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ON COMPLEX ANALYTIC PROPERTIES OF LIMIT SETS AND JULIA SETS

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To the memory of Professor Nobuyuki Suita

Abstract

It is known that Kleinian groups and complex dynamics have many properties in common. We shall show that the structure of the Martin boundaries of the region of discontinuity and the Fatou set are similar under a certain condition. We also show that any conformal mapping defined on the region of discontinuity (resp. the Fatou set) has a continuous extension on the conical limit set (resp. the conical Julia set in the sense of Lyubich and Minsky).

1. Introduction and statements of results

Let $f(z)$ be a rational function of degree $d \geq 2$ and Γ a non-elementary Kleinian group. A lot of computational experiences show that the Julia set $J(f)$ of f and the limit set $\Lambda(\Gamma)$ of Γ seem to be similar. In this paper, we shall prove that Julia sets and limit sets have some complex analytic properties in common. Especially, we focus on conical limit sets of Fuchsian groups and conical Julia sets of rational functions in the sense of Lyubich and Minsky. In a study of Hausdorff dimensions, it is observed that some statements are hold for these sets simultaneously ([8], [9]). We show that several common phenomena are observed in both sets from the view point of the Martin compactification and functions theoretic null sets.

First, we consider the Martin compactification of the Fatou sets $F(f)$ of a rational map f and the region of discontinuity $\Omega(\Gamma)$ of a Kleinian group Γ, a discrete subgroup of $PSL(2, \mathbf{C})$.

Suppose that both $F(f)$ and $\Omega(\Gamma)$ are connected. We denote by $F(f)^*$ and $\Omega(\Gamma)^*$ the Martin compactification of the Fatou set and the region of discontinuity, respectively. In the theory of the Martin compactification, the Martin boundary consists of two disjoint sets, minimal points and non-minimal points. Important is the set of minimal points because the representing measure for a positive harmonic function is supported on the set of minimal points (Proposition 2.1).

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For $\zeta \in \partial F(f)$ (resp. $\in \partial\Omega(\Gamma)$), we denote by $\Delta(\zeta)$ the set of the Martin boundary over ζ (see Sec. 2 for the terminologies).

THEOREM 1.1. *Let $\Delta(\zeta)$ be the same one as above.*

(A): *Suppose that Γ be a Fuchsian group of the second kind. If $\zeta \in \Lambda(\Gamma)$ is a parabolic fixed point of Γ, then $\Delta(\zeta)$ contains exactly two minimal boundary points. On the other hand, if ζ is a conical limit point, then $\Delta(\zeta)$ consists of a single minimal point. Furthermore, if Γ is a Schottky group, then the Martin compactification $\Omega(\Gamma)^*$ is homeomorphic to the Riemann sphere $\hat{\mathbf{C}}$.*

(B): *Let f be a rational function of degree $d \geq 2$ with totally disconnected Julia set $J(f)$. Then for every ζ in the conical Julia set $J_{con}(f)$ of f, $\Delta(\zeta)$ consists of a single minimal point.*

From this theorem, we have:

COROLLARY 1.1. *Let $P(z)$ be a polynomial of degree $d \geq 2$. Suppose that all critical points are attracted to ∞. Then for every $\zeta \in J(P)$, $\Delta(\zeta)$ consists of a single minimal point. Therefore, the Martin compactification $F(P)^*$ of the Fatou set is homeomorphic to $\hat{\mathbf{C}}$.*

COROLLARY 1.2. *Suppose that $B(z)$ is a finite Blaschke product of degree $d \geq 2$ with the disconnected Julia set $J(B)$ on the unit circle. If $\zeta \in J(B)$ does not belong to the backward orbit of the parabolic fixed point of B, then $\Delta(\zeta)$ consists of a single minimal point. In particular, if $B(z)$ does not have a parabolic fixed point and if $J(B)$ is totally disconnected, then the Martin compactification $F(B)^*$ of the Fatou set is homeomorphic to $\hat{\mathbf{C}}$.*

Next, we consider a continuous extendability of quasiconformal mappings defined on $\Omega(\Gamma)$ and $F(f)$.

A compact subset E of $\mathbf{C}$ is called *holomorphically removable* if any homeomorphism on a neighbourhood U of E which is holomorphic on $U - E$ is holomorphically extended to U. It is known that the Julia set $J(f)$ is holomorphically removable if f satisfies a certain condition (cf. [3], [15]).

Here, we consider a strong removability for a totally disconnected compact subset. Namely, we consider the extendability of holomorphic functions defined *only on the outside* of E. In general, there exists a totally disconnected compact set $E \subset \mathbf{C}$ such that a conformal mapping on the complement of E does not have a limit at some point of E (cf. [11] XI. 3L). But for conical limit sets and conical Julia set, the situation is quite satisfactory if they are totally disconnected.

THEOREM 1.2. **(A):** *Let Γ be a Fuchsian group of the second kind. Then, for any neighbourhood U of the limit set $\Lambda(\Gamma)$ every quasiconformal mapping defined on $U \cap \Omega(\Gamma)$ has a limit at every conical limit point* *of* Γ. *Furthermore, if* Γ *is finitely generated and has no parabolic elements, then every quasiconformal (resp. conformal) mapping on* $U \cap \Omega(\Gamma)$ *is extended to a quasiconformal (resp. conformal) map of* U.

(B1): *Let* f *be a rational function of degree* $d \geq 2$ *with totally disconnected Julia set* $J(f)$. *Then, for any neighbourhood* U *of the Julia set every quasiconformal mapping defined on* $U \cap F(f)$ *has a limit at every point in the conical Julia set of* f. *Furthermore, if every point of the Julia set of* f *belongs to the conical Julia set of* f, *then every quasiconformal (resp. conformal) mapping defined on* $U \cap F(f)$ *is extended to a quasiconformal (resp. conformal) mapping on* U.

(B2): *Let* $B(z)$ *be a finite Blaschke product of degree* $d \geq 2$ *with the disconnected Julia set* $J(B)$. *Then, for any neighbourhood* U *of* $J(B)$, *every bounded holomorphic function on* $U \cap F(B)$ *is extended to a holomorphic function on* U. *Furthermore, every quasiconformal mapping defined on* $U \cap F(B)$ *is extended to a quasiconformal mapping on* U. *In particular, any conformal mapping on* $F(B)$ *is a Möbius transformation.*

From the above theorem, we see that if Γ is a Schottky group, then any quasiconformal (resp. conformal) mapping on $\Omega(\Gamma)$ is extended to a quasiconformal (resp. conformal) map of $\hat{\mathbf{C}}$. Also we see that if f is a hyperbolic rational map with the totally disconnected Julia set, then every quasiconformal mapping on the Fatou set is extended to a quasiconformal mapping on the Riemann sphere and every conformal mapping on the Fatou set must be a Möbius transformation.

Finally, we shall consider the extendability of quasiconformal mappings to the Martin boundary, which is one of interesting problems in the theory of the Martin compactification. It is known ([14]) that there are a plane region D and a quasiconformal mapping w on D such that the mapping w *does not* have a continuous extension to the Martin compactification D^* of D.

Combining our results, we have:

Corollary 1.3. *Let* D *be either a region of discontinuity of a Schottky group or the Fatou set of a finite Blaschke product with no parabolic fixed point or a rational function* f *with* $J_{con}(f) = J(f)$. *Then, every quasiconformal mapping* w *on* D *has a homeomorphic extension to the Martin compactification* D^* *of* D.

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2. Preliminaries

2.1. Conical limit points and conical Julia sets. We define conical limit points for Fuchsian groups and conical Julia sets for rational functions which is defined in [8].

A Fuchsian group Γ acting on $\mathbf{H}$ is called of *the first kind* if the limit set $\Lambda(\Gamma) = \hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. If Γ is not of the first kind, it is called of the *second kind*. The limit set of a non-elementary Fuchsian group of the second kind is totally disconnected.

Definition 2.1. Let x be a limit point of a Fuchsian group Γ acting on the upper half plane $\mathbf{H}$. The point x is called a *conical limit point* if there exist a non-tangential cone C in $\mathbf{H}$ and a sequence $\{\gamma_n\} \subset \Gamma$ such that $C \cap \partial\mathbf{H} = \{x\}$ and $\gamma_n(i) \in C$ for all n.

It is known that the limit set of any finitely generated Fuchsian group consists of parabolic fixed points and conical limit points.

Let f be a rational function of degree $d \geq 2$. We define the conical Julia set of f in the sense of Lyubich and Minsky.

Definition 2.2. We take any $r > 0$ and fix it. We say a point $x \in J(f)$ belongs to $J_{p-con}(f, r)$ for some $p \in \mathbf{N}$ if for any $\varepsilon > 0$, there exist a simply connected neighbourhood U of x and $n \in \mathbf{N}$ such that $\operatorname{diam}(U) < \varepsilon$ and

$$f^n : U \to \Delta(f^n(x), r)$$

is a p to one analytic mapping, where $\Delta(a, r)$ stands for a disk of center a with radius r. We define the conical Julia set $J_{con}(f)$ of f by

$$J_{con}(f) = \bigcup_{r>0,\, p \in \mathbf{N}} J_{p-con}(f, r).$$

Remark 2.1. Actually, the set $J_{con}(f)$ above is denoted by Δ_1 in [8]. It is slightly bigger that the "conical" set defined in the paper (see [8] p. 71).

Remark 2.2. The set $J_{rad}(f) = \bigcup_{r>0} J_{1-con}(f, r)$ is called the radial Julia set of f (cf. [9]). A set $X \subset J(f)$ is called hyperbolic if f is expanding there. The union of hyperbolic sets is called the hyperbolic Julia set and it is denoted by $J_{hyp}(f)$. It is observed that

$$J_{hyp}(f) \subset J_{rad}(f) \subset J_{con}(f) \subset J(f).$$

2.2. Martin compactification. In this section, we give a brief introduction of the Martin compactification of a Riemann surface. See [2] for the detail.

Let S be an open Riemann surface with Green's function. We denote by $g(\cdot, q)$ Green's function with pole at $q \in S$. For a fixed point $a_0 \in S$, we set

$$k_q(p) = k(p, q) = \frac{g(p, q)}{g(a_0, q)}. \quad (p, q \in S)$$

The Martin compactification S^* of S is the smallest compactification such that $k(p, q)$ has a continuous extension as a function of q. The set $\Delta = \Delta(S) = S^* - S$ is called the *Martin boundary* of S. Note that $k_q(\cdot)$ is a positive harmonic function on S when q is on the Martin boundary. It is known that S^* is metrizable.

A point $q \in \Delta$ is called a *minimal point* if there exists a constant $c > 0$ such that $u = ck_q$ whenever a positive harmonic function u on S satisfies $0 \le u(\cdot) \le k_q(\cdot)$. The set of minimal points is denoted by $\Delta_1 = \Delta_1(S)$. The following is a fundamental results of the theory of Martin compactification.

Proposition 2.1. *Let u be a positive harmonic function on S. Then there exists a unique positive measure μ on $\Delta_1(S)$ such that*

$$u(p) = \int_{\Delta_1(S)} k_q(p)\, d\mu(q).$$

Now we consider a planar domain $\Omega \subset \hat{\mathbf{C}}$ whose complement has positive logarithmic capacity. For each $\zeta \in \partial\Omega$, we denote by $\Delta(\zeta)$ the set of points in Δ over ζ, that is, the set of points in Δ obtained by all sequences $\{q_n\}_{n=1}^{\infty} \subset \Omega$ converging to ζ. The cardinal number of $\Delta(\zeta)$ is denoted by $\dim \Delta(\zeta)$.

As for the structure of $\Delta(\zeta)$, the following is known (cf. [2] Satz 13. 2).

Proposition 2.2. *The set $\Delta(\zeta)$ contains at least one minimal point. If $\zeta \in \partial\Omega$ is an irregular point of the Dirichlet problem on Ω, then* $\dim \Delta(\zeta) = 1$.

Therefore, if $\dim \Delta(\zeta) = 1$, then $\Delta(\zeta)$ consists of a single minimal point and we only consider regular points on the boundary. For a regular point ζ on the boundary, we have also a sufficient condition for $\dim \Delta(\zeta) = 1$. Before giving it, we shall introduce a notion of coarseness for compact subsets of $\mathbf{C}$.

Definition 2.3. Let E be a compact subset of $\mathbf{C}$. A point $\zeta \in E$ is called *annularly coarse for E* if there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of annuli with the following conditions.

- Every A_n is in $\mathbf{C} - E$.
- $\{A_n\}_{n=1}^{\infty}$ is nested to ζ, that is, A_{n+1} separates ζ from A_n for each $n \in \mathbf{N}$ and $A_n \to \zeta$ as $n \to \infty$.
- $\inf_n \operatorname{mod}(A_n) > 0$.

A compact set E is called *annularly coarse* if any point $\zeta \in E$ is annularly coarse.

Then the following is known.

Proposition 2.3 ([13] Lemma 3). *Let ζ be a regular point on $\partial\Omega$. If $\zeta \in \partial\Omega$ is annularly coarse for $\partial\Omega$, then* $\dim \Delta(\zeta) = 1$.

2.3. Function theoretic null sets. In this section, we describe the null sets appeared in the theorems of Sec. 1. Throughout this section, E is a totally disconnected compact subset of $\mathbf{C}$.

Definition 2.4. A point $\zeta \in E$ is called *weak for* E if any conformal mapping on $\mathbf{C} - E$ has a limit at ζ.

Proposition 2.4 (cf. [11] VII. 5C). *Let E be a compact subset of $\mathbf{C}$. Suppose that every point of E is weak for E. Then, E belongs to N_{SD}, that is, $\mathbf{C} - E$ has no non-constant injective analytic functions with finite Dirichlet integrals.*

It is known that there exists a totally disconnected compact subset E such that any point of E is not weak (cf. [11] XI).

From the definition, we have the following.

Lemma 2.1. *Let E be a compact subset of $\mathbf{C}$. If $\zeta \in E$ is weak for E, then any quasiconformal mapping w on $\mathbf{C} - E$ has the limit at ζ.*

Proof. There exists a measurable function μ on $\mathbf{C}$ such that $\|\mu\|_\infty < 1$ and $f = w^\mu \circ w$ is conformal on $\mathbf{C} - E$, where w^μ is a quasiconformal mapping on $\mathbf{C}$ with

$$(w^\mu)_{\bar{z}}(z) = \mu(z)(w^\mu)_z(z). \quad (a.\ e.)$$

The existence of w^μ is guaranteed by the Ahlfors-Bers theory (cf. [7]). From the weakness of ζ, f has a limit at ζ. Since w^μ is a homeomorphism on $\mathbf{C}$, $w = (w^\mu)^{-1} \circ f$ also has a limit at ζ. □

A lot of conditions for weakness are known. Here, we note a test called Poincaré's metric test ([11] XI. Theorem 1E).

Proposition 2.5. *Let E be a compact subset of $\mathbf{C}$ and $\zeta \in E$. Suppose that there exists a sequence $\{c_n\}_{n=1}^\infty$ of simple closed curves in $\mathbf{C} - E$ such that*

- *c_{n+1} is contained in a bounded domain of the complement of c_n $(n = 1, 2, \ldots)$,*
- *c_{n+1} separates ζ from c_n,*
- *c_n converges to ζ as $n \to \infty$, and*
- *the hyperbolic lengths of c_n in $\mathbf{C} - E$ $(n = 1, 2, \ldots)$ are bounded.*

Then, ζ is weak for E.

Finally in this section, we shall note a recent work of Gotoh and Taniguchi ([5]) which is a generalization of a result in [6].

Proposition 2.6 (Gotoh and Taniguchi). *Let E be a compact subset of the plane. Suppose that the set E is annularly coarse. Then, the area of E is zero and every quasiconformal mapping defined on $\mathbf{C} - E$ is uniquely extended to a quasiconformal mapping on $\mathbf{C}$.*

3. Proofs of the theorems (limit sets)

In this section, we prove the statements of the theorems for limit sets.

Proofs of Theorem 1.1 (*A*) *and Theorem* 1.2 (*A*). Let ζ be a parabolic fixed point of $\gamma \in \Gamma$. We may assume that Γ acts on the upper half plane, $\gamma(z) = z + 1$ and $\zeta = \infty$. It is known that for a non-elementary Fuchsian group Γ, the limit set $\Lambda(\Gamma)$ has positive capacity. Hence, our assertion for a parabolic fixed point is immediately obtained from the following fact.

Proposition 3.1 ([12] Theorem 2). *Let E_0 be a closed subset in* $[0, 1]$ *with positive capacity. Set $E_n = g_n(E_0)$ and $E = \bigcup_{n \in \mathbf{N}} E_n \cup \{\infty\}$, where $g_n(z) = z + n$* ($n \in \mathbf{N}$). *Then, for the Martin compactification of $\Omega = \hat{\mathbf{C}} - E$, $\Delta(\infty)$ contains exactly two minimal points.*

As for a conical limit point ζ, it suffices to show that the existence of annuli $\{A_n\}_{n=1}^{\infty}$ satisfying the conditions in Proposition 2.3.

We may assume that $\zeta = 0$. Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence in Γ which converges conically to ζ. Take hyperbolic geodesics c, c' from $z = i$ so that the closures $\bar{c}$, $\overline{c'}$ are contained in $\Omega(\Gamma)$ and their intersection angle at $z = i$ is almost π. Put $C = c \cup c'$.

First, we suppose that $C_n = \gamma_n(C)$ intersects with $I = \{z = iy \mid 0 < y < 1\}$ for all $n \in \mathbf{N}$. Then we have a domain H_n bounded by C_n and the real axis such that the closure contains $z = 0$. Furthermore, if $C_{n+1} \cap I \neq \emptyset$ and $C_{n+1} \subset H_n$ for all $n \in \mathbf{N}$, then we obtain a sequence $\{A_n\}_{n=1}^{\infty}$ of annuli in $\Omega(\Gamma)$ satisfying the desired condition. Indeed, in this case, C together with its mirror image C' for the real axis becomes a simple closed curve α in $\Omega(\Gamma)$. By fatting α, we have an annulus A in $\Omega(\Gamma)$. Then $A_n = \gamma_n(A)$ $(n = 1, 2, \ldots)$ give a nested sequence of annuli for $z = 0$.

Since γ_n is a conformal mapping, $\mathrm{mod}(A_n) = \mathrm{mod}(A)$. Hence, $\{A_n\}_{n=1}^{\infty}$ is our desired sequence of annuli (Figure 1).

A modification of the above argument gives the proof for general case. To do so, we note a result of the hyperbolic geometry.

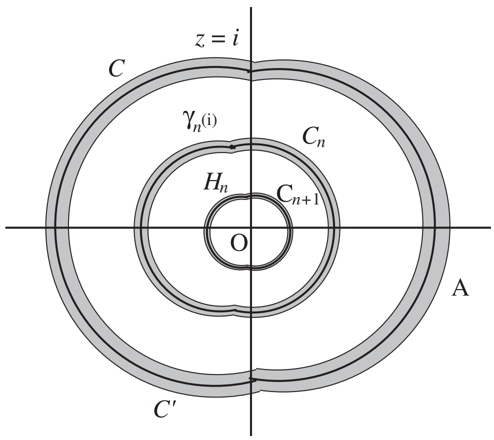

Figure 1

LEMMA 3.1. *Let* $S = \left\{ z \in \mathbf{H} \,\middle|\, \frac{\pi}{2} - \phi < \arg z < \frac{\pi}{2} + \phi \right\}$ *be a cone for some* ϕ $\left(0 < \phi < \frac{\pi}{2}\right)$. *For a constant* $M > 1$, *we set*

$$A_M = \{ z \in \mathbf{H} \mid M^{-1} < |z| < M \}.$$

Then there exists a constant $\theta(\phi, M) > 0$ *depending only on* ϕ *and* M *such that the intersection angle of any two geodesic rays from any* $z_0 \in S \cap \{ z \in \mathbf{H} \mid |z| = 1, \operatorname{Re} z > 0\}$ *is less than* $\theta(\phi, M)$ *if they are entirely contained in* $A_M \cap \{\operatorname{Re} z > 0\}$.

Proof. An elementary argument of the hyperbolic geometry gives the proof of the lemma. So, we omit it (see Figure 2). □

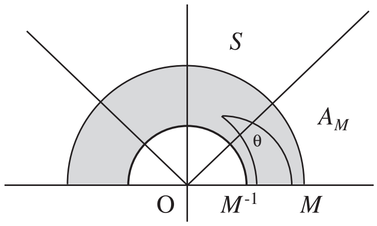

FIGURE 2

Suppose that the cone S in Lemma 3.1 contains all $\gamma_n(i)$ $(n = 1, 2, \ldots)$. By taking a subsequence, we may assume that

$$\left| \frac{\gamma_n(i)}{\gamma_{n+1}(i)} \right| > M^3 \quad (n = 1, 2, \ldots) \tag{1}$$

for some $M > 0$. Now, we take geodesic rays $c_1, c_2, \ldots, c_2k$ from $z = i$ satisfying the following conditions for $k \in \mathbf{N}$ (see Figure 3 below).

- c_j intersects c_{k+j} at $z = i$ with an angle almost π.
- c_j (resp. c_{k+j}) intersects c_{j+1} (resp. c_{k+j+1}) at $z = i$ with an angle less than $\theta(\phi, M)$ (here we set $c_{2k+1} = c_1$).
- The closures $\overline{c_j}$ and $\overline{c_{k+j}}$ are contained in $\Omega(\Gamma)$.

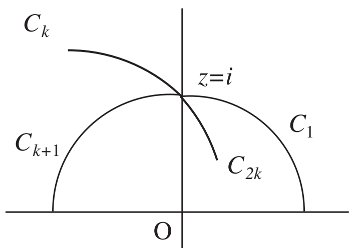

FIGURE 3

It is always possible if we take the number k sufficiently large because the limit set $\Lambda(\Gamma)$ is nowhere dense in the real axis. Then, from Lemma 3.1 some $C_n^{(j_n)} = \gamma_n(c_j \cup c_{k+j})$ $(1 \le j_n \le k)$ is entirely contained in an annulus $\{|\gamma_n(i)|M^{-1} < |z| < |\gamma_n(i)|M\}$. By taking a subsequence and renumbering $\{c_j, c_{k+j}\}_{j=1}^{k}$, we may assume that $j_n = 1$ for all n. Moreover, from (1)

$$d(C_n^{(1)}, C_{n+1}^{(1)}) > \delta > 0 \tag{2}$$

holds for some δ which does not depend on n, where $d(\cdot\,,\cdot)$ is the hyperbolic distance. Therefore, by fatting a closed curve formed by $c_1 \cup c_{k+1}$ and the mirror image for the real axis, we have an annulus A_0 so that for $A_n = \gamma_n(A_0)$,

(1) A_{n+1} is contained a bounded component of $\mathbf{C} - A_n$

(2) $\mathrm{mod}(A_n) = \mathrm{mod}(A_0)$.

(3) $\lim_{n\to\infty} A_n = \{0\}$.

Hence, the sequence $\{A_n\}_{n=1}^{\infty}$ of annuli satisfies the conditions of Proposition 2.3. So, dim $\Delta(0) = 1$. Therefore we conclude from Proposition 2.2 that $\Delta(0)$ consists of a single minimal point.

As for a Schottky group, from a canonical fundamental region for the group, we may easily obtain a sequence of annuli as above for each limit point. Thus, every limit point is weak. The proof of Theorem 1.1 (A) is completed.

The proof of Theorem 1.2 (A) is obtained simultaneously. Indeed, the sequence $\{C_n^{(1)}\}_{n=1}^{\infty}$ of the above proof satisfies the conditions of Proposition 2.5 for $\zeta = 0$ because each γ_n is the isometry for the hyperbolic distance on $\Omega(\Gamma)$. Hence every conical limit point is weak.

If Γ is a Schottky group, then it is a quasiconformal deformation of a finitely generated Fuchsian group G of the second kind with no parabolic element. Let f be the quasiconformal mapping, that is, $\Gamma = fGf^{-1}$. It is known that every point of $\Lambda(G)$ is a conical limit point. Thus, for any $\zeta \in \Lambda(G)$, we have a sequence $\{C_n\}_{n=1}^{\infty}$ of simple closed curves in $\Omega(G)$ for ζ as in the proof of Theorem 1.1 (A). Since $C_n = g_n(C_1)$ for some $g_n \in G$, $f(C_n) = \gamma_n(f(C_1))$ $(\gamma_n = fg_nf^{-1}; n = 1, 2, \ldots)$ satisfy the conditions of Proposition 2.5 for $f(\zeta)$. Therefore, $f(\zeta)$ is weak and the proof of Theorem 1.2 (A) is completed.

4. Proofs of the theorems (Julia sets)

In this section, we prove the statements of the theorems for Julia sets.

4.1. Proofs of Theorem 1.1 (B) and Theorem 1.2 (B1). Let $x \in J_{con}(f)$ be an arbitrary point of the conical Julia set of f. Then, the point x belongs to $J_{p-con}(f, r)$ for some $r > 0$ and $p \in \mathbf{N}$. We may assume that $x \ne \infty$.

Take a monotone decreasing sequence $\{\varepsilon_m\}_{m=1}^{\infty}$ of positive number ε_m with $\varepsilon_m \to 0$ as $m \to \infty$. From the definition of $J_{p-con}(f, r)$, there exist $n_m \in \mathbf{N}$ and a simply connected neighbourhood U_m of x for each m such that diam $U_m < \varepsilon_m$ and

$$f^{n_m} : U_m \to \Delta(f^{n_m}(x), r)$$

is a p to one analytic mapping. Therefore, the number of critical points of f^{n_m} in U_m is $p-1$.

Furthermore, we may assume that $\{f^{n_m}(x)\}_{m=1}^{\infty}$ converges to a point $x_0 \in J(f)$. We may also assume that $x_0 \neq \infty$. Hence, for sufficiently large m,

$$f^{n_m}(U_m) = \Delta(f^{n_m}(x), r) \supset \Delta\left(x_0, \frac{1}{2}r\right).$$

Since $\Delta\left(x_0, \frac{1}{2}r\right)$ contains at most $(p-1)$ critical values of f^{n_m}, we may find an annulus $A'_k = \left\{\frac{1}{2^{k+1}}r < |z - x_0| < \frac{1}{2^k}r\right\}$ $(1 \leq k \leq p)$ which contains no critical values of f^{n_m}. Moreover, taking a subsequence of $\{f^{n_m}\}_{m=1}^{\infty}$ if necessary, we may assume that A'_k does not contain any critical values of f^{n_m} $(m = 1, 2, \ldots)$.

Noting that $J(f)$ is totally disconnected, we may take an annulus $A \subset A'_k \cap F(f)$ so that the bounded component of A^c contains both $f^{n_m}(x)$ and x_0 for sufficiently large m. The annulus A does not contain any critical value of f^{n_m} $(m = 1, 2, \ldots)$. Thus, all components of $f^{-n_m}(A)$ are annuli and we see that there exists an annulus $\tilde{A}_m$ in U_m such that $f^{n_m}(\tilde{A}_m) = A$ and the bounded component of $\tilde{A}_m^c$ contains x.

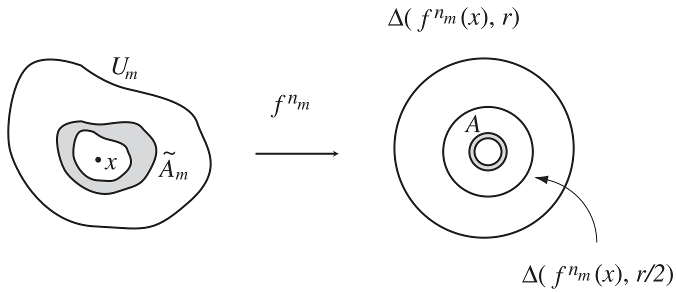

FIGURE 4

Indeed, if for any component $\tilde{A}$ of $f^{-n_m}(A)$, the bounded component of $\tilde{A}^c$ does not contain x, then there exists a curve C in $U_m - f^{-n_m}(A)$ such that the curve C connects x and ∂U_m. Hence $f^{n_m}(C)$ is a curve in $\Delta(f^{n_m}(x), r)$ connecting $f^{n_m}(x)$ and $\partial\Delta(f^{n_m}(x), r)$ with $f^{n_m}(C) \cap A \neq \emptyset$. It is absurd.

Here, we note the following.

LEMMA 4.1. *Let $\varphi : A_1 \to A_2$ be a k-sheeted smooth holomorphic mapping of an annulus A_1 onto an annulus A_2. Then*

$$\text{mod}(A_1) = \frac{1}{k}\,\text{mod}(A_2). \tag{3}$$

Proof. We may assume that $A_j = \{1 < |z| < e^{R_j}\}$, where $R_j = \text{mod}(A_j)$ $(j = 1, 2)$. We may also assume that the mapping φ is continuous on $\overline{A_1}$ and

$\varphi(\{|z| = 1\}) = \{|w| = 1\}$ and $\varphi(\{|z| = e^{R_1}\}) = \{|w| = e^{R_2}\}$. Hence, by using the reflection principle repeatedly, we have a holomorphic mapping Φ which is an extension of φ of $\hat{\mathbf{C}}$ onto itself with $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. Therefore, we conclude that $\Phi(z) = z^k$ and $R_2 = kR_1$ which implies (3). □

The mapping f^{n_m} on $\tilde{A}_m$ is at most p-sheeted over A. Hence, we have

$$\mathrm{mod}(\tilde{A}_m) \geq \frac{1}{p}\,\mathrm{mod}(A).$$

Therefore, we verify that the sequence $\{\tilde{A}_m\}_{m=1}^{\infty}$ of annuli satisfies the conditions of Proposition 2.3 and $\Delta(x)$ consists of a single minimal point.

As we have seen in Sec. 3, the above argument proves Theorem 1.2 (B1) as well as Theorem 1.1 (B).

4.2. Proofs of Corollaries. In this section, we prove Corollary 1.1, 1.2 and 1.3.

Corollary 1.1 is easily verified because it is well known (cf. [4]) that any polynomial as in the corollary is hyperbolic.

Similar consideration gives a proof of Corollary 1.2. Here, we present a direct proof for convenience of the readers.

Proof of Corollary 1.2. Let $B(z)$ be a finite Blaschke product of degree $d \geq 2$. It follows from Denjoy-Wolff theorem (cf. [4]) and from local dymanics at a parabolic periodic point that B has at most one parabolic periodic point and that it is a fixed point.

Suppose that the Blaschke product has a parabolic fixed point $z_0 \in J(B)$. We denote by $O_-(z_0)$ the set of backward orbits of z_0. For any $\zeta \in J(B) - O_-(z_0)$, there exists a monotone sequence $\{n_p\}_{p=1}^{\infty} \subset \mathbf{N}$ such that $\{B^{n_p}(\zeta)\}_{p=1}^{\infty}$ has a limit $\zeta_0 \neq z_0$ as $p \to \infty$. Indeed, it is easily seen from local dynamics of B near the parabolic fixed point z_0 (cf. [4]). Since every critical point of B is attracted to z_0, there exists a neighbourhood $\Delta(\zeta_0, r) = \{z \in \mathbf{C} \mid |z - \zeta_0| < r\}$ $(r > 0)$ of ζ_0 such that $\Delta(\zeta_0, r) \cap P_B = \emptyset$, where P_B is the post-critical set of B, that is, the set of critical points and their forward orbits of B.

Lemma 4.2. *There exist local inverses f_n of B^n $(n = 1, 2, \dots)$ in $\Delta(\zeta_0, r)$ so that $f_{n_p}(\zeta_0) \to \zeta$ as $p \to \infty$.*

Proof. Since $\Delta(\zeta_0, r) \cap P_B = \emptyset$, any branch of of B^{-n} exists for any $n \in \mathbf{N}$ on $\Delta(\zeta_0, r)$. Take a branch f_{n_p} of B^{-n_p} as $f_{n_p}(B^{n_p}(\zeta)) = \zeta$. The map f_{n_p} is a holomorphic map of $\Delta(\zeta_0, r)$ to $S_B = \hat{\mathbf{C}} - P_B$ and P_B is an infinite set. Hence, S_B is hyperbolic as a Riemann surface and it follows from Schwartz's lemma that

$$d_{S_B}(\zeta, f_{n_p}(\zeta_0)) = d_{S_B}(f_{n_p}(B^{n_p}(\zeta)), f_{n_p}(\zeta_0)) \leq d_{\Delta(\zeta_0, r)}(B^{n_p}(\zeta), \zeta_0),$$

where $d_S(\cdot\,,\cdot)$ stands for the hyperbolic distance on a hyperbolic Riemann surface S. Since $B^{n_p}(\zeta) \to \zeta_0$, we have

$$\lim_{p\to\infty} f_{n_p}(\zeta_0) = \zeta.$$

Hence, the proof of the lemma is completed. □

Since f_{n_p} is a holomorphic map of $\Delta(\zeta_0, r)$ to $\hat{\mathbf{C}} - P_B$, $\{f_{n_p}\}_{p=1}^{\infty}$ becomes a normal family of univalent functions. Thus, a limit function $f = \lim_{p\to\infty} f_{n_p}$ is either an univalent function or a constant. If it is not a constant, the image $f(\Delta(\zeta_0, r))$ of $\Delta(\zeta_0, r)$ contains an open set $U \ni \zeta$ with $\bar{U} \subset f(\Delta(\zeta_0, r))$. Since $\{f_{n_p}\}_{p=1}^{\infty}$ converges to f uniformly on every compact subset of $\Delta(\zeta_0, r)$, we verify that there exist a compact subset K of $\Delta(\zeta_0, r)$ such that $f_{n_p}(K) \supset U$ for sufficiently large n_p. This implies $K \supset B^{n_p}(U)$ and it contradicts to $U \cap J(B) \neq \emptyset$. Therefore, we conclude that $f \equiv \zeta$, i.e., $\{f_{n_p}\}_{p=1}^{\infty}$ converges to a constant function.

Since the Julia set $J(B)$ of B is totally disconnected, we may take an annulus A in $\Delta(\zeta_0, r) \cap F(B)$ so that the closure $\bar{A}$ is contained in $\Delta(\zeta_0, r)$ and the bounded component of $\bar{A}^c$ contains ζ_0. As $B^{n_p}(\zeta) \to \zeta_0$, $B^{n_p}(\zeta)$ and ζ_0 belong to the same component of $\bar{A}^c$ for sufficiently large n_p. Hence, the bounded component U_{n_p} of $\bar{A}_{n_p}^c$ contains ζ. Since $\{f_{n_p}\}_{p=1}^{\infty}$ converges to $f \equiv \zeta$ uniformly on A, we may assume that

$$U_{n_{p+1}} \subset \bar{A}_{n_{p+1}} \subset U_{n_p} \quad (p = 1, 2, \ldots).$$

Noting that $\mathrm{mod}(A_{n_p}) = \mathrm{mod}(A)$ for $A_{n_p} = f_{n_p}(A)$, we verify that the sequence $\{A_{n_p}\}_{p=1}^{\infty}$ of annuli satisfies the conditions of Proposition 2.3 and $\dim \Delta_1(\zeta) = 1$.

It is easily seen that the above argument also works if B does not have a parabolic fixed point. Hence, we complete the proof of the corollary.

Proof of Corollary 1.3. From Theorem 1.2, any quasiconformal mapping w on D is extended to a quasiconformal mapping on $\hat{\mathbf{C}}$. We denote it by the same letter w.

Let ζ be an arbitrary point of ∂D. According to the proof of Theorem 1.2, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of annuli in D such that it satisfies the conditions of Proposition 2.3 for $\zeta \in \partial D$ and D. Since w is a quasiconformal mapping, we have

$$K(w)^{-1} \,\mathrm{mod}(A_n) \le \mathrm{mod}(w(A_n)) \le K(w)\,\mathrm{mod}(A_n)$$

for each $n \in \mathbf{N}$, where $K(w)$ is the maximal dilatation of w. Therefore, $\{w(A_n)\}_{n=1}^{\infty}$ also satisfies the conditions of Proposition 2.3 and we verify that both D^* and $w(D)^*$ are homeomorphic to $\hat{\mathbf{C}}$. Hence, we conclude that w gives a homeomorphism from D^* onto $w(D)^*$.

4.3. Proof of Theorem 1.2 (B2). First, we note the following lemma.

Lemma 4.3. *Let E be a compact subset of $\mathbf{C}$. If E is of linear measure zero, then it is AB-removable, that is, for any open neighbourhood U of E, any bounded holomorphic function on $U - E$ is extended holomorphically to whole U.*

Proof. The proof is done as an elementary application of Cauchy's integral formula. See [11], for example. □

Therefore, it is sufficient to show that the Julia set $J(B)$ on the unit circle $\partial\Delta(0,1)$ is of linear measure zero. If not, the harmonic measure $\omega_{J(B)}$ in the unit disk $\Delta(0,1)$ for $J(B)$ is a positive harmonic function in $\Delta(0,1)$.

Lemma 4.4. *For any* $z \in \Delta(0,1)$,

$$\omega_{J(B)}(B(z)) = \omega_{J(B)}(z). \tag{4}$$

Proof. From Perron's method for the Dirichlet problem (cf. [2]), we see that

$$\omega_{J(B)}(z) = \inf_{\bar{s} \in \overline{\mathscr{S}}} \bar{s}(z) = \sup_{\underline{s} \in \underline{\mathscr{S}}} \underline{s}(z),$$

where $\overline{\mathscr{S}}$ (resp. $\underline{\mathscr{S}}$) is the family of superharmonic (resp. subharmonic) functions in $\Delta(0,1)$ such that for

$$\liminf_{z \to \zeta \in \partial\Delta(0,1)} \bar{s}(z) \geq \chi_{J(B)}(\zeta) \quad \left(\text{resp. } \limsup_{z \to \zeta \in \partial\Delta(0,1)} \underline{s}(z) \leq \chi_{J(B)}(\zeta) \right),$$

where $\chi_{J(B)}$ is the characteristic function of $J(B)$ in $\partial\Delta(0,1)$.

Since $B^{-1}(J(B)) = J(B)$, we see that $\bar{s} \circ B \in \overline{\mathscr{S}}$ for any $\bar{s} \in \overline{\mathscr{S}}$. Hence, we have

$$\omega_{J(B)} \circ B(z) \geq \omega_{J(B)}(z).$$

Similarly, we obtain

$$\omega_{J(B)} \circ B(z) \leq \omega_{J(B)}(z).$$

Thus, we have (4). □

For $z \in F(B)$, *the grand orbit* of z is the set of $y \in F(B)$ with

$$B^n(z) = B^m(y)$$

for some $n, m \in \mathbf{N}$.

The above equation (4) shows that the harmonic measure $\omega_{J(B)}$ is regarded as a function of the set of grand orbits of $\Delta(0,1)$. Since $J(B)$ is not connected, $F(B)$ is either a parabolic component or an attractive component which is not super attractive. Thus, the set of grand orbits of $F(B)$ forms a Riemann surface of type $(1,k)$ or $(0,2+k)$ for some $k \geq 1$ ([10]). Noting that the Riemann surface is a double of the space of grand orbits in $\Delta(0,1)$, we see that the grand orbits of $\Delta(0,1)$ forms a finite bordered Riemann surface S and $\omega_{J(B)}$ is considered as a bounded harmonic function u on S which vanishes identically on the relative boundary ∂S. Hence, the maximal principle of harmonic functions yields that

$$u = \omega_{J(B)} = 0.$$

We have a contradiction and the linear measure of $J(B)$ is zero. From Lemma 4.3, we complete the proof of Theorem 1.2 (B2).

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