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FANO-MORI ELEMENTARY CONTRACTIONS WITH REDUCIBLE GENERAL FIBER

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Abstract

Let $\varphi : X \to W$ be an elementary divisorial Fano-Mori contraction from a smooth projective variety, defined by a linear system $|m(K_X + \tau L)|$, with L a φ-ample line bundle in $\mathrm{Pic}(X)$, τ a positive integer and $m \gg 0$.

General fibers of such contractions are known to be irreducible if $\tau \geq \dim X - 3$ (and so if $\dim X \leq 4$). We prove that, if $\tau \geq \dim X - 4$, except possibly for one case, a general non trivial fiber is irreducible.

The special case, which can occur when $\dim X = 5$, is effective, as we show by an example in the last section of the paper.

1. Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers; a *contraction* $\varphi : X \to W$ is a proper surjective map with connected fibers onto a normal variety W. If the canonical bundle K_X is not nef then the negative part of the cone of effective 1-cycles $\mathrm{NE}(X)$ is locally polyhedral, by the Cone Theorem, and to every face in this part of the cone, by the Contraction Theorem, is associated a contraction; such contractions are called *Fano-Mori contractions* or *extremal contractions*. A Fano-Mori contraction is called *elementary* if $\rho(X/W) = 1$ or equivalently if it is associated to an extremal ray, i.e. to a face of dimension one in $\mathrm{NE}(X)_{K_X<0}$; in this case we define the *length* of the ray to be the minimum anticanonical degree of contracted curves.

A Fano-Mori contraction $\varphi : X \to W$ is defined by a linear system $|m(K_X + \tau L)|$, with L a φ-ample line bundle, τ a positive integer and $m \gg 0$; the divisor $K_X + \tau L$ is called a *supporting divisor* of the contraction.

The integer τ is bounded above by $n+1$ if φ is of fiber type, i.e. if $\dim W < \dim X$ and by $n-1$ if φ is birational; elementary contractions with values of τ close to the maximum were studied by general adjunction theory [6].

The situation, which is quite simple for the maximum values of τ (φ is the contraction of a projective space to a point in the fiber type case or the blow-up

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of a smooth point in the birational case) becomes more and more complicated as values of τ decrease.

Elementary contractions with $\tau \geq n-2$ were classified by Mori [13] in the case of smooth threefolds and by Fujita [8] and Ionescu [9] in the general case, while more recently Kawamata [10], Andreatta and Wiśniewski [4] dealt with the case $n = 4$ and $\tau = 1$ and gave a complete classification.

To proceed in the classification, i.e. to study either the case $\tau = n-3$ with $n > 4$ or $\tau < n-3$, the first step is to consider the general non trivial (i.e. non 0-dimensional) fiber of the contraction. Nakamura [14] considered the case of Fano-Mori elementary contractions either of fiber type or divisorial, supported by $K_X + (n-3)L$.

A new problem arises in the divisorial case: the exceptional divisor of the contraction is known to be irreducible by [11, Proposition 5.1.6], but it is no longer easy to prove that the general non trivial fiber of the contraction is irreducible. Nevertheless in the case $\tau = n-3$ general non trivial fibers of divisorial contractions are actually irreducible, as proved in [14].

In the subsequent case (i.e. $\tau = n-4$) proving the irreducibility of a general fiber becomes an hard problem. Actually a new phenomenon occurs: there exist elementary divisorial contractions with reducible general non trivial fibers. This is the subject of our paper; more precisely, we prove the following:

Theorem 1.1. *Let X be a smooth complex projective variety, let $\varphi_X : X \to W$ be a divisorial elementary Fano-Mori contraction supported by $K_X + \tau L$ and denote by E the exceptional divisor. Suppose that $\tau \geq \dim X - 4$. Then the general non trivial fiber of φ_X is irreducible, except in the following effective case*:

1. $\dim X = 5$;
2. $\tau = 1$;
3. *the length of the extremal ray contracted by φ_X is one*;
4. *the image of E, $\varphi_X(E)$ is a curve*;
5. *a general non trivial fiber is the union of two irreducible components isomorphic to $\mathbf{P}_{\mathbf{P}^1}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(2))$ which meet along a smooth quadric surface.*

In section 2 and 3 we recall general definitions and properties of families of rational curves and Fano-Mori contractions; then, in section 4, we explain the vertical slicing costruction which allows us to reduce to the case of a non elementary divisorial contraction $\varphi : Y \to Z$ whose exceptional locus is a reducible divisor contracted to a point and such that $\dim Y < \dim X$.

In section 5 we study families of rational curves which cover the irreducible components of the exceptional locus of φ; the key observation is to consider deformation of curves in the irreducible components of the exceptional locus rather than in the ambient variety. As a byproduct we obtain in this section a different proof of the irreducibility of the general fiber in the case studied in [14]. Section 6 contains the proof of theorem 1.1, while in the last section is presented an example, suggested by Jaroslaw Wiśniewski, which shows the effectiveness of the result.

2. Families of rational curves

Throughout this section our main reference is [12], with which our notation is coherent.

Let X be a projective scheme and let $\mathrm{Hom}(\mathbf{P}^1, X)$ be the scheme parametrizing morphisms from $\mathbf{P}^1$ to X. Let $\mathrm{Hom}_{bir}(\mathbf{P}^1, X) \subset \mathrm{Hom}(\mathbf{P}^1, X)$ be the open subscheme corresponding to those morphisms which are birational onto their image and let $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X)$ be the normalization of $\mathrm{Hom}_{bir}(\mathbf{P}^1, X)$: the group $\mathrm{Aut}(\mathbf{P}^1)$ acts on $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X)$ and the quotient exists.

Definition 2.1. We define the space $\mathrm{RatCurves}^n(X)$ to be the quotient of $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X)$ by the action of $\mathrm{Aut}(\mathbf{P}^1)$ and the space $\mathrm{Univ}(X)$ to be the quotient of the product action of $\mathrm{Aut}(\mathbf{P}^1)$ on $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X) \times \mathbf{P}^1$.

We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X) \times \mathbf{P}^1 \arrow[r, "U"] \arrow[d] \arrow[rr, bend left, "\mathrm{ev}"] & \mathrm{Univ}(X) \arrow[r, "\iota"'] \arrow[d, "\pi"'] & X \\
\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X) \arrow[r, "u"] & \mathrm{RatCurves}^n(X) &
\end{tikzcd} \tag{2.2}$$

where u and U are principal $\mathrm{Aut}(\mathbf{P}^1)$-bundles, π is a $\mathbf{P}^1$-bundle and ev is the evaluation map.

There exists a "pointed" version of this construction: let $x \in X$ be a point and let $\mathrm{Hom}_{bir}(\mathbf{P}^1, X, 0 \mapsto x)$ be the scheme that parametrizes morphisms $f : \mathbf{P}^1 \to X$ which send the point $0 \in \mathbf{P}^1$ to $x \in X$. Let $\mathrm{Aut}(\mathbf{P}^1, 0)$ be the group of the automorphisms of $\mathbf{P}^1$ which fix a point $0 \in \mathbf{P}^1$ and let $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X, 0 \mapsto x)$ be the normalization of $\mathrm{Hom}_{bir}(\mathbf{P}^1, X, 0 \mapsto x)$: the group $\mathrm{Aut}(\mathbf{P}^1, 0)$ acts on $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X, 0 \mapsto x)$ and the quotient exists.

Definition 2.3. The space $\mathrm{RatCurves}^n(x, X)$ is the quotient of $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X, 0 \to x)$ by the action of $\mathrm{Aut}(\mathbf{P}^1, 0)$ and the space $\mathrm{Univ}(x, X)$ is the quotient of the product action of $\mathrm{Aut}(\mathbf{P}^1, 0)$ on $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, X, 0 \to x) \times \mathbf{P}^1$.

Definition 2.4. A *family of rational curves* V on X is an irreducible subvariety of $\mathrm{RatCurves}^n(X)$. Given a family of rational curves V, we can consider the curves of V passing through a fixed point $x \in X$ and call it $V_x := V \cap \mathrm{RatCurves}^n(x, X)$.

To each family of rational curves V we can associate its universal family U, which is the restriction of $\mathrm{Univ}(X)$.

$$\begin{array}{ccc} U & \xrightarrow{\iota} & X \\ {\scriptstyle \pi}\downarrow & & \\ V & & \end{array}$$

We denote by $\mathrm{Locus}(V)$ the closure of $\iota(U)$ and we call it the *locus* of the family; finally we denote by $\mathrm{Locus}(V_x)$ the locus of $V \cap \mathrm{RatCurves}^n(x, X)$, i.e. the locus of the curves in the family which pass through x.

Definition 2.5. Let V be a family of rational curves on X. Then
1. V is *unsplit* if it is proper;
2. V is *locally unsplit* if every component of V_x is unsplit for a general $x \in \mathrm{Locus}(V)$;
3. V is *generically unsplit* if there is at most a finite number of curves of V passing through two general points of $\mathrm{Locus}(V)$.

Definition 2.6 [18, Definition 4]. Let X be a projective variety, H an ample divisor on X and $E \subseteq X$ a closed subset.

A family of rational curves V is a *minimal dominating family* for E if $E = \mathrm{Locus}(V)$ and $H \cdot V$ is minimal among the families whose locus is E.

Remark 2.7. With the above notation, if V is a minimal dominating family for E, then V is locally unsplit.

Proposition 2.8 [12, IV.2.6 and II.1.3]. *Let X be a projective variety which is a local complete intersection and let V be a family of rational curves whose locus meets the smooth locus of X.*

Assume either that V is generically unsplit and x is a general point in $\mathrm{Locus}(V)$ *or that V is unsplit and x is any point in* $\mathrm{Locus}(V)$; *then*

$$\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1 \geq \dim X - K_X \cdot V.$$

What follows is a variation of a classical construction of Mori theory (see for instance [7, Proof of 1.4.5] or [12, II.4.19]).

Definition 2.9. Let X be a projective variety, let V be an unsplit family of rational curves and let Y be a subset of X. We define

$$\mathrm{Locus}(V)_Y := \{x \in X \mid \exists C \text{ in } V \text{ with } C \cap Y \neq \emptyset, \, x \in C\}$$

i.e. $\mathrm{Locus}(V)_Y$ is the set of points that can be joined to Y by a curve of V.

Lemma 2.10 [15, Lemma 1]. *Let X be a projective variety; let Y be a closed subset of X and let V be an unsplit family of rational curves. Then* $\mathrm{Locus}(V)_Y$ *is closed in X and every curve in* $\mathrm{Locus}(V)_Y$ *is numerically equivalent to a linear combination with rational coefficients*

$$\alpha C_Y + \beta C_V,$$

where $C_Y \subset Y$, C_V *belongs to the family* V *and* $\alpha \geq 0$.

Corollary 2.11. *Let* V *be a family of rational curves on a projective variety* X, *and let* $x \in X$ *be a point such that an irreducible component of* V_x *is proper and dominant. Then* $\rho(X) = 1$.

3. Fano-Mori contractions

Our notation is coherent with [11], to which we refer for the details.

Let X be a complex projective variety and let

$$N_1(X) = ((1 - cycles)/\equiv) \otimes \mathbf{R}, \quad N^1(X) = (\mathrm{Pic}(X)/\equiv) \otimes \mathbf{R},$$

where $\equiv$ denotes numerical equivalence. These vector spaces are dual to each other via the intersection pairing and they are of finite dimension $\rho(X)$, the *Picard number* of X. Let also $\overline{NE(X)} \subset N_1(X)$ be the closure of the cone of effective 1-cycles.

From now on we will assume that X is smooth and that K_X is not nef, that is there exists an effective curve C such that $K_X \cdot C < 0$; thus by the Cone Theorem the negative part (with respect to the canonical bundle of X) of $\overline{NE(X)}$ is locally polyhedral. We will call any face σ in the negative part of the cone an *extremal face* of X. By the Rationality Theorem, given an extremal face σ of X, there exists a nef divisor on X such that $\sigma = \{z \in \overline{NE(X)} \mid H \cdot z = 0\}$; H is called a *supporting divisor* for σ. Moreover, for each extremal face σ, the Contraction Theorem gives a normal projective variety W and a surjective morphism $\varphi : X \to W$ with connected fibers such that

1. for every curve C in X, $\varphi(C)$ is a point if and only if the class $[C] \in N_1(X)$ is in σ;
2. $H = \varphi^*(A)$, with A an ample Cartier divisor on W.

The map φ is called *Fano-Mori* (*or extremal*) *contraction*, the contraction of the face σ, and H is called a *supporting divisor* for the map φ. We denote by $\mathrm{Exc}(\varphi)$ the largest subset such that φ is an isomorphism on $X \backslash \mathrm{Exc}(\varphi)$ and we call it the *exceptional locus* of φ. If the map φ is birational, it can be *divisorial* (if the exceptional locus is a divisor on X) or *small* (if the codimension of the exceptional locus in X is greater or equal than 2). If $\dim_{\mathbf{R}} \sigma = 1$ the face σ is called an *extremal ray*, while φ is called an *elementary contraction*.

Remark 3.1. We have also (see [13]) that if X has an extremal ray R then there exists a rational curve Γ on X such that $0 < -K_X \cdot \Gamma \leq \dim X + 1$ and $R = \mathbf{R}_+[\Gamma]$. A rational curve C in R whose intersection number with $-K_X$ is minimal is called a *minimal extremal curve*, while the intersection number $l(R) = -K_X \cdot C$ is called *length* of the ray R.

Remark 3.2. We can always choose a supporting divisor of the form $K_X + rL$, where L is an ample Cartier divisor and r is an integer.

Lemma 3.3. *Let $\varphi : X \to W$ be the contraction of an extremal ray R; then there exists a locally unsplit family of rational curves V whose numerical class is in R, and such that* $\mathrm{Locus}(V) = \mathrm{Exc}(\varphi)$.

Proof. $\mathrm{Exc}(\varphi)$ is covered by rational curves whose numerical class is in R, so there exists at least an irreducible component of $\mathrm{RatCurves}^n(X)$ with these properties; let V be one of these components whose degree with respect to L is minimal. We claim that V is locally unsplit: in fact, if for a general $x \in \mathrm{Exc}(\varphi)$ curves in V_x degenerate to a reducible cycle, then the numerical class of every irreducible component of this cycle belongs to R by the extremality of R. Then through a general point of $\mathrm{Exc}(\varphi)$ we would have a rational curve of the extremal ray R of degree strictly less than $L \cdot V$, and so a family which covers $\mathrm{Exc}(\varphi)$ with this property, a contradiction. □

4. Divisorial elementary contractions

In this paper we deal with elementary divisorial contractions, a special case of Fano-Mori contractions, which have the following fundamental property:

Proposition 4.1 [11, Proposition 5.1.6]. *Let $\varphi : X \to W$ be an elementary divisorial Fano-Mori contraction with exceptional divisor E; then E is irreducible.*

In order to fix more precisely our setup, we give the following

Definition 4.2. Let X be a smooth n-fold and let $\varphi_X : X \to W$ be the contraction of an extremal ray R, supported by $K_X + \tau L$, with L a φ_X-ample line bundle on X.

We say that L is *numerically reduced on X with respect to R* if for every φ_X-ample line bundle $L' \in \mathrm{Pic}(X)$ we have $L \cdot C \leq L' \cdot C$ for any curve C whose numerical class is in R.

Let X be a smooth n-fold and let

$$\varphi_X : X \to W$$

be the contraction of an extremal ray R. Suppose that φ_X is divisorial and it is supported by $K_X + \tau L$, where L is a φ_X-ample line bundle which is numerically reduced on X with respect to R. We denote by E the exceptional divisor, and by G a *general* non trivial fiber of φ_X, both considered as subschemes of X with the reduced structure.

Lemma 4.3 (See [1, Theorem 2.1] and [14, Lemma 4.2]). *The image of the restriction map* $\mathrm{Pic}(X) \to \mathrm{Pic}(G)$ *is of rank one, generated by $L_G = L|_G$, and we have*

1. $K_X|_G = -\tau L_G$;
2. $E|_G = (N_{E/X})|_G = -qL_G$.

The assumption that L is numerically reduced with respect to R implies that τ and q are (positive) integers.

To study a general fiber of the contraction $\varphi_X : X \to W$, we will use the vertical slicing technique, following Ando [1]:

4.4 (Construction: Vertical slicing). Let $\varphi_X : X \to W$ be an elementary divisorial Fano-Mori contraction with exceptional divisor E and let $r = \dim \varphi_X(E)$. If $r = 0$ then the contraction φ_X maps E to a point and so there is a unique fiber E, otherwise we take r general very ample divisors $Z_1, \dots, Z_r$ on W, we set $Y_i = \varphi_X^*(Z_i)$, and we consider the two varieties

$$Y := \bigcap_i Y_i, \quad Z := \bigcap_i Z_i,$$

with $\dim Y = \dim Z = n - r := m$.

From Bertini theorem we know that Y is smooth and Z is normal; moreover a connected component of $E \cap Y$ is a general fiber G of φ_X. It is straightforward to prove that the restriction of φ_X to Y is a Fano-Mori contraction $\varphi : Y \to Z$, supported by $K_Y + \tau L_Y$, which maps G (a divisor in Y) to a point in Z.

Remark 4.5. If φ_X contracts E to a point (i.e. if $Y = X$), then theorem 1.1 follows immediately from proposition 4.1, so from now on we assume that $m = \dim Y < n = \dim X$.

The normal bundle $N_{G/Y}$ in Y is well defined and we have:

$$N_{G/Y} = (N_{E/X})|_G.$$

Combining this with lemma 4.3, we have

$$K_X|_G = -\tau L_G, \quad G|_G = N_{G/Y} = (N_{E/X})|_G = -qL_G. \tag{4.6}$$

Suppose that G is reducible:

$$G = G_1 + \cdots + G_s \quad \text{with } s > 1;$$

in this case, by proposition 4.1, φ can not be elementary, so φ is the contraction of an extremal face $\sigma \subset \mathrm{NE}(Y)$ such that $\dim \sigma \geq 2$. Let $R_1, \dots, R_t$ be the extremal rays in σ and for every $i = 1, \dots, t$ denote by $\varphi_i : Y \to Z_i$ the contraction of the ray R_i. We have commutative diagrams:

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] \arrow[dr, "\varphi_i"'] & & Z \\ & Z_i \arrow[ur, "\psi_i"'] & \end{tikzcd}$$

so the contractions φ_i are birational and supported by $K_Y + \tau L_i'$, with L_i' a φ_i-ample divisor on Y which can be written as $L_i' = L + \varphi_i^*(A_i)$ for a suitable ample divisor $A_i \in \mathrm{Pic}(Z_i)$.

For every $i = 1, \dots, s$ we can consider the decomposition of G

$$G = G_i + \tilde{G}_i,$$

where we denote by $\tilde{G}_i$ the sum of all the components of G different from G_i. Since G is connected and reducible, $\tilde{G}_i$ is non empty and the restriction

$$D_i := \tilde{G}_i|_{G_i}.$$

is an *effective* divisor on G_i.

Using the formulas 4.6, we can compute the normal bundle of G_i in Y:

$$N_{G_i/Y} = G_i|_{G_i} = (G - \tilde{G}_i)|_{G_i} = -qL_{G_i} - D_i \tag{4.7}$$

and so, by adjunction, we have

$$K_{G_i} = -(\tau + q)L_{G_i} - D_i. \tag{4.8}$$

5. Rational curves and reducible fibers

Using the vertical slicing construction we have reduced the study of the general fiber of $\varphi_X : X \to W$ to the study of the exceptional divisor G of a Fano-Mori contraction $\varphi : Y \to Z$, from a smooth m-fold Y, supported by $K_Y + \tau L$, whose exceptional locus is a (possibly reducible) divisor G which is mapped to a point. Moreover

$$K_G = -(\tau + q)L \quad N_{G/Y} = -qL$$

for some positive integers τ and q.

In order to understand the geometry of G, in this section we will analyze the properties of the families of rational curves which cover it. The main idea is to use deformations of rational curves in the irreducible components of G, as they can give stronger restrictions than deformations in Y.

Let V be a generically unsplit family of rational curves in G_i, such that there exists a curve C in V which meets the smooth locus of G_i. By proposition 2.8, we have

$$\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1 \geq \dim G_i - K_{G_i} \cdot C,$$

so, recalling formula 4.8, we obtain the following inequality:

$$\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1 \geq m - 1 + (\tau + q)L \cdot C + D_i \cdot C. \tag{5.1}$$

Remark 5.2. Let V be an unsplit family of rational curves in G_i, C a curve of the family and x a point of C. Since D_i is an effective divisor on G_i, if $D_i \cdot C = 0$, we have that $\mathrm{Locus}(V_x) \subset D_i$.

PROPOSITION 5.3. *Let $\varphi : Y \to Z$ be a Fano-Mori contraction from a smooth m-fold Y, supported by $K_Y + \tau L$, whose exceptional locus is a divisor G which is mapped to a point and such that*

$$K_G = -(\tau + q)L \quad N_{G/Y} = -qL.$$

Suppose moreover that G *is reducible and that* $\tau \geq m - 3$; *then*

1. *each irreducible component* G_i *of* G *is covered by an unsplit family of rational curves* W^i *such that* $L \cdot W^i = 1$. *Moreover the families* W^j *and* W^k *are independent in* $N_1(Y)$ *if* $j \neq k$;
2. *if* $G_{j,k}$ *is an irreducible component of* $G_j \cap G_k$, *and* $x \in G_{j,k}$ *is a general point then there exist curves of* W^j_x *and* W^k_x *which are contained in* $G_{j,k}$;
3. *every curve in* $G_{j,k}$ *is numerically equivalent in* Y *to a linear combination with nonnegative rational coefficients*

$$a\Gamma_j + b\Gamma_k,$$

where Γ_j *belongs to* W^j *and* Γ_k *belongs to* W^k.

Proof of 1. Let W^i be a minimal dominating family for G_i, let x be a general point in G_i, not contained in $\operatorname{Sing} G_i \cup D_i$ and let Γ_i be a curve in W^i through x; since W^i is dominating for G_i we have $\dim \operatorname{Locus}(W^i) = \dim G_i = m - 1$, and so inequality (5.1) gives

$$m \geq \dim \operatorname{Locus}(W^i_x) + 1 \geq (\tau + q)L \cdot \Gamma_i + D_i \cdot \Gamma_i. \tag{5.4}$$

Since D_i is effective and $\Gamma_i \not\subset D_i$ we have $D_i \cdot \Gamma_i \geq 0$ and so we obtain that

$$m \geq \dim \operatorname{Locus}(W^i_x) + 1 \geq (\tau + q)L \cdot \Gamma_i \geq (m-2)L \cdot \Gamma_i. \tag{5.5}$$

Suppose, by contradiction, that $L \cdot \Gamma_i \geq 2$; it is easy to show that this can happen only if $m = 4$, $q = 1$, $D_i \cdot \Gamma_i = 0$ and $\operatorname{Locus}(W^i_x) = G_i$.

Since W is locally unsplit, the last condition, together with corollary 2.11, gives $\rho(G_i) = 1$; in particular D_i, which is an effective divisor on G_i, is ample, against the condition $D_i \cdot C_i = 0$. Hence $L \cdot \Gamma_i = 1$ and W^i is an unsplit family.

To prove the independence of W^j and W^k we note that $G \cdot \Gamma_i = -qL \cdot \Gamma_i = -q$ and $\operatorname{Locus}(W^i) = G_i$ for every i; hence G_i is negative on Γ_i, and it is the only irreducible component of G with this property. Therefore $G_j \cdot \Gamma_k < 0$ if $j = k$ and $G_j \cdot \Gamma_k \geq 0$ if $j \neq k$. $\square$

Proof of 2. The irreducible components of $G_j \cap G_k$ are the common components of D_j and D_k, so it is enough to show that for every i and for a general x belonging to an irreducible component $\overline{D_i}$ of D_i there exists a curve of W^i_x contained in $\overline{D_i}$. We will show that this is the case if x is not contained in any other irreducible component of D_i.

By inequality 5.5 we have three cases, according to the dimension of $\operatorname{Locus}(W^i_x)$.

a) $\dim \operatorname{Locus}(W^i_x) = m - 3$.
If Γ_i is a curve in W^i_x, inequality 5.4 gives that $D_i \cdot \Gamma_i = 0$; from remark 5.2 we get $\operatorname{Locus}(W^i_x) \subset D_i$ and so, by our choice of x, $\operatorname{Locus}(W^i_x) \subset \overline{D_i}$ and we are done.

b) $\dim \operatorname{Locus}(W^i_x) = m - 2$.
If Γ_i is a curve in W^i_x, inequality 5.4 gives that either $D_i \cdot \Gamma_i = 0$ or $D_i \cdot \Gamma_i = 1$.

In the first case, $\mathrm{Locus}(W^i_x) \subset D_i$ by remark 5.2 and we conclude as in a).
If $D_i \cdot \Gamma_i = 1$, let G_l be an irreducible component of G, different from G_i and containing $\overline{D_i}$; since $G_i \cap G_l \subset D_i$ and $\mathrm{Locus}(W^i_x) \subset G_i$,

$$\mathrm{Locus}(W^i_x) \cap G_l \subset \mathrm{Locus}(W^i_x) \cap D_i.$$

By inequality 5.4 we have $m \geq 4$, hence $\dim \mathrm{Locus}(W^i_x) + \dim G_l > \dim Y$, and from Serre's inequality we have that $\mathrm{Locus}(W^i_x) \cap D_i$ contains a curve; in particular there exists a point $y \neq x \in \mathrm{Locus}(W^i_x) \cap D_i$, hence a curve Γ in W^i_x which passes through y. Being $D_i \cdot \Gamma = 1$, Γ must be contained in D_i (hence in $\overline{D_i}$, by our choice of x) and we are done.

Claim. Case c), i.e. $\dim \mathrm{Locus}(W^i_x) = m - 1$ cannot happen for any x in G_i.

Suppose by contradiction that for some $x \in G_i$ we have $\dim \mathrm{Locus}(W^i_x) = m - 1$. This implies $\mathrm{Locus}(W^i_x) = G_i$ and, since W^i_x is unsplit, $\rho(G_i) = 1$ by corollary 2.11. In particular every curve in G_i (and so in $\overline{D_i}$) is numerically proportional to curves of W^i.

Let G_l be an irreducible component of G containing $\overline{D_i}$ and different from G_i, let $x' \in \overline{D_i}$ be a generic point and consider $\mathrm{Locus}(W^l_{x'})$.

If $\dim \mathrm{Locus}(W^l_{x'}) = m - 3$ or $m - 2$ then $\overline{D_i}$ contains curves of $W^l_{x'}$ by cases a) and b), while if $\dim \mathrm{Locus}(W^l_{x'}) = m - 1$ every curve in $\overline{D_i}$ is numerically proportional to curves of $W^l_{x'}$ as before.

In all cases this leads to a contradiction with the fact that every curve in $\overline{D_i}$ is numerically proportional to curves of W^i, while W^i and W^l are numerically independent. □

Proof of 3. Let x be a general point of $G_{j,k}$ (i.e. a point which is not contained in any other irreducible component of $G_j \cap G_k$) and let Γ_j be a curve of W^j_x contained in $G_{j,k}$; by our choice of x and Γ_j, the generic point x' of Γ_j is not contained in any other irreducible component of $G_j \cap G_k$.

Consider $\mathrm{Locus}(W^k)_{\Gamma_j} \subseteq G_k$ (see Definition 2.9): this is a closed subset of X by lemma 2.10 and, by [3, Lemma 5.4] we have

$$\dim \mathrm{Locus}(W^k)_{\Gamma_j} = \dim \mathrm{Locus}(W^k_x) + 1.$$

Therefore according to the proof of part 2 we have two possibilities, depending on the dimension of $\mathrm{Locus}(W^k_x)$:
if $\dim \mathrm{Locus}(W^k_x) = m - 2$, we have that $\dim \mathrm{Locus}(W^k)_{\Gamma_j} = m - 1$, and so $\mathrm{Locus}(W^k)_{\Gamma_j} = G_k$;
if $\dim \mathrm{Locus}(W^k_x) = m - 3$, we have that $\dim \mathrm{Locus}(W^k)_{\Gamma_j} = m - 2$ and, if x' is a generic point of Γ_j, $\mathrm{Locus}(W^k_{x'})$ is contained in $G_{j,k}$ by proof of part 2 case a). Thus $\mathrm{Locus}(W^k)_{\Gamma_j} \cap G_{j,k}$ contains a dense subset of $G_{j,k}$ (that is $\bigcup_{x'} \mathrm{Locus}(W^k_{x'})$ with x' general in Γ_j) and so $G_{j,k} \subset \mathrm{Locus}(W^k)_{\Gamma_j}$.
In both cases, we have that $G_{j,k} \subseteq \mathrm{Locus}(W^k)_{\Gamma_j}$.

Let C be a curve in $G_{j,k}$; by Lemma 2.10 C is numerically equivalent to a linear combination

$$C \equiv a_j \Gamma_j + b_j \Gamma_k,$$

of curves in W^j and W^k, with $a_j, b_j \in \mathbf{Q}$ and $a_j \geq 0$. We can repeat all the argument exchanging j and k to show that there exist $a_k, b_k \in \mathbf{Q}$, with $b_k \geq 0$, such that

$$C \equiv a_k \Gamma_j + b_k \Gamma_k.$$

Since $[\Gamma_j], [\Gamma_k] \in N_1(Y)$ are linearly independent, the decomposition of $[C]$ is unique, and so

$$a = a_j = a_k \geq 0, \quad b = b_j = b_k \geq 0$$

as claimed. □

Corollary 5.6. *With the assumptions and the notations of proposition* 5.3 *we have* $\tau = m - 3$ *and* $G|_G = -L_G$ *i.e.* $q = 1$.

Proof. By the claim in the proof of proposition 5.3, part 2, for every $x \in G_i$ we have $\dim \mathrm{Locus}(W_x^i) \leq m - 2$; therefore we can rewrite inequality 5.4 as follows:

$$m - 1 \geq \dim \mathrm{Locus}(W_x^i) + 1 \geq \tau + q + D_i \cdot \Gamma_i.$$

Suppose, by contradiction, that $q > 1$, or $\tau \geq m - 2$; this can happen only if

$$\dim \mathrm{Locus}(W_x^i) = m - 2 \quad \text{and} \quad D_i \cdot \Gamma_i = 0.$$

In particular, if we choose $x' \in D_i$, by remark 5.2 we have that $\mathrm{Locus}(W_{x'}^i)$ is a closed subscheme of D_i with the same dimension and so it is an irreducible component $\overline{D_i}$ of D_i and has Picard number one by corollary 2.11.

By proposition 5.3, part 2 we have that each component of D_i contains curves of two independent unsplit families and so its Picard number cannot be one, a contradiction. □

Corollary 5.7 [14, Lemma 8.1]. *Let X be a smooth complex projective variety of dimension n and let $\varphi_X : X \to W$ be a divisorial elementary Fano-Mori contraction supported by $K_X + \tau L$, with $\tau \geq n - 3$. Then the general fiber G of φ_X is irreducible.*

Proof. Suppose by contradiction that G is reducible. By vertical slicing we reduce to a Fano-Mori contraction $\varphi : Y \to Z$ from a smooth m-fold Y which satisfies the assumptions of proposition 5.3.

As noted in remark 4.5, we can assume that $m < n$, so, by corollary 5.6 we have $\tau = m - 3 < n - 3$, a contradiction. □

6. Proof of theorem 1.1

We are now ready to use the results of the previous sections to prove the main theorem. By vertical slicing we reduce to the study of the following situation:

6.1 (Setup). Let $\varphi : Y \to Z$ be a Fano-Mori contraction from a smooth m-fold Y, with $m = \dim Y < \dim X = n$, supported by $K_Y + \tau L$, with $\tau \geq n - 4$, whose exceptional locus is a reducible divisor G which is mapped to a point and such that $K_G = -(\tau + q)L$ and $N_{G/Y} = -qL$.

Since $m < n$ we have that $\tau \geq m - 3$, so, by corollary 5.6 we have $\tau = m - 3$ and $q = 1$, hence

$$K_G = -(m-2)L \quad N_{G/Y} = -L.$$

Step 1. Elementary contractions of a vertical section

As we have seen in section four, φ is the contraction of an extremal face $\sigma = \langle R_1, \dots, R_t \rangle$ of dimension ≥ 2. The contraction $\varphi_i : Y \to Z_i$ associated to the ray R_i is supported by $K_Y + (m-3)L_i'$ where L_i' is φ_i-ample and $L_i' = L + \varphi_i^* A_i$ for a suitable A_i ample on Z_i.

The following lemmata describe some properties of these contractions.

Lemma 6.2. *In the assumptions of Setup* 6.1 *suppose that one of the contractions φ_i, call it φ_1, is divisorial; then every fiber of φ_1 has dimension $\leq m-2$.*

Proof. By proposition 4.1 the exceptional locus $\mathrm{Exc}(\varphi_1)$ of φ_1 is an irreducible divisor, so, being contained in G, it coincides with one of its irreducible components, call it G_1.

Suppose by contradiction that $\dim \varphi_1(G_1) = 0$; let G_2 be an irreducible component of G meeting G_1; by part 2 of proposition 5.3, $G_1 \cap G_2$ contains curves of W^2, the unsplit family which covers of G_2 whose existence is guaranteed by part 1 of proposition 5.3.

Since a general curve of W^2 is not contained in $\mathrm{Exc}(\varphi_1)$, curves of W^2 are not contracted by φ_1; in particular the curves of W^2 contained in $G_1 \cap G_2$ are not contracted by φ_1, a contradiction. □

Lemma 6.3. *In the assumptions of Setup* 6.1 *suppose that one of the contractions φ_i, call it φ_1, is a small contraction, and denote by C_1 a curve in R_1 of minimal anticanonical degree. Then $G_i \cdot C_1 \leq 0$ for every $i = 1, \dots, s$.*

Proof. Since φ_1 is a small contraction supported by $K_Y + (m-3)L_1'$, by [2, Theorem A], we have that φ_1 contracts a finite number of disjoint $\mathbf{P}^{m-2}$ to points and that $L_1' \cdot C_1 = 1$.

Fix one of these projective spaces and call it P: of course $P \subset G$ and so there exists a component G_1 such that $P \subset G_1$; we claim that P does not meet any other irreducible component of G.

Suppose by contradiction that P meets $G_2 \neq G_1$; then by Serre's inequality $G_2 \cap P$ contains a curve Γ of P. Hence Γ is contained in an irreducible component $G_{1,2}$ of $G_1 \cap G_2$.

Thus from part 3 of proposition 5.3 there exist $\alpha, \beta \in \mathbf{Q}$, with $\alpha, \beta \geq 0$, and $(\alpha, \beta) \neq (0,0)$ such that

$$\Gamma \equiv \alpha\Gamma_1 + \beta\Gamma_2,$$

where Γ_i is a curve in an unsplit family of rational curves W^i which dominates G_i.

Since Γ is contracted by φ_1 it is extremal in $\mathrm{NE}(Y)$, so at least one among Γ_1 and Γ_2 belongs to the extremal ray R_1 and this is not possible, since $\mathrm{Locus}(W^i) = G_i$ is a divisor for every i.

Thus we have shown that each irreducible component of the locus of a small ray meets only one component of G (the one in which it is included); then $G_i \cdot C_1 = 0$ for $i \neq 1$. Hence $G_1 \cdot C_1 = G \cdot C_1 = -L \cdot C_1 = -L_1' \cdot C_1 = -1$. □

Step 2. Bounding the number of irreducible components

Proposition 6.4. *Let $\varphi : Y \to Z$ and G be as in Setup 6.1. Then $m = \dim Y = 4$, $\tau = 1$ and G has exactly two irreducible components G_1 and G_2. Moreover G_1 and G_2 are the exceptional loci of two elementary (divisorial) contractions of length one with every non trivial fiber of dimension two.*

Proof. Fix one component G_1 of G. Since G is connected we can find a curve $\Gamma \subset G$ which has positive intersection with G_1 and is not contained in it. The numerical class of Γ belongs to the face σ, so it is a positive linear combination of numerical classes of extremal rational curves. Therefore we can find a minimal extremal curve, say C_2, which has positive intersection with G_1.

From lemma 6.3 we have that $R_2 = \mathbf{R}_+[C_2]$ is a divisorial ray; moreover it is clear that $\mathrm{Locus}(R_2)$ meets G_1. We claim that $\mathrm{Locus}(R_2) \neq G_1$.

In fact, since C_2 is contained in G we have that $G \cdot C_2 = -L \cdot C_2 < 0$ and so there exists an irreducible component of G, call it G_2 which is negative on C_2. In particular we have $\mathrm{Locus}(R_2) \subset G_2$.

Obviously $G_1 \neq G_2$, since the intersection numbers with C_2 have opposite sign. Since R_2 is divisorial we have $\mathrm{Locus}(R_2) = G_2$ and so $G_i \cdot C_2 \geq 0$ for every $i \neq 2$; in particular since $G_1 \cdot C_2 > 0$ we have that

$$D_2 \cdot C_2 = (G_1 + G_3 + \cdots + G_s) \cdot C_2 > 0. \tag{6.5}$$

Let V^2 be a locally unsplit family of rational curves in R_2 which covers G_2 (see lemma 3.3) and let C be a curve of V^2. Denote by F_2 a general fiber of φ_2 and let x be a general point of F_2; by inequality 5.1 and lemma 6.2 we have

$$m - 2 \geq \dim F_2 \geq \mathrm{Locus}(V_x^2) \geq (m-2)L \cdot C + D_2 \cdot C - 1 \tag{6.6}$$

The curve C belongs to R_2, so $D_2 \cdot C > 0$, forcing $L \cdot C = D_2 \cdot C = 1$, and $[C] = [C_2]$ in $N_1(Y)$. Therefore

$$L \cdot C_2 = 1 = D_2 \cdot C_2. \tag{6.7}$$

Inequality 6.6 also gives $\dim F_2 = m-2$ for a general fiber of φ_2; combining this with lemma 6.2 and with upper semicontinuity of the dimension of the fibers we have that $\dim F_2 = m-2$ for every non trivial fiber of φ_2.

Note that, by the first equality of 6.7 the extremal ray R_2 has length one, while from the second equality and from the fact that $G_1 \cdot C_2 > 0$ inequality 6.5 gives $G_i \cdot C_2 = 0$ for all $i \neq 1, 2$.

Claim. G_1 is the only component of G which meets G_2.

Suppose by contradiction that there exists another irreducible component G_3 of G which meets G_2; since $G_3 \cdot C_2 = 0$, if x is a general point of $G_2 \cap G_3$ then $\mathrm{Locus}(V_x^2) \subset G_2 \cap G_3$ and, being of the same dimension, it is an irreducible component $G_{2,3}$ of $G_2 \cap G_3$. This forces $\rho(G_{2,3}) = 1$, against part 2 of proposition 5.3, and the claim follows.

Now we fix G_2 and we repeat the first argument of the proof in step 2 to find a divisorial ray R_1 whose locus is an irreducible component of G different from G_2 which meets G_2; from the claim we have $\mathrm{Locus}(R_1) = G_1$.

We can repeat now for R_1 all the steps of the proof; in the end we find that G_2 is the only component of G which meets G_1, so, since G is connected we have that G has exactly two irreducible components.

We have also proved that each of this components is the exceptional locus of an elementary divisorial contraction of length one. To prove that $m = \dim Y = 4$ (and so that $\tau = 1$) we consider a point x in $G_1 \cap G_2$; denoted by F_1 and F_2 the fibers of φ_1 and φ_2 containing x, by Serre's inequality we have

$$\dim(F_1 \cap F_2) \geq \dim F_1 + \dim F_2 - \dim Y \geq m - 4.$$

Thus if $m \geq 5$ there exists a curve whose numerical class belongs to two different extremal rays, and this is impossible. □

Step 3. Description of the irreducible components

Proposition 6.8. *Let $\varphi : Y \to Z$ and G be as in Setup* 6.1; *then the two irreducible components G_1 and G_2 of G are $\mathbf{P}^2$-bundles isomorphic to $\mathbf{P}_{\mathbf{P}^1}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(2))$ which meet along a smooth quadric.*

Proof. Denote by φ_i the divisorial contraction whose locus is G_i and by C_i a curve in R_i which has minimal anticanonical degree; by [5, Theorem 4.1], or [17, Main Theorem], we have two possibilities: φ_i is either a $\mathbf{P}^2$-bundle or a quadric bundle over a smooth curve.

The second case is ruled out because, applying inequality 6.6 to a general point x in a general fiber of φ_i, we have $\dim \mathrm{Locus}(V_x^i) = 2$. Thus φ_i gives to G_i a $\mathbf{P}^2$-bundle structure over a smooth curve B_i.

Let $G_{1,2}$ be an irreducible component of $G_1 \cap G_2$ (and so of D_i); since $G_{1,2}$ is an effective divisor on G_i we have that $G_{1,2} \cdot C_i \geq 0$ in G_i $(i = 1, 2)$, equality holding if and only if $G_{1,2}$ is a fiber of φ_i, but this possibility is ruled out by part 2 of proposition 5.3.

Therefore $G_{1,2} \cdot C_i > 0$; since, by formula 6.7, we have $D_i \cdot C_i = 1$ we conclude that $D := D_1 = D_2 = G_{1,2}$ is irreducible.

Since $D_1 \cdot C_1 = D_2 \cdot C_2 = 1$ and D is irreducible we see that D restricts to a line on every fiber of φ_i and so D, being a surface with two $\mathbf{P}^1$-bundle structures, is isomorphich to $\mathbf{P}^1 \times \mathbf{P}^1$.

Moreover $\mathscr{E}_i = \varphi_{i*}D$ is a locally free sheaf of rank three on B_i, which is rational and $G_i = \mathbf{P}_{B_i}(\mathscr{E}_i)$. In G_i we have

$$(D \cdot C_j)_{G_i} = G_j|_{G_i} \cdot C_j = (G - G_i) \cdot C_j = -2,$$

so, pushing forward the exact sequence

$$0 \to \mathcal{O}_{G_i} \to \mathcal{O}_{G_i}(D) \to \mathcal{O}_D(D) \to 0$$

to $B_i \simeq \mathbf{P}^1$ we obtain the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1} \to \mathscr{E}_i \to \mathcal{O}(-2) \oplus \mathcal{O}(-2) \to 0.$$

This sequence splits, hence we have $G_i = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus^2 \mathcal{O}_{\mathbf{P}^1}(-2))$. $\square$

Conclusion of proof of theorem 1.1

We have only to show that $\dim X = 5$, since this will imply also that $\dim \varphi_X(E) = 1$, but the assertion is clear, since we have proved that $\tau = 1$ and we are assuming $\tau \geq \dim X - 4$ and $\dim X > \dim Y = 4$.

7. Example

We will now present an example, due to Jaroslaw A. Wiśniewski, of an elementary Fano-Mori contraction of a smooth fivefold contracting a divisor to a curve and such that any positive dimensional fiber is reducible (and its structure is as described in theorem 1.1). The construction is divided in two steps: first, using toric geometry, we construct a Fano-Mori divisorial contraction of a fourfold with reducible exceptional locus (corresponding to the situation described in proposition 6.8) and then we fit it in a suitable five dimensional manifold.

7.1. A Fano Mori contraction with a $\mathbf{Z}_2$-action

Let e_1, e_2, v_1, v_2 be a basis of a 4-dimensional lattice, and let $w_1 = 2e_1 - e_2 - v_1$ and $w_2 = 2e_2 - e_1 - v_2$. Let Δ be the fan generated by these six vectors and containing the following maximal cones:

$$\begin{array}{cccc} \langle e_1, v_1, w_1, v_2 \rangle & \langle e_1, v_1, w_1, w_2 \rangle & \langle e_1, e_2, v_1, w_2 \rangle & \langle e_1, e_2, v_1, v_2 \rangle \\ \langle e_2, v_2, w_2, v_1 \rangle & \langle e_2, v_2, w_2, w_1 \rangle & \langle e_1, e_2, v_2, w_1 \rangle & \langle e_1, e_2, w_1, w_2 \rangle \end{array}$$

Let Y be the variety associated to this fan and let Z be the affine toric variety associated to the cone $\Sigma = \langle v_1, v_2, w_1, w_2 \rangle$. The fan Δ is a subdivision of Σ, obtained introducing the vectors e_1 and e_2; let $\varphi : Y \to Z$ be the proper birational morphism associated to this subdivision (see figure 1).

It is straightforward to prove, using basic toric geometry, that

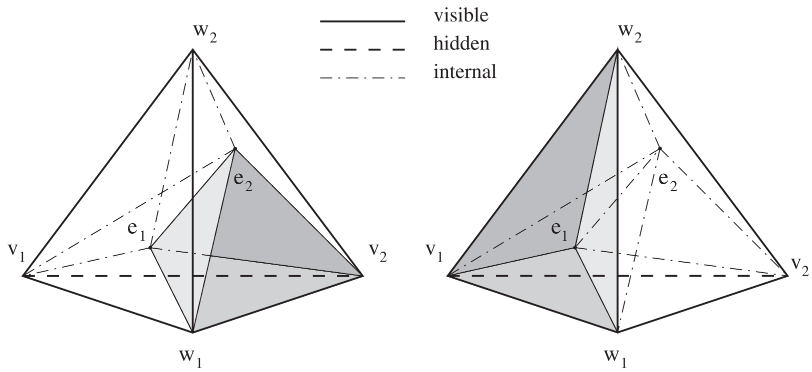

FIGURE 1. The fan Δ and two elements of the subdivision of Δ

1. Y is smooth;
2. Exc(φ) consists of two divisors G_1 and G_2, isomorphic to $\mathbf{P}_{\mathbf{P}^1}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(2))$, which intersect along a quadric $\mathbf{P}^1 \times \mathbf{P}^1$ and are mapped to a point by φ;
3. φ is a Fano-Mori contraction;
4. there is a $\mathbf{Z}_2$-action which interchanges e_1 and e_2, v_1 and v_2, w_1 and w_2.

$N_1(G)$ is generated by two classes of rational curves $[C_1]$ and $[C_2]$: in G_1, $[C_1]$ is the class of lines in the fibers and $[C_2]$ is the class of a minimal section corresponding to a surjection $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(2) \to \mathcal{O}$, while in G_2 these classes are exchanged; moreover the $\mathbf{Z}_2$-action exchanges G_1 with G_2 and $[C_1]$ with $[C_2]$.

7.2. Moebius strip construction

Let C' be a smooth curve with a free $\mathbf{Z}_2$-action, so that the action induces an étale covering $C' \to C$ of degree 2. Take the product actions $\pi : Y \times C' \to X$ and $\pi' : Z \times C' \to W$, where X and W are the quotients; these product actions are free and so X is smooth.

By the universal property of group actions there exists a morphism $\varphi_X : X \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} Y \times C' & \xrightarrow{\varphi \times 1} & Z \times C' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{\varphi_X} & W \end{array} \tag{7.1}$$

CLAIM. The map $\varphi_X : X \to W$ is an elementary Fano-Mori contraction, whose exceptional locus is $E = \pi(G \times C')$ (where G is the exceptional locus of φ and $\varphi(G) = z \in Z$), which is mapped to $C \subset W$; moreover, a vertical slicing of φ_X is the contraction $\varphi : Y \to Z$ constructed in the previous subsection.

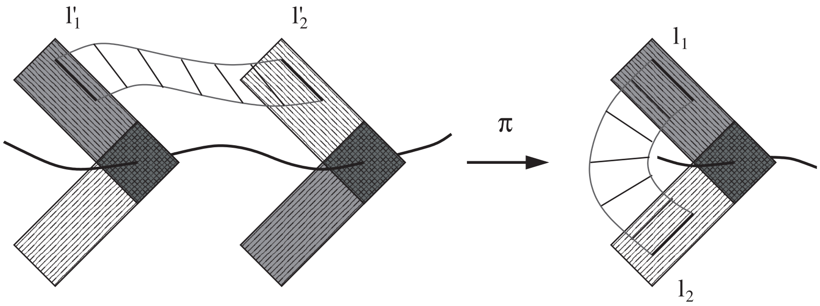

FIGURE 2. ℓ_1 and ℓ_2 are algebraically equivalent

We prove only the non trivial fact that φ_X is elementary.

Fix $c \in C$, and let $F = \varphi_X^{-1}(c)$ be the fiber of φ_X over c; F is isomorphic to G via π and we denote by F_1 and F_2 its irreducible components, which are rational $\mathbf{P}^2$-bundles.

We will show that a line in a fiber of F_1 is algebraically equivalent to a line in a fiber of F_2; since $N_1(F)$ is generated by the classes of such lines, φ_X is elementary.

Let ℓ be a line in a fiber of G_1 and consider the product $\ell \times C' \subset G \times C'$: it is a flat family of rational curves and it is mapped by π into the exceptional locus of φ_X. Let $z \times \{c_1', c_2'\} = \pi'^{-1}(c)$, let $\ell_i' = \ell \times \{c_i'\} \subset G_1 \times \{c_i'\}$ and consider the restriction of the previous diagram

$$\begin{array}{ccc} G \times \{c_1', c_2'\} & \xrightarrow{\varphi \times 1} & z \times \{c_1', c_2'\} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ F & \xrightarrow{\varphi_X} & c \end{array}$$

Since the product action identifies $G_1 \times \{c_1'\}$ with $G_2 \times \{c_2'\}$ and $G_2 \times \{c_1'\}$ with $G_1 \times \{c_2'\}$ we have that $\ell_1 = \pi(\ell_1')$ is a line in a fiber of F_1 and $\ell_2 = \pi(\ell_2')$ is a line in a fiber of F_2.

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