $0<|c_0|<1$). Therefore, for any point $x_1=(\boldsymbol{y}_1, \boldsymbol{u}_1, \boldsymbol{v}_1)\in K_{x_0}^{m_1}$ and any point $x=(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v})\in K_{x_1}^{m_2+m_3}$, we have

$$\boldsymbol{y}=\boldsymbol{y}_1,$$

$$\langle \boldsymbol{u}, \boldsymbol{u}\rangle=\langle \boldsymbol{u}_1, \boldsymbol{u}_1\rangle=\sum_{\alpha_2=1}^{m_2}(u_{0\alpha_2})^2+(u_{1(m_2+1)})^2$$

$$\langle \boldsymbol{v}, \boldsymbol{v}\rangle=\langle \boldsymbol{v}_1, \boldsymbol{v}_1\rangle=\sum_{\alpha_3=1}^{m_3}(v_{0\alpha_3})^2+(v_{1(m_3+1)})^2 .$$

By a suitable change of orthogonal coordinates of $R_1^{m_1}$, we may put

$$\boldsymbol{a}=(a_1, \cdots, a_{m_1})=(c_1, 0, \cdots, 0) .$$

Thus, setting $u_{1(m_2+1)}=t_2$, $v_{1(m_3+1)}=t_3$, we can represent locally and explicitly M^n as:

$$(6.15)\qquad \begin{cases} \langle \boldsymbol{y}, \boldsymbol{y}\rangle+\langle \boldsymbol{u}, \boldsymbol{u}\rangle+\langle \boldsymbol{v}, \boldsymbol{v}\rangle=1 , \\ \langle \boldsymbol{u}, \boldsymbol{u}\rangle=t_2^2+b_2^2,\ \langle \boldsymbol{v}, \boldsymbol{v}\rangle=t_3^2+b_3^2 , \\ c_1y_1+c_2t_2+c_3t_3=c_0 \end{cases}$$

where b_2, b_3, c_1, c_2, c_3 and c_0 are all constants such that

$$(6.16)\qquad 0\leqq b_2\leqq 1 ,\quad 0\leqq b_3\leqq 1 ,\quad c_1^2+c_2^2+c_3^2=1 ,\quad 0<c_0<1$$

and t_2 and t_3 are auxiliary variables.

Now, we compute the normal vector M^n at x. By differentiating (6.16), we get

$$\langle \boldsymbol{y}, d\boldsymbol{y}\rangle+\langle \boldsymbol{u}, d\boldsymbol{u}\rangle+\langle \boldsymbol{v}, d\boldsymbol{v}\rangle=0 ,$$

$$\langle \boldsymbol{u}, d\boldsymbol{u}\rangle=t_2dt_2 ,\qquad \langle \boldsymbol{v}, d\boldsymbol{v}\rangle=t_3dt_3 ,$$

$$c_1dy_1+c_2dt_2+c_3dt_3=0 ,$$

from which we obtain

$$dt_2=-\frac{1}{c_3t_2-c_2t_3}\langle c_3\boldsymbol{y}-t_3\boldsymbol{a}, d\boldsymbol{y}\rangle ,\qquad dt_3=\frac{1}{c_3t_2-c_2t_3}\langle c_2\boldsymbol{y}-t_2\boldsymbol{a}, d\boldsymbol{y}\rangle$$

and, supposing $c_3t_2-c_2t_3\neq 0$, and

$$\frac{t_2}{c_3t_2-c_2t_3}\langle c_3\boldsymbol{y}-t_3\boldsymbol{a}, d\boldsymbol{y}\rangle+\langle \boldsymbol{u}, d\boldsymbol{u}\rangle=0 ,$$

$$-\frac{t}{c_3t_2-c_2t_3}\langle c_2\boldsymbol{y}-t_2\boldsymbol{a}, d\boldsymbol{y}\rangle+\langle \boldsymbol{v}, d\boldsymbol{v}\rangle=0 .$$

Hence, the normal unit vector $N(x)$ at x is represented as

$$N(x)=\lambda(t_2(c_3\boldsymbol{y}-t_3\boldsymbol{a}),\ (c_3t_2-c_2t_3)\boldsymbol{u},\ 0)$$
$$+\mu(-t_3(c_2\boldsymbol{y}-t_2\boldsymbol{a}),\ 0,\ (c_3t_2-c_2t_3)\boldsymbol{v}) .$$

Since $\langle x, N(x)\rangle=0$, we have

$$\lambda\{c_3t_2\langle \boldsymbol{y}, \boldsymbol{y}\rangle-t_2t_3\langle \boldsymbol{a}, \boldsymbol{y}\rangle+(c_3t_2-c_2t_3)\langle \boldsymbol{u}, \boldsymbol{u}\rangle\}$$
$$+\mu\{-c_2t_3\langle \boldsymbol{y}, \boldsymbol{y}\rangle+t_2t_3\langle \boldsymbol{a}, \boldsymbol{y}\rangle+(c_3t_2-c_2t_3)\langle \boldsymbol{v}, \boldsymbol{v}\rangle\}=0 .$$

The coefflcients of λ and μ of the above equality are

$$c_3(1-b_3^2)t_2-c_2b_2^2t_3-c_0t_2t_3 \quad \text{and} \quad -c_2(1-b_2^2)t_3+c_3b_3^2t_2+c_0t_2t_3$$

respectively. Therefore, $N(x)$ is proportional to

$$\begin{aligned}&\{-c_2(1-b_2^2)t_3+c_3b_3^2t_2+c_0t_2t_3\}(t_2(c_3\boldsymbol{y}-t_3\boldsymbol{a}),\ (c_3t_2-c_2t_3)\boldsymbol{u},\ 0)\\&+\{-c_3(1-b_3^2)t_2+c_2b_2^2t_3+c_0t_2t_3\}(-t_3(c_2\boldsymbol{y}-t_2\boldsymbol{a}),\ 0,\ (c_3t_2-c_2t_3)\boldsymbol{v})\\&=(c_3b_3^2t_2+c_2b_2^2t_3+c_0t_2t_3)(c_3t_2-c_2t_3)(\boldsymbol{y},\boldsymbol{u},\boldsymbol{v})\\&\quad-(c_3t_2-c_2t_3)(t_2t_3\boldsymbol{a},\ c_2t_3\boldsymbol{u},\ c_3t_2\boldsymbol{v}).\end{aligned}$$

Hence, supposing $t_2t_3\neq 0$, $N(x)$ is proportional to

$$\tilde{N}(x)=\left(c_0+\frac{b_2^2c_2}{t_2}+\frac{b_3^2c_3}{t_3}\right)(\boldsymbol{y},\boldsymbol{u},\boldsymbol{v})-\left(\boldsymbol{a},\ \frac{c_2}{t_2}\boldsymbol{u},\ \frac{c_3}{t_3}\boldsymbol{v}\right), \tag{6.17}$$

$$x=(\boldsymbol{y},\boldsymbol{u},\boldsymbol{v}).$$

Now, we compute the 1st and 2nd fundamental forms of M^n at x, where $t_2\neq 0$, $t_3\neq 0$. From the above argument, we may still put

$$\boldsymbol{u}=(0,\cdots,0,p_2), \qquad \boldsymbol{v}=(0,\cdots,0,p_3)$$

by changing the coordinate axises of $R_2^{m_2+1}$ and $R_3^{m_3+1}$. We get easily

$$du_{m_2+1}=\frac{t_2}{p_2}dt_2, \qquad dv_{m_3+1}=\frac{t_3}{p_3}dt_3$$

and

$$dt_2=-\frac{1}{c_3t_2-c_2t_3}\{c_3\langle\boldsymbol{y},d\boldsymbol{y}\rangle-c_1t_3dy_1\}$$

$$dt_3=\frac{1}{c_3t_2-c_2t_3}\{c_2\langle\boldsymbol{y},d\boldsymbol{y}\rangle-c_1t_2dy_1\}.$$

Hence we have

$$\begin{aligned}ds^2&=\langle d\boldsymbol{y},d\boldsymbol{y}\rangle+\langle d\boldsymbol{u},d\boldsymbol{u}\rangle+\langle d\boldsymbol{v},d\boldsymbol{v}\rangle\\&=[\langle d\boldsymbol{y},d\boldsymbol{y}\rangle+(du_{m_2+1})^2+(dv_{m_3+1})^2]\\&\quad+\sum_{\alpha_2=1}^{m_2}du_{\alpha_2}du_{\alpha_2}+\sum_{\alpha_3=1}^{m_3}dv_{\alpha_3}dv_{\alpha_3}.\end{aligned} \tag{6.18}$$

By means of (6.17), setting $\varphi=c_0+\dfrac{b_2^2c_2}{t_2}+\dfrac{b_3^2c_3}{t_3}$, we have

$$\begin{aligned}&-\langle d^2x,\tilde{N}(x)\rangle=\langle dx,d\tilde{N}(x)\rangle=\varphi\langle dx,dx\rangle\\&-\left\langle d\boldsymbol{u},\ \frac{c_2}{t_2}d\boldsymbol{u}-\frac{c_2}{t_2^2}\boldsymbol{u}dt_2\right\rangle+\left\langle d\boldsymbol{v},\ \frac{c_3}{t_3}d\boldsymbol{v}-\frac{c_3}{t_3^2}\boldsymbol{v}dt_3\right\rangle\end{aligned}$$

$$=\varphi\langle d\boldsymbol{y}, d\boldsymbol{y}\rangle+\left(\varphi-\frac{c_2}{t_2}\right)\langle d\boldsymbol{u}, d\boldsymbol{u}\rangle+\left(\varphi-\frac{c_3}{t_3}\right)\langle d\boldsymbol{v}, d\boldsymbol{v}\rangle+\frac{c_2}{t_2}dt_2dt_2+\frac{c_3}{t_3}dt_3dt_3$$

i. e.

$$\begin{aligned}-\langle d^2x, \tilde{N}(x)\rangle=&\left[\varphi\langle d\boldsymbol{y}, d\boldsymbol{y}\rangle+\left(\varphi-\frac{c_2}{t_2}+\frac{c_2p_2^2}{t_2^3}\right)(du_{m_2+1})^2\right.\\ &\left.+\left(\varphi-\frac{c_3}{t_3}+\frac{c_3p_3^2}{t_3^3}\right)(dv_{m_3+1})^2\right]\\ &+\left(\varphi-\frac{c_2}{t_2}\right)\sum_{\alpha_2=1}^{m_2}du_{\alpha_2}du_{\alpha_2}+\left(\varphi-\frac{c_3}{t_3}\right)\sum_{\alpha_3=1}^{m_3}dv_{\alpha_3}dv_{\alpha_3}\,.\end{aligned}\tag{6.19}$$

(6.18) shows that the parameter submanifolds corresponding to $(y_1, \cdots, y_{m_1})$, $(u_1, \cdots, u_{m_2})$ and $(v_1, \cdots, v_{m_3})$ are orthogonal to each others at x. And (6.19) show that the tangent spaces to the submanifolds corresponding to $(u_1, \cdots, u_{m_2})$ and $(v_1, \cdots, v_{m_3})$ must be $E(\mu_2, x)$ and $E(\mu_3, x)$ respectively. Therefore, the form in the brackets of (6.19) must be proportional to the one of (6.18).

Since we have

$$\begin{aligned}&\langle d\boldsymbol{y}, d\boldsymbol{y}\rangle+(du_{m_2+1})^2+(dv_{m_3+1})^2\\ &=\langle d\boldsymbol{y}, d\boldsymbol{y}\rangle+\left(\frac{t_2t_3}{(c_3t_2-c_2t_3)p_2p_3}\right)^2\left[p_2^2\left(c_1dy_1-\frac{c_2}{t_2}\langle \boldsymbol{y}, d\boldsymbol{y}\rangle\right)^2\right.\\ &\left.+p_3^2\left(c_1dy_1-\frac{c_2}{t_3}\langle \boldsymbol{y}, d\boldsymbol{y}\rangle\right)^2\right]\end{aligned}\tag{6.20}$$

and

$$\begin{aligned}&\varphi\langle d\boldsymbol{y}, d\boldsymbol{y}\rangle+\left(\varphi-\frac{c_2}{t_2}+\frac{c_2p_2^2}{t_2^3}\right)(du_{m_2+1})^2+\left(\varphi-\frac{c_3}{t_3}+\frac{c_3p_3^2}{t_3^3}\right)(dv_{m_3+1})^2\\ &=\varphi\{\langle d\boldsymbol{y}, d\boldsymbol{y}\rangle+(du_{m_2+1})^2+(dv_{m_3+1})^2\}\\ &+\left(\frac{t_2t_3}{(c_3t_2-c_2t_3)p_2p_3}\right)^2\left[\frac{b_2^2c_2p_2^2}{t_2^3}\left(c_1dy_1-\frac{c_2}{t_2}\langle \boldsymbol{y}, d\boldsymbol{y}\rangle\right)^2\right.\\ &\left.+\frac{b_3^2c_3p_3^2}{t_3^3}\left(c_1dy_1-\frac{c_3}{t_3}\langle \boldsymbol{y}, d\boldsymbol{y}\rangle\right)^2\right],\end{aligned}$$

the differential form in $dy_1, \cdots, dy_m$

$$\frac{b_2^2c_2p_2^2}{t_2^3}\left(c_1dy_1-\frac{c_2}{t_2}\langle \boldsymbol{y}, d\boldsymbol{y}\rangle\right)^2+\frac{b_3^2c_3p_3^2}{t_3^3}\left(c_1dy_1-\frac{c_3}{t_3}\langle \boldsymbol{y}, d\boldsymbol{y}\rangle\right)^2$$

must be proportional to (6.20). This is generally impossible. Thus, we reach a contradiction. Q. E. D.

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