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REGIONS OF VARIABILITY FOR FUNCTIONS OF BOUNDED DERIVATIVES

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To the memory of Professor Nobuyuki Suita

1. Introduction

For a complex number α with $|\alpha| \le 1$ let $\mathscr{B}(\alpha)$ be the class of analytic functions f in the unit disc $\mathbf{D}$ with $f(0) = f'(0) - \alpha = 0$ satisfying $|f'(z)| \le 1$ in $\mathbf{D}$. Similarly for $\operatorname{Re} \alpha > 0$ let $\mathscr{P}(\alpha)$ be the class of analytic functions f in $\mathbf{D}$ with $f(0) = f'(0) - \alpha = 0$ satisfying $\operatorname{Re} f'(z) > 0$ in $\mathbf{D}$. For each $z_0 \in \mathbf{D}$ let

$$V_{\mathscr{B}}(z_0, \alpha) = \{f(z_0) : f \in \mathscr{B}(\alpha)\} \quad \text{for } \alpha \in \overline{\mathbf{D}}, \tag{1}$$

$$V_{\mathscr{P}}(z_0, \alpha) = \{f(z_0) : f \in \mathscr{P}(\alpha)\} \quad \text{for } \operatorname{Re} \alpha > 0. \tag{2}$$

In this paper we shall determine the variability regions $V_{\mathscr{B}}(z_0, \alpha)$ and $V_{\mathscr{P}}(z_0, \alpha)$ explicitly.

Theorem 1. *If $z_0 = 0$ or $|\alpha| = 1$, then $V_{\mathscr{B}}(z_0, \alpha) = \{\alpha z_0\}$. If $z_0 \ne 0$ and $|\alpha| < 1$, then $V_{\mathscr{B}}(z_0, \alpha)$ is the convex closed Jordan domain surrounded by the simple closed curve $\partial\mathbf{D} \ni c \mapsto f_c(z_0)$, where*

$$f_c(z) = \int_0^z \frac{c\zeta + \alpha}{1 + \bar{\alpha}c\zeta}\, d\zeta = \frac{z}{\bar{\alpha}} - \frac{1 - |\alpha|^2}{\bar{\alpha}^2 c} \log(1 + \bar{\alpha}cz), \quad z \in \mathbf{D}.$$

Furthermore if $f(z_0) = f_c(z_0)$ for some $f \in \mathscr{B}(\alpha)$ and $c \in \partial\mathbf{D}$, then $f = f_c$.

As a simple application of Theorem 1 we have the sharp growth estimate as follows.

Corollary 2. *Let $\alpha, z_0 \in \mathbf{D}\backslash\{0\}$. Then for $f \in \mathscr{B}(\alpha)$,*

$$|f(z_0)| \le |f_{c_0}(z_0)| = \frac{|z_0|}{|\alpha|} - \frac{(1 - |\alpha|^2)}{|\alpha|^2} \log(1 + |\alpha|\,|z_0|),$$

where $c_0 = \alpha\overline{z_0}/(|\alpha|\,|z_0|)$ with equality if and only if $f = f_{c_0}$.

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THEOREM 3. *Let* $z_0 \in \mathbf{D}$ *and* $\operatorname{Re} \alpha > 0$. *If* $z_0 = 0$, *then* $V_{\mathscr{P}}(z_0, \alpha) = \{0\}$. *If* $z_0 \neq 0$, *then* $V_{\mathscr{P}}(z_0, \alpha)$ *is the convex closed Jordan domain surrounded by the simple closed curve* $\partial\mathbf{D} \ni c \mapsto \tilde{f}_c(z_0)$, *where*

$$\tilde{f}_c(z) = \int_0^z \left(\frac{\alpha + \bar{\alpha}c\zeta}{1 - c\zeta}\right) d\zeta = -\bar{\alpha}z + \frac{2 \operatorname{Re} \alpha}{c} \log \frac{1}{1 - cz}, \quad z \in \mathbf{D}.$$

Furthermore if $f(z_0) = \tilde{f}_c(z_0)$ *for some* $f \in \mathscr{P}(\alpha)$ *and* $c \in \partial\mathbf{D}$, *then* $f = \tilde{f}_c$.

COROLLARY 4. *For* $z_0 \in \mathbf{D}\backslash\{0\}$ *and* $f \in \mathscr{P}(\alpha)$ *we have*

$$\operatorname{Re} \alpha \left\{\frac{2}{|z_0|} \log(1 + |z_0|) - 1\right\} \leq \operatorname{Re}\left(\frac{f(z_0)}{z_0}\right)$$

with equality if and only if $f = \tilde{f}_{-\bar{z}_0/|z_0|}$. *Also we have*

$$\operatorname{Re}\left(\frac{f(z_0)}{z_0}\right) \leq \operatorname{Re} \alpha \left\{\frac{2}{|z_0|} \log \frac{1}{1 - |z_0|} - 1\right\}$$

with equality if and only if $f = \tilde{f}_{\bar{z}_0/|z_0|}$.

Let S^* be the class of analytic functions f in $\mathbf{D}$ with $f(0) = f'(0) - 1 = 0$ which map $\mathbf{D}$ conformally onto a starlike domain with respect to the origin. A function $f \in S^*$ is said to be univalent starlike. For a positive integer p let $(S^*)^p = \{f = f_0^p : f_0 \in S^*\}$. The proofs of Theorems 1 and 3 heavily rely on the following.

LEMMA 5. *Let* f *be an analytic function in* $\mathbf{D}$ *with* $f(z) = z^p + \cdots$. *If*

$$\operatorname{Re}\left(z\frac{f''(z)}{f'(z)}\right) > -1, \quad z \in \mathbf{D},$$

then $f \in (S^*)^p$.

Although we could not find any references for proofs of the above lemma, it might be well known. See [3] and [1]. For completeness, by making use of Libera's lemma (see [4] and [2]), we shall give an analytic proof of Lemma 5 in Section 3.

2. Regions of variability

In this section, assuming Lemma 5 for the moment, we prove the theorems and corollaries.

Proof of Theorem 1. Clearly we have $V_{\mathscr{B}}(0, \alpha) = \{0\} = \{\alpha \cdot 0\}$. Let $f \in \mathscr{B}(\alpha)$. If $|\alpha| = 1$, then from the maximum modulus theorem we have $f(z) \equiv \alpha z$ and hence $V_{\mathscr{B}}(z_0, \alpha) = \{\alpha z_0\}$.

Suppose that $z_0 \in \mathbf{D}\backslash\{0\}$ and $|\alpha| < 1$. We shall show that $\partial V_{\mathscr{B}}(z_0, \alpha)$ is a simple closed curve and $V_{\mathscr{B}}(z_0, \alpha)$ is the closed domain surrounded by $\partial V_{\mathscr{B}}(z_0, \alpha)$. To this end, since $V_{\mathscr{B}}(z_0, \alpha)$ is a compact and convex subset of $\mathbf{C}$, it suffices to show $V_{\mathscr{B}}(z_0, \alpha)$ contains an open set.

For $|c| \leq 1$ and $z \in \mathbf{D}$ put

$$f_c(z) = \int_0^z \frac{c\zeta + \alpha}{1 + \bar{\alpha}c\zeta}\, d\zeta = \frac{z}{\bar{\alpha}} - \frac{1 - |\alpha|^2}{\bar{\alpha}^2 c} \log(1 + \bar{\alpha}cz). \tag{3}$$

Then it is easy to see that $f_c \in \mathscr{B}(\alpha)$. Since for any fixed $z_0 \in \mathbf{D}\backslash\{0\}$, the function $\mathbf{D} \ni c \mapsto f_c(z_0)$ is nonconstant analytic, it is an open mapping. Thus $V_{\mathscr{B}}(z_0, \alpha)$ contains the open set $\{f_c(z_0) : |c| < 1\}$.

Next we shall show that $f_c(z_0) \in \partial V_{\mathscr{B}}(z_0, \alpha)$ for all $c \in \partial\mathbf{D}$. For $f \in \mathscr{B}(\alpha)$ we have from Schwarz's lemma

$$\left| \frac{f'(z) - \alpha}{1 - \bar{\alpha}f'(z)} \right| \leq |z|,$$

which is equivalent to

$$\left| f'(z) - \frac{\alpha(1 - |z|^2)}{1 - |\alpha|^2|z|^2} \right| \leq \frac{(1 - |\alpha|^2)|z|}{1 - |\alpha|^2|z|^2}. \tag{4}$$

Thus for any C^1-curve $\gamma : z = z(t)$, $0 \leq t \leq 1$, with $z(0) = 0$ and $z(1) = z_0$ we have

$$\begin{aligned}
&\left| f(z_0) - \int_0^1 \frac{\alpha(1 - |z(t)|^2)}{1 - |\alpha|^2|z(t)|^2} z'(t)\, dt \right| \\
&\quad = \left| \int_0^1 \left\{ f'(z(t)) - \frac{\alpha(1 - |z(t)|^2)}{1 - |\alpha|^2|z(t)|^2} \right\} z'(t)\, dt \right| \\
&\quad \leq \int_0^1 \left| f'(z(t)) - \frac{\alpha(1 - |z(t)|^2)}{1 - |\alpha|^2|z(t)|^2} \right| |z'(t)|\, dt \\
&\quad \leq \int_0^1 \frac{(1 - |\alpha|^2)|z(t)|}{1 - |\alpha|^2|z(t)|^2} |z'(t)|\, dt.
\end{aligned}$$

This implies $f(z_0) \in \overline{\mathbf{D}}(C(\alpha, \gamma), R(\alpha, \gamma)) = \{z \in \mathbf{C} : |z - C(\alpha, \gamma)| \leq R(\alpha, \gamma)\}$, where

$$C(\alpha, \gamma) = \int_0^1 \frac{\alpha(1 - |z(t)|^2)}{1 - |\alpha|^2|z(t)|^2} z'(t)\, dt, \tag{5}$$

$$R(\alpha, \gamma) = \int_0^1 \frac{(1 - |\alpha|^2)|z(t)|}{1 - |\alpha|^2|z(t)|^2} |z'(t)|\, dt. \tag{6}$$

Thus we have $V_{\mathscr{B}}(z_0, \alpha) \subset \overline{\mathbf{D}}(C(\alpha, \gamma), R(\alpha, \gamma))$.

While we have for $|c| = 1$

$$
f_c'(z) - \frac{\alpha(1-|z|^2)}{1-|\alpha|^2|z|^2} = \frac{(1-|\alpha|^2)z(c+\alpha\bar{z})}{(1-|\alpha|^2|z|^2)(1+\bar{\alpha}cz)} = \frac{c(1-|\alpha|^2)}{1-|\alpha|^2|z|^2}\frac{|z|g(z)}{|g(z)|}, \tag{7}
$$

where

$$
g(z) = \frac{z}{(1+\bar{\alpha}cz)^2}. \tag{8}
$$

Since for $z \in \mathbf{D}$

$$
\mathrm{Re}\left(z\frac{g'(z)}{g(z)}\right) = \mathrm{Re}\left(\frac{1-\alpha cz}{1+\alpha cz}\right) > 0,
$$

by Lemma 5 we have $G(z) = 2\int_0^z g(\zeta)\,d\zeta \in (S^*)^2$. Thus there exists $G_0 \in S^*$ such that $G = G_0^2$. Put $\gamma_0 : z = z(t) = G_0^{-1}(t^{1/2}G_0(z_0))$, $0 \le t \le 1$. Then we have $G(z(t)) = tG(z_0)$ and hence $2g(z(t))z'(t) = G'(z(t))z'(t) = G(z_0)$. Thus from (7) we have

$$
\left(f_c'(z(t)) - \frac{\alpha(1-|z(t)|^2)}{1-|\alpha|^2|z(t)|^2}\right)z'(t) = \frac{c(1-|\alpha|^2)|z(t)|}{1-|\alpha|^2|z(t)|^2}\frac{g(z(t))z'(t)}{|g(z(t))|} = \frac{cG(z_0)}{|G(z_0)|}\frac{(1-|\alpha|^2)|z(t)|}{1-|\alpha|^2|z(t)|^2}|z'(t)| \tag{9}
$$

and

$$
f_c(z_0) - C(\alpha,\gamma_0) = \frac{cG(z_0)}{|G(z_0)|}\int_0^1 \frac{(1-|\alpha|^2)|z(t)|}{1-|\alpha|^2|z(t)|^2}|z'(t)|\,dt = \frac{cG(z_0)}{|G(z_0)|}R(\alpha,\gamma_0). \tag{10}
$$

Specially we have $f_c(z_0) \in \partial\mathbf{D}(C(\alpha,\gamma_0), R(\alpha,\gamma_0))$. Since $f_c(z_0) \in V_{\mathscr{B}}(z_0,\alpha) \subset \overline{\mathbf{D}}(C(\alpha,\gamma_0), R(\alpha,\gamma_0))$, we have $f_c(z_0) \in \partial V_{\mathscr{B}}(z_0,\alpha)$.

We deal with uniqueness and show the injectivity of the mapping $\partial\mathbf{D} \ni c \mapsto f_c(z_0)$. Suppose that $f(z_0) = f_c(z_0)$ for some $f \in \mathscr{B}(\alpha)$ and $c \in \partial\mathbf{D}$. Put

$$
k(t) = \frac{|G(z_0)|}{cG(z_0)}\left(f'(z(t)) - \frac{\alpha(1-|z(t)|^2)}{1-|\alpha|^2|z(t)|^2}\right)z'(t).
$$

Then $k(t)$ is a continuous function of $t \in [0,1]$. And we have from (4) and (9)

$$\begin{aligned}\operatorname{Re} k(t) \le |k(t)| &\le \frac{(1-|\alpha|^2)|z(t)|}{1-|\alpha|^2|z(t)|^2}|z'(t)| \\ &= \frac{|G(z_0)|}{cG(z_0)}\left(f_c'(z(t)) - \frac{\alpha(1-|z(t)|^2)}{1-|\alpha|^2|z(t)|^2}\right)z'(t).\end{aligned}$$

Thus we have from (9) and (10)

$$\begin{aligned}R(\alpha,\gamma_0) &= \operatorname{Re}\left\{\frac{|G(z_0)|}{cG(z_0)}(f_c(z_0) - C(\alpha,\gamma_0))\right\} \\ &= \operatorname{Re}\left\{\frac{|G(z_0)|}{cG(z_0)}(f(z_0) - C(\alpha,\gamma_0))\right\} \\ &= \int_0^1 \operatorname{Re} k(t)\, dt \\ &\le \int_0^1 \frac{|G(z_0)|}{cG(z_0)}\left(f_c'(z(t)) - \frac{\alpha(1-|z(t)|^2)}{1-|\alpha|^2|z(t)|^2}\right)z'(t)\, dt \\ &= R(\alpha,\gamma_0).\end{aligned}$$

This implies

$$k(t) = \frac{|G(z_0)|}{cG(z_0)}\left(f_c'(z(t)) - \frac{\alpha(1-|z(t)|^2)}{1-|\alpha|^2|z(t)|^2}\right)z'(t).$$

for all $t \in [0,1]$. Hence we have $f' = f_c'$ on γ_0. From the identity theorem for analytic functions we have $f' = f_c'$ and hence $f = f_c$ by normalization.

Finally suppose that the mapping $\partial\mathbf{D} \ni c \mapsto f_c(z_0)$ is not injective. Then there exist $c_1, c_2 \in \partial\mathbf{D}$ with $c_1 \neq c_2$ such that $f_{c_1}(z_0) = f_{c_2}(z_0)$. Since $f_{c_1}, f_{c_2} \in \mathscr{B}(\alpha)$, we have $f_{c_1} = f_{c_2}$ from uniqueness. This contradicts $c_1 \neq c_2$.

We have shown that the simple closed curve $\partial V_{\mathscr{B}}(z_0,\alpha)$ contains the curve $\partial\mathbf{D} \ni c \mapsto f_c(z_0)$. Since a simple closed curve can not contain any simple closed curve other than itself, $\partial V_{\mathscr{B}}(z_0,\alpha)$ coincides with the curve $\partial\mathbf{D} \ni c \mapsto f_c(z_0)$. □

Proof of Corollary 2. For $w, \beta \in \mathbf{D}\backslash\{0\}$ we have

$$\left|\frac{w+\beta}{1+\bar{\beta}w}\right| \le \frac{|w|+|\beta|}{1+|\beta|\,|w|}$$

with equality if and only if $\bar{\beta}w$ is positive. Thus for $|c| = 1$ we have

$$\begin{aligned}|f_c(z_0)| &= \left|z_0\int_0^1 \frac{cz_0t+\alpha}{1+\bar{\alpha}cz_0t}\,dt\right| \\ &\le |z_0|\int_0^1 \frac{|z_0|t+|\alpha|}{1+|\alpha|\,|z_0|t}\,dt = \frac{|z_0|}{|\alpha|} - \frac{1-|\alpha|^2}{|\alpha|^2}\log(1+|\alpha|\,|z_0|) = |f_{c_0}(z_0)|\end{aligned}$$

with equality if and only if $c = c_0 = \alpha\overline{z_0}/(|\alpha|\,|z_0|)$. Combining this and Theorem 1 we have for any $f \in \mathscr{B}(\alpha)$

$$|f(z_0)| \le \max_{|c|=1} |f_c(z_0)| = |f_{c_0}(z_0)|$$

with equality if and only if $f = f_{c_0}$. $\square$

Our proof of Theorem 3 is quite similar to that of Theorem 1. We only outline the proof, details of which may be supplied by the interested reader.

Proof of Theorem 3. Clearly $V_{\mathscr{P}}(0,\alpha) = \{0\}$. Suppose $z_0 \in \mathbf{D}\backslash\{0\}$. It is easy to see that $V_{\mathscr{P}}(z_0,\alpha)$ is compact and convex subset of $\mathbf{C}$. For $|c| \le 1$ put

$$\tilde{f}_c(z) = \int_0^z \frac{\alpha + \bar{\alpha}c\zeta}{1 - c\zeta}\, d\zeta = -\bar{\alpha}z + \frac{2(\mathrm{Re}\,\alpha)}{c} \log \frac{1}{1 - cz} \in \mathscr{P}(\alpha). \tag{11}$$

Since the mapping $\mathbf{D} \ni c \mapsto \tilde{f}_c(z_0)$ is open, $V_{\mathscr{P}}(z_0,\alpha)$ contains the nonempty open set $\{\tilde{f}_c(z_0) : |c| < 1\}$. Thus $V_{\mathscr{P}}(z_0,\alpha)$ is a closed Jordan domain surrounded by the simple closed curve $\partial V_{\mathscr{P}}(z_0,\alpha)$.

Let $f \in \mathscr{P}(\alpha)$. From the Schwarz's lemma we have

$$\left|\frac{f'(z) - \alpha}{f'(z) + \bar{\alpha}}\right| \le |z|,$$

which is equivalent to

$$\left| f'(z) - \frac{\alpha + \bar{\alpha}|z|^2}{1 - |z|^2} \right| \le \frac{2(\mathrm{Re}\,\alpha)|z|}{1 - |z|^2}. \tag{12}$$

This implies

$$V_{\mathscr{P}}(z_0,\alpha) \subset \overline{\mathbf{D}}(\tilde{C}(\alpha,\gamma), \tilde{R}(\alpha,\gamma)), \tag{13}$$

where $\gamma : z(t),\ 0 \le t \le 1$, is any C^1-curve with $z(0) = 0,\ z(1) = z_0$ and

$$\tilde{C}(\alpha,\gamma) = \int_0^1 \frac{\alpha + \bar{\alpha}|z(t)|^2}{1 - |z(t)|^2} z'(t)\, dt, \tag{14}$$

$$\tilde{R}(\alpha,\gamma) = \int_0^1 \frac{2(\mathrm{Re}\,\alpha)|z(t)|}{1 - |z(t)|^2} |z'(t)|\, dt. \tag{15}$$

While we have

$$\tilde{f}_c'(z) - \frac{\alpha + \bar{\alpha}|z|^2}{1 - |z|^2} = \frac{2(\mathrm{Re}\,\alpha)c|z|}{1 - |z|^2} \frac{\tilde{g}(z)}{|\tilde{g}(z)|}, \tag{16}$$

where

(17) $$\tilde{g}(z) = \frac{z}{(1-cz)^2}.$$

Since $\operatorname{Re}(z\tilde{g}'(z)/\tilde{g}(z)) = \operatorname{Re}\{(1+cz)(1-cz)^{-1}\} > 0$ for $|z| < 1$, there exists $\tilde{G}_0 \in S^*$ satisfying $\tilde{G}(z) = 2\int_0^z g(\zeta)\,d\zeta = \tilde{G}_0(z)^2$ by Lemma 5. Putting $\gamma_0 : z(t) = \tilde{G}_0^{-1}(\sqrt{t}\tilde{G}_0(z_0))$, we have $2\tilde{g}(z(t))z'(t) = \tilde{G}(z_0)$ and

(18) $$\left\{\tilde{f}_c'(z(t)) - \frac{\alpha + \bar{\alpha}|z(t)|^2}{1-|z(t)|^2}\right\}z'(t) = \frac{c\tilde{G}(z_0)}{|\tilde{G}(z_0)|}\frac{2(\operatorname{Re}\alpha)|z(t)|}{(1-|z(t)|^2)}|z'(t)|.$$

By integrating the above equality we have

(19) $$\tilde{f}_c(z_0) - \tilde{C}(\alpha, \gamma_0) = \frac{c\tilde{G}(z_0)}{|\tilde{G}(z_0)|}\tilde{R}(\alpha, \gamma_0).$$

Since $\tilde{f}_c(z_0) \in V_{\mathscr{P}}(z_0, \alpha) \subset \overline{\mathbf{D}}(\tilde{C}(\alpha, \gamma_0), \tilde{R}(\alpha, \gamma_0))$ and $\tilde{f}_c(z_0) \in \partial\mathbf{D}(\tilde{C}(\alpha, \gamma_0), \tilde{R}(\alpha, \gamma_0))$, we have $\tilde{f}_c(z_0) \in \partial V_{\mathscr{P}}(z_0, \alpha)$.

Suppose that $f(z_0) = \tilde{f}_c(z_0)$ for some $f \in \mathscr{P}(\alpha)$ and $|c| = 1$. Put

(20) $$\tilde{k}(t) = \frac{|\tilde{G}(z_0)|}{c\tilde{G}(z_0)}\left\{f'(z(t)) - \frac{\alpha + \bar{\alpha}|z(t)|^2}{1-|z(t)|^2}\right\}z'(t).$$

Then we have from (12) and (18)

(21) $$\operatorname{Re}\tilde{k}(t) \le |\tilde{k}(t)| \le \frac{|\tilde{G}(z_0)|}{c\tilde{G}(z_0)}\left\{\tilde{f}_c'(z(t)) - \frac{\alpha + \bar{\alpha}|z(t)|^2}{1-|z(t)|^2}\right\}z'(t).$$

Thus we have from (18), (19) and $f(z_0) = \tilde{f}_c(z_0)$

$$\begin{aligned}\tilde{R}(\alpha, \gamma_0) &= \operatorname{Re}\tilde{R}(\alpha, \gamma_0)\\ &= \int_0^1 \operatorname{Re}\tilde{k}(t)\,dt\\ &\le \int_0^1 \frac{|\tilde{G}(z_0)|}{c\tilde{G}(z_0)}\left\{\tilde{f}_c'(z(t)) - \frac{\alpha + \bar{\alpha}|z(t)|^2}{1-|z(t)|^2}\right\}z'(t)\,dt\\ &= \tilde{R}(\alpha, \gamma_0).\end{aligned}$$

Since $\tilde{k}(t)$ is a continuous function of $t \in [0,1]$, this implies $f' = \tilde{f}_c'$ on γ_0 and hence $f = \tilde{f}_c$ by normalization.

As in the proof of Theorem 2 it easily follows from uniqueness that the closed curve $\partial\mathbf{D} \ni c \mapsto \tilde{f}_c(z_0)$ is simple and coincides with $\partial V_{\mathscr{B}}(z_0, \alpha)$. □

Proof of Corollary 4. From (11) we have

$$\tilde{f}_c(z_0) - i(\mathrm{Im}\,\alpha)z_0 = \mathrm{Re}\,\alpha \int_0^{z_0} \frac{1 + c\zeta}{1 - c\zeta}\,d\zeta$$
$$= (\mathrm{Re}\,\alpha)z_0 \int_0^1 \frac{1 + cz_0t}{1 - cz_0t}\,dt.$$

Thus we have

$$\mathrm{Re}\left(\frac{\tilde{f}_c(z_0)}{z_0}\right) = (\mathrm{Re}\,\alpha) \int_0^1 \mathrm{Re}\left(\frac{1 + cz_0t}{1 - cz_0t}\right) dt$$
$$= (\mathrm{Re}\,\alpha) \int_0^1 \frac{1 - |z_0|^2t^2}{|1 - cz_0t|^2}\,dt.$$
$$\geq (\mathrm{Re}\,\alpha) \int_0^1 \frac{1 - |z_0|t}{1 + |z_0|t}\,dt = (\mathrm{Re}\,\alpha)\left\{\frac{2}{|z_0|}\log(1 + |z_0|) - 1\right\}$$

with equality if and only if $c = -\bar{z}_0/|z_0|$. From this and Theorem 3 we have for $f \in \mathscr{P}(\alpha)$

$$\mathrm{Re}\left(\frac{f(z_0)}{z_0}\right) \geq \min_{|c|=1} \mathrm{Re}\left(\frac{\tilde{f}_c(z_0)}{z_0}\right)$$
$$= (\mathrm{Re}\,\alpha)\left\{\frac{2}{|z_0|}\log(1 + |z_0|) - 1\right\}$$

with equality if and only if $f = \tilde{f}_{-\bar{z}_0/|z_0|}$.

Since the proof of the other half of the corollary is quite similar, we omit it. □

3. Sufficient conditions for Multivalent starlikeness

Let p be a positive integer and $\mathscr{A}_p$ be the class of analytic functions f in $\mathbf{D}$ with $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$.

Lemma 6. *Let $f \in \mathscr{A}_p$. Then $f \in (S^*)^p$ if and only if*

$$\mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right) > 0, \quad z \in \mathbf{D}. \tag{22}$$

In the case that $p = 1$ the condition (22) implies that f is starlike univalent. When $p \geq 2$, see [5] for a proof.

Proof of Lemma 5. Let $p = 1, 2, \ldots$ and $f \in \mathscr{A}_p$ with $1 + \mathrm{Re}(zf''(z)/f'(z)) > 0$ in $\mathbf{D}$. From Lemma 6 it suffices to prove $\mathrm{Re}(zf'(z)/f(z)) > 0$ in $\mathbf{D}$.

First we show that $p^{-1}zf'(z) \in (S^*)^p$. Since $f'(z) \neq 0$ for all $z \in \mathbf{D}\backslash\{0\}$, there exists an analytic function $f_1 \in \mathscr{A}_1$ such that $zf'(z) = pf_1(z)^p$ in $\mathbf{D}$. The fact that $f_1 \in S^*$ easily follows from

$$p \operatorname{Re}\left(z \frac{f_1'(z)}{f_1(z)}\right) = \operatorname{Re}\left(z \frac{f''(z)}{f'(z)}\right) + 1 > 0.$$

For any $z_1 \in \mathbf{D} \backslash \{0\}$ put $z(t) = f_1^{-1}(t^{1/p} f_1(z_1))$, $0 \le t \le 1$. Then we have $z(t) f'(z(t)) = pf_1(z(t))^p = ptf_1(z_1)^p = tz_1 f'(z_1)$ and hence

$$\{f'(z(t)) + z(t) f''(z(t))\} z'(t) = \frac{d}{dt}\{z(t) f'(z(t))\} = z_1 f'(z_1).$$

Thus we have for any $z_1 \in \mathbf{D} \backslash \{0\}$

$$\begin{aligned}
\operatorname{Re}\left(\frac{f(z_1)}{z_1 f'(z_1)}\right) &= \operatorname{Re}\left\{\int_0^1 \frac{f'(z(t)) z'(t)}{z_1 f'(z_1)}\, dt\right\} \\
&= \int_0^1 \operatorname{Re}\left\{\frac{f'(z(t))}{f'(z(t)) + z(t) f''(z(t))}\right\} dt \\
&= \int_0^1 \operatorname{Re}\left\{\frac{1}{1 + \dfrac{z(t) f''(z(t))}{f'(z(t))}}\right\} dt \\
&= \int_0^1 \left\{\frac{1 + \operatorname{Re}\left(\dfrac{z(t) f''(z(t))}{f'(z(t))}\right)}{\left|1 + \dfrac{z(t) f''(z(t))}{f'(z(t))}\right|^2}\right\} dt > 0.
\end{aligned}$$

Combining this and $\lim_{z \to 0} zf'(z)/f(z) = p$ we have $\operatorname{Re}(zf'(z)/f(z)) > 0$ in $\mathbf{D}$. □

Following Goodman [1] let $\mathscr{C}(p)$ be the class of analytic functions f in $\mathbf{D}$ such that there exists $r \in (0,1)$ with the following properties:

(i)

$$\operatorname{Re}\left(z \frac{f''(z)}{f'(z)} + 1\right) > 0 \quad \text{for } r < |z| < 1,$$

(ii)

$$\int_{-\pi}^{\pi} \operatorname{Re}\left(\rho e^{i\theta} \frac{f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})} + 1\right) d\theta = 2\pi p \quad \text{for } r < \rho < 1.$$

Theorem 7. *Let $f \in \mathscr{C}(2)$. Then there exists $z_0 \in \mathbf{D}$ such that $f'(z_0) = 0$, $f''(z_0) \neq 0$ and $f'(z) \neq 0$ in $\mathbf{D} \backslash \{z_0\}$, and that*

$$\frac{1}{(1 - |z_0|^2)^2 f''(z_0)} \left\{ f\left(\frac{z + z_0}{1 + \bar{z}_0 z}\right) - f(z_0) \right\} \in (S^*)^2.$$

Proof. Put

$$h(z) = z\frac{f''(z)}{f'(z)} + 1, \quad z \in \mathbf{D}.$$

Since for $r < \rho < 1$

$$4\pi = \int_0^{2\pi} \mathrm{Re}\{h(\rho e^{i\theta})\}\, d\theta = \mathrm{Re}\left\{\int_{|z|=\rho}\left(z\frac{f''(z)}{f'(z)} + 1\right)\frac{dz}{iz}\right\}$$
$$= 2\pi + \mathrm{Im}\left\{\int_{|z|=\rho}\frac{f''(z)}{f'(z)}\, dz\right\},$$

there exists $z_0 \in \mathbf{D}$ such that $f'(z_0) = 0$, $f''(z_0) \neq 0$ and $f'(z) \neq 0$ for all $z \in \mathbf{D}\backslash\{z_0\}$. Thus the function

$$g(z) = f\left(\frac{z + z_0}{1 + \overline{z_0}z}\right) - f(z_0), \quad z \in \mathbf{D}$$

satisfies $g(0) = g'(0) = 0$, $g''(0) = (1 - |z_0|^2)^2 f''(z_0) \neq 0$ and $g'(z) \neq 0$ for all $z \in \mathbf{D}\backslash\{0\}$. Hence the function $1 + (zg''(z)/g'(z))$ is analytic in $\mathbf{D}$ and its real part is harmonic in $\mathbf{D}$.

We claim that the inequality

$$\mathrm{Re}\left(z\frac{g''(z)}{g'(z)} + 1\right) > 0$$

holds in $\mathbf{D}$.

From an elementary calculation we have

$$z\frac{g''(z)}{g'(z)} + 1 = 1 - \frac{2\overline{z_0}z}{1 + \overline{z_0}z} + \frac{(1 - |z_0|^2)z}{(1 + \overline{z_0}z)^2}\frac{f''\left(\dfrac{z + z_0}{1 + \overline{z_0}z}\right)}{f'\left(\dfrac{z + z_0}{1 + \overline{z_0}z}\right)} \tag{23}$$
$$= \frac{z_0 - \overline{z_0}z^2}{(z + z_0)(1 + \overline{z_0}z)} + \frac{(1 - |z_0|^2)z}{(z + z_0)(1 + \overline{z_0}z)} h\left(\frac{z + z_0}{1 + \overline{z_0}z}\right).$$

Without loss of generality we may assume h is continuous on $\overline{\mathbf{D}}$. Since for $|z| = 1$ we have from the identities

$$\frac{z_0 - \overline{z_0}z^2}{(z + z_0)(1 + \overline{z_0}z)} = -2i\frac{\mathrm{Im}(\overline{z_0}z)}{|z + z_0|^2},$$
$$\frac{(1 - |z_0|^2)z}{(z + z_0)(1 + \overline{z_0}z)} = \frac{1 - |z_0|^2}{|z + z_0|^2},$$

it follows that

$$\liminf_{\mathbf{D} \ni z \to \zeta} \operatorname{Re}\left(z \frac{g''(z)}{g'(z)} + 1\right) \geq 0$$

for all $\zeta \in \partial \mathbf{D}$. From this and $\lim_{z \to 0} zg'(z)/g(z) = 1$ we have $\operatorname{Re}(zg''(z)/g'(z)) + 1 > 0$ in $\mathbf{D}$.

Now the fact that $2(1 - |z_0|^2)^{-2} f''(z_0)^{-1} g \in (S^*)^2$ easily follows from Lemma 5. □

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