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k-NORMALITY OF WEIGHTED PROJECTIVE SPACES

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Abstract

It is known that a complete linear system on a projective variety in a projective space is generated from the linear system of the projective space by restriction if its degree is sufficiently large. We obtain a bound of degree of linear systems on weighted projective spaces when they are generated from those of the projective spaces. In particular, we show that a weighted projective 3-space embedded by a complete linear system is projectively normal. We treat more generally $\mathbf{Q}$-factorial toric varieties with the Picard number one, and obtain the same bounds for them as those of weighted projective spaces.

Introduction

Let X be a nondegenerate projective variety of dimension n in $\mathbf{P}^r$. It is well known that the homomorphism

$$H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(k)) \to H^0(X, \mathcal{O}_X(k))$$

is surjective for large enough k. We say that X is *k-normal* if this homomorphism is surjective. It is of interest to find an explicit bound k_0 such that all nonsingular, nondegenerate, projective varieties of dimension n and degree d in $\mathbf{P}^r$ are k-normal for all $k \geq k_0$. This was done for curves in $\mathbf{P}^3$ by Castelnuovo [C], and for reduced irreducible curves in $\mathbf{P}^r$, $r \geq 3$ by Gruson, Lazarsfeld and Peskine [GLP]. They showed that the best possible $k_0 = d + 1 - r$. This suggests the equality

$$k_0 = d + n - r.$$

According to Mumford [M1], [M2], we say that X is *k-regular* if $H^i(\mathbf{P}^r, \mathcal{I}_X(k-i)) = 0$ for all $i \geq 1$, where $\mathcal{I}_X$ is the sheaf of ideals of X in $\mathbf{P}^r$. It is easy to see that X is $(k+1)$-regular if and only if X is k-normal and $H^i(X, \mathcal{O}_X(k-i)) = 0$ for all $i \geq 1$. Eisenbud and Goto [EG] conjectured that X is k-regular for all $k \geq d + n - r + 1$. For nonsingular surfaces, Pinkham [P] obtained a bound, and Lazarsfeld [L] obtained the full conjecture. Kwak [Kw1], [Kw2] obtained a good bound for $n = 3, 4$.

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In this paper we obtain a bound of k-normality for a class of toric varieties containing weighted projective spaces. A weighted projective space of dimension n is a quotient of the projective n-space by a finite abelian group. We treat a class of toric varieties that are quotients of the projective n-space by finite abelian groups, in other words, a class of $\mathbf{Q}$-factorial toric varieties with the Picard number one. These toric varieties are defined by integral simplices (see [F], [Od]). We use combinatorics of polytopes corresponding to toric varieties. Herzog and Hibi [HH] also obtain a result on the Castelnuovo regularity of affine semigroup rings defined by integral simplices.

A projective toric variety of dimension one is the projective line. It is known [Ko] that an ample line bundle on a toric surface X is normally generated, i.e., it is very ample and X is k-normal for all $k \geq 1$. In general, it is known [NO] that for an ample line bundle L on a projective toric variety X of dimension n (>1) the multiplication map

$$H^0(X, L^{\otimes i}) \otimes H^0(X, L) \to H^0(X, L^{\otimes i+1})$$

is surjective for all $i \geq n-1$.

Theorem 1. *Let X be a projective toric variety of dimension n which is a quotient of the projective n-space by a finite abelian group, and let L a very ample line bundle on X. Then we have that*

$$H^0(X, L^{\otimes i}) \otimes H^0(X, L) \to H^0(X, L^{\otimes i+1})$$

is surjective for all $i \geq [n/2]$. In particular, any weighted projective 3-space embedded by a very ample line bundle is projectively normal.

Theorem 2. *Let X be a projective toric variety of dimension n $(n > 3)$ which is a quotient of the projective n-space by a finite abelian group embedded by a very ample line bundle in $\mathbf{P}^r$. Then X is k-normal for all $k \geq n - 1 + [n/2]$.*

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1. Polarized toric varieties

First we mention the fact about toric varieties needed in this paper following Oda's book [Od], or Fulton's book [F].

Let N be a free $\mathbf{Z}$-module of rank n, M its dual and $\langle\, , \rangle : M \times N \to \mathbf{Z}$ the canonical pairing. By scalar extension to the field $\mathbf{R}$ of real numbers, we have real vector spaces $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$. Let $T_N := N \otimes_{\mathbf{Z}} \mathbf{C}^* \cong (\mathbf{C}^*)^n$ be the algebraic torus over the complex number field $\mathbf{C}$, where $\mathbf{C}^*$ is the multiplicative group of $\mathbf{C}$. Then $M = \mathrm{Hom}_{\mathrm{gr}}(T_N, \mathbf{C}^*)$ is the character group of T_N. For $m \in M$ we denote $\mathbf{e}(m)$ as the character of T_N. Let Δ be a complete finite fan of N consisting strongly convex rational polyhedral cones σ, that is, with a finite number of elements $v_1, v_2, \ldots, v_s$ in N we can denote

$$\sigma = \mathbf{R}_{\geq 0}v_1 + \cdots + \mathbf{R}_{\geq 0}v_s$$

and it satisfies that $\sigma \cap \{-\sigma\} = \{0\}$. Then we have a complete toric variety $X = T_N \operatorname{emb}(\Delta) := \bigcup_{\sigma \in \Delta} U_\sigma$ of dimension n (see Section 1.2 [Od], or Section 1.4 [F]). Here $U_\sigma = \operatorname{Spec} \mathbf{C}[\sigma^\vee \cap M]$ and $\sigma^\vee$ is the dual cone of σ with respect to the paring $\langle\, , \rangle$. For the origin $\{0\}$, the affine open set $U_{\{0\}} = \operatorname{Spec} \mathbf{C}[M]$ is the unique dense T_N-orbit. We note that a toric variety is always normal.

Let L be an ample T_N-invariant invertible sheaf on X. Then the polarized variety (X, L) corresponds to an integral convex polytope. We call the convex hull $\operatorname{Conv}\{u_0, u_1, \ldots, u_r\}$ in $M_{\mathbf{R}}$ of a finite subset $\{u_0, u_1, \ldots, u_r\} \subset M$ an *integral convex polytope* in $M_{\mathbf{R}}$. The correspondence is given by the isomorphism

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} \mathbf{C}\mathbf{e}(m), \tag{1.1}$$

where $\mathbf{e}(m)$ are considered as rational functions on X because they are functions on an open dense subset T_N of X (see Section 2.2 [Od], or Section 3.5 [F]).

Let P_1 and P_2 be integral convex polytopes in $M_{\mathbf{R}}$. Then we can consider the Minkowski sum $P_1 + P_2 := \{x_1 + x_2 \in M_{\mathbf{R}}; x_i \in P_i \ (i = 1, 2)\}$ and the multiplication by scalars $rP_1 := \{rx \in M_{\mathbf{R}}; x \in P_1\}$ for a positive real number r. If l is a natural number, then lP_1 coincides with the l times sum of P_1, i.e., $lP_1 = \{x_1 + \cdots + x_l \in M_{\mathbf{R}}; x_1, \ldots, x_l \in P_1\}$. The l times twisted sheaf $L^{\otimes l}$ corresponds to the convex polytope $lP := \{lx \in M_{\mathbf{R}}; x \in P\}$. Moreover the multiplication map

$$H^0(X, L^{\otimes l}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (l+1)}) \tag{1.2}$$

transforms $\mathbf{e}(u_1) \otimes \mathbf{e}(u_2)$ for $u_1 \in lP \cap M$ and $u_2 \in P \cap M$ to $\mathbf{e}(u_1 + u_2)$ through the isomorphism (1.1). Therefore the equality $lP \cap M + P \cap M = (l+1)P \cap M$ means the surjectivity of (1.2). For the case of dimension two Koelman [Ko] proved that $lP \cap M + P \cap M = (l+1)P \cap M$ for all natural number l. Nakagawa and Ogata generalize this in the higher dimension.

Proposition 1.1 (Nakagawa-Ogata [NO]). *Let P be an integral polytope of dimension n (>1). Then*

$$iP \cap M + P \cap M = (i+1)P \cap M$$

for all $i \geq n - 1$.

For a proof see Proposition 1.2 in [NO].

In this article we assume that L is *very ample*, that is, the global sections of L defines an embedding of X into the projective space $\mathbf{P}(H^0(X, L)) \cong \mathbf{P}^r$. Since $H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1)) \cong H^0(X, L)$, the k-normality of X implies the surjectivity of the multiplication map $\operatorname{Sym}^k H^0(X, L) \to H^0(X, L^{\otimes k})$. We denote the subset of $kP \cap M$ consisting of sums of k elements in $P \cap M$ by $\sum^k P \cap M$. Then the k-normality means the equality

$$\sum^k P \cap M = kP \cap M. \tag{1.3}$$

Next we may explain how to describe a weighted projective space as a toric variety according to Fulton's book [F]. Let $q_0, q_1, \dots, q_n$ be positive integers with g.c.d.$\{q_0, q_1, \dots, q_n\} = 1$. Then we define the weighted projective n-space with the weight $(q_0, q_1, \dots, q_n)$ as the quotient $\mathbf{P}(q_0, q_1, \dots, q_n) := (\mathbf{C}^{n+1} \backslash \{0\})/\mathbf{C}^*$, where the action of $t \in \mathbf{C}^*$ is defined by $t \cdot (x_0, x_1, \dots, x_n) = (t^{q_0}x_0, t^{q_1}x_1, \dots, t^{q_n}x_n)$. We know that the space can be expressed as the quotient of the projective n-space by an action of a finite abelian group as $\mathbf{P}(q_0, q_1, \dots, q_n) \cong \mathbf{P}^n/(\mathbf{Z}/(q_0) \times \mathbf{Z}/(q_1) \times \cdots \times \mathbf{Z}/(q_n))$. Let $m :=$ l.c.m.$\{q_0, q_1, \dots, q_n\}$ and $d_i = m/q_i$ for $i = 0, 1, \dots, n$. Set $u_0 = (d_0, 0, \dots, 0)$, $u_1 = (0, d_1, 0, \dots, 0), \dots, u_n = (0, \dots, d_n)$ in $\tilde{M} := \mathbf{Z}^{n+1}$. Let $P = \mathrm{Conv}\{u_0, u_1, \dots, u_n\}$ be a convex hull of this $n+1$ points in $\tilde{M}_{\mathbf{R}}$. Let H be the affine hyperplane containing P, and let $M := H \cap \tilde{M}$. Then $P \subset M_{\mathbf{R}} = H$ is an integral convex polytope of M. The integral convex polytope P defines a polarized toric variety $(\mathbf{P}(q_0, q_1, \dots, q_n), \mathcal{O}(m))$. We can easily see that on $\mathbf{P}(1, 6, 10, 15)$ the invertible sheaf $\mathcal{O}(30)$ is ample, but not very ample.

In this paper we treat an integral n-simplex P in $M = \mathbf{Z}^n$, which corresponds not only to a weighted projective space but also to a toric variety defined as a quotient of the projective n-space by a finite abelian group. For example, set $n = 3$ and $P = \mathrm{Conv}\{(0,0,0), (1,0,0), (0,1,0), (3,3,4)\}$. Then the corresponding toric variety X is isomorphic to $\mathbf{P}^3/\langle\zeta\rangle$, where ζ is a primitive 4-th root of unity, and the corresponding embedding is $X \cong \{z_0z_1z_2z_3 = z_4^4\} \subset \mathbf{P}^4$.

2. k-normality

Let n be an integer greater than two and $M = \mathbf{Z}^n$. Let $P = \mathrm{Conv}\{u_0, u_1, \dots, u_n\}$ be an integral n-simplex with its vertices $u_0, u_1, \dots, u_n \in M$. We assume that L is very ample for the polarized toric variety (X, L) corresponding to P. We may say that P is *very ample* when L is very ample.

Lemma 2.1. *Let $P = \mathrm{Conv}\{u_0, u_1, \dots, u_n\}$ be a very ample integral n-simplex. Let s be an integer greater than one and let $x \in sP \cap M$. Then for any u_i there exist $x_1, \dots, x_{2s-1} \in P \cap M$ with $(s-1)u_i + x = x_1 + \cdots + x_{2s-1}$.*

Proof. Since $sP = \mathrm{Conv}\{su_0, su_1, \dots, su_n\}$, any $x \in sP$ can be expressed uniquely as a linear combination $x = \sum_{i=0}^n \mu_i(su_i)$ with $0 \leq \mu_i \leq 1$. We may write as $x = \sum_{i=0}^n \lambda_i u_i$ with $\lambda_i = s\mu_i$. For simplicity we may take u_i as u_0. By an affine transformation of M we may put u_0 as the origin. Then $x = \sum_{i=0}^n \lambda_i u_i$ is contained in tP if and only if $\sum_{i=1}^n \lambda_i \leq t$. Now since $x \in sP$, we have $\sum_{i=1}^n \lambda_i \leq s$. Since P is very ample, the equality (1.3) holds for a sufficiently large k. Hence, for $(k-s)u_0 + x \in kP \cap M$ there exist $x_1, \dots, x_k \in P \cap M$ such that $(k-s)u_0 + x = x_1 + \cdots + x_k$. If $x_1 + x_2 \in P$, then by setting $y_1 = x_1 + x_2$ we have $(k-1-s)u_0 + x = y_1 + x_3 + \cdots + x_k$ with $y_1 \in P \cap M$. If we write as $x_1 + x_2 = \sum_{i=0}^n \lambda_i' u_i$ and if $x_1 + x_2 \notin P$, then $\sum_{i=1}^n \lambda_i' > 1$. Hence, if $x_i + x_j \notin P$ for every i and j, then $\sum_{i=1}^n \lambda_i > \dfrac{k}{2}$. This implies $k < 2s$.

PROPOSITION 2.2. *Let* $P = \mathrm{Conv}\{u_0, u_1, \ldots, u_n\}$ *be an integral n-simplex. If P is very ample, then we have*

$$lP \cap M = (l-1)P \cap M + P \cap M$$

for all $l > n/2$.

In particular, if P is a very ample integral 3*-simplex, then it is normally generated.*

Proof. Set $l \geq 2$. Assume that $lP \cap M \neq (l-1)P \cap M + P \cap M$. Take x in $lP \cap M$ but not in $(l-1)P \cap M + P \cap M$. We can express uniquely as $x = \sum_{i=0}^{n} \lambda_i u_i$ with $\lambda_i \geq 0$ and $\sum_{i=0}^{n} \lambda_i = l$. From Lemma 2.1 there exist $x_1, \ldots, x_{2l-1} \in P \cap M$ such that $(l-1)u_0 + x = x_1 + \cdots + x_{2l-1}$. Move u_0 to the origin. Set $y_j := x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_{2l-1}$. Each y_j is not contained in $(l-1)P$ by the assumption. Since $x = \dfrac{1}{2(l-1)} \sum_{j=1}^{2l-1} y_j$, the point x is not contained in $\dfrac{2l-1}{2(l-1)}(l-1)P = (l-1/2)P$, that is, $\sum_{i=1}^{n} \lambda_i > l - 1/2$. Thus we have $\lambda_0 < 1/2$. This estimate holds for other u_i. Hence we have $\lambda_i < 1/2$ for $i = 0, 1, \ldots, n$. Thus we have $l = \sum_{i=0}^{n} \lambda_i < (n+1)/2$. The inequality $n/2 < l < (n+1)/2$ does not hold. Hence we have $lP \cap M = (l-1)P \cap M + P \cap M$.

LEMMA 2.3. *Let* $P = \mathrm{Conv}\{u_0, u_1, \ldots, u_n\}$ *be an integral n-simplex. For* $l \geq n+1$ *we have*

$$lP = \bigcup_{i=0}^{n} \{u_i + (l-1)P\}.$$

PROPOSITION 2.4. *Let* $n \geq 4$ *and let* $P = \mathrm{Conv}\{u_0, u_1, \ldots, u_n\}$ *a very ample integral n-simplex. For* $l \geq n - 1 + [n/2]$ *we have*

$$\sum^{l} P \cap M = lP \cap M.$$

Proof of Proposition 2.4. Set $t = [n/2]$. Then $l \geq n - 1 + t$. Take $x \in lP \cap M$. We shall find $x_1, \ldots, x_l \in P \cap M$ with $x = x_1 + \cdots + x_l$. If we successively $l - n$ times apply Lemma 2.3, then we can find nonnegative integers $a_0, a_1, \ldots, a_n$ with $\sum_{i=0}^{n} a_i = l - n$ $(\geq t-1)$ and an $x' \in nP \cap M$ such that $x = \sum_{i=0}^{n} a_i u_i + x'$. By applying Proposition 2.2 $n - t$ times to $x' \in nP \cap M$, there exist $x_1, \ldots, x_{n-t} \in P \cap M$ and a $y \in tP \cap M$ such that $x' = y + x_1 + \cdots + x_{n-t}$. If we could find $x_{n-t+1}, \ldots, x_l \in P \cap M$ with $\sum_{i=0}^{n} a_i u_i + y = x_{n-t+1} + \cdots + x_l$, then we complete the proof. It is obtained by the following lemma.

LEMMA 2.5. *Set* $t = [n/2]$. *For nonnegative integers* $a_0, a_1, \ldots, a_n$ *with* $\sum_{i=0}^{n} a_i = t - 1$ *and* $y \in tP \cap M$ *there exist* $y_1, y_2, \ldots, y_{2t-1} \in P \cap M$ *such that*

$$\sum_{i=0}^{n} a_i u_i + y = y_1 + \cdots + y_{2t-1}.$$

Proof. Take a_i to be positive. From Lemma 2.1 there exist $y_1, \dots, y_{2t-1} \in P \cap M$ such that $(t-1)u_i + y = y_1 + \cdots + y_{2t-1}$. Move u_i to the origin. If sum of any two among y_j's is contained in P, then we may write as $y = z_1 + \cdots + z_{t-1} + y_{2t-1}$ with $z_j = y_{2j-1} + y_{2j}$ for $j = 1, \dots, t-1$. Thus y is in $\sum^t P \cap M$. In this case we proved the lemma since $\sum_{i=0}^n a_i u_i \in \sum^{t-1} P \cap M$. If $y_1 + y_2 \notin P$, then $z := y_3 + \cdots + y_{2t-1}$ is containd in $(t-1)P$. Thus we have $u_i + y = y_1 + y_2 + z$ in $(t+1)P \cap M$. Next we consider $\sum_{j \neq i} a_j u_j + (a_i - 1)u_i + z$ for $z \in (t-1)P \cap M$. By induction we obtain a proof.

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