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A NOTE ON ENTIRE PSEUDO-HOLOMORPHIC CURVES AND THE PROOF OF CARTAN-NOCHKA'S THEOREM

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To the memory of Professor Nobuyuki Suita

1. Introduction

The purpose of this note is twofold. The first is to prove a lemma on differentials for entire pseudo-holomorphic curves in a compact almost complex manifold (see Lemma 2.1), which is an analogue to Nevanlinna's lemma on logarithmic derivative and also to a lemma on holomorphic 1-forms which had been conjectured by A. Bloch [Bl26] and was proved by T. Ochiai [Oc77].

The second is to give a complete proof of Cartan-Nochka's Theorem with truncated counting functions and with small error term "$S_f(r) = O(\log^+ r + \log^+ T_f(r))||$" by a simple Cartan method (see Theorem 3.1). This was what Nochka [Nc83] stated the theorem with a sketch of the proof. So far there has been no literature of the complete full proof which is accessible to wider audience, while there are, I learned orally, a longer paper of Nochka in Russian other than [Nc82a], [Nc82b] and [Nc83], and a proof based on the same idea as the present one in the book [Ng03] in Japanese. We will show the theorem in a form slightly more general than those in the above mentioned references.

We also give some generalization to the case where the domain is an analytic ramified covering space over $\mathbf{C}^m$ (see Theorem 3.18).

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2. Entire pseudo-holomorphic curves

The problem of Kobayashi hyperbolicity for pseudo-holomorphic curves are intensively studied by several authors (cf., e.g., R. Debalme and S. Ivashkovich [DI01] S. Kobayashi [Ko01], [Ko03], [Ko04]). The analogue of Brody's Theorem holds for pseudo-holomorphic curves (cf. [Ko03], [Ko04]). Henceforth the Kobayashi hyperbolicity of a compact almost complex manifold M is in-

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ferred from the non-existence of a non-constant entire pseudo-holomorphic curve $f : \mathbf{C} \to M$.

In the complex analytic case, Nevanlinna theory seems to be the best approach for the non-existence problem of f (cf. S. Kobayashi [Ko98] Introduction), where the so-called *Nevanlinna's lemma on logarithmic derivative* plays an essential role and provides a counter part to Schwarz' Lemma in the theory of Kobayashi hyperbolicity. In the proof of Bloch-Ochiai's Theorem (cf. [NO$\frac{84}{90}$]) a *lemma on holomorphic differentials*, which had been conjectured by [Bl26] and was proved by [Oc77], was important and played an exactly similar role to Nevanlinna's lemma on logarithmic derivative. The purpose of this section is to show such a lemma for entire pseudo-holomorphic curves (see Lemma 2.1).

Let M be a compact almost complex manifold and let $h = \sum h_{\nu\bar{\mu}}\, dx^{\nu} d\bar{x}^{\mu}$ be a fixed hermitian metric on M with the associated $(1,1)$-form $\omega = \sum h_{\nu\bar{\mu}} \frac{i}{2}\, dx^{\nu} \wedge d\bar{x}^{\mu}$. Let $f : \mathbf{C} \to M$ be an entire pseudo-holomorphic curve. We define the order function of f with respect to h by

$$T_f(r; h) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^*\omega.$$

For a smooth differential 1-form η on M we have the decomposition $\eta = \eta' + \eta''$ to the $(1,0)$-form η' and $(0,1)$-form η''. We set

$$f^*\eta = \eta'_f\, dz + \eta''_f\, d\bar{z},$$

$$m_f(r; \eta) = \int_{|z|=r} \log^+ \sqrt{|\eta'_f|^2 + |\eta''_f|^2}\, \frac{d\theta}{2\pi}.$$

Here $\log^+ s = \max\{\log s, 0\}$ for $s \in \mathbf{R}^+ = \{s \in \mathbf{R}; s \geqq 0\}$.

LEMMA 2.1. *Let $f : \mathbf{C} \to M$ be a pseudo-holomorphic curve and let η be a smooth differential 1-form on M. Then we have*

$$m_f(r; \eta) \leqq \delta \log r + 2 \log^+ T_f(r; h) \|_{\delta} \quad (0 < \delta < 1),$$

where the symbol "$\|_{\delta}$" stands for the stated inequality to hold for all $r > 0$ outside a Borel subset dependent on $\delta > 0$ with finite Lebesgue measure.

We need the following, called Borel's Lemma (cf., e.g., [NO$\frac{84}{90}$]).

LEMMA 2.2. *Let $\phi(r)$ be a continuous, increasing function on $\mathbf{R}^+$ such that $\phi(r_0) > 0$ for some $r_0 \in \mathbf{R}^+$. Then for an arbitrary small $\delta > 0$ we have*

$$\frac{d}{dr}\phi(r) < \phi(r)^{1+\delta} \|_{\delta}.$$

To prove Lemma 2.1 we may assume that η is a $(1,0)$-form. We set

$$f^*\eta = \eta_f(z)\,dz, \quad f^*\omega = s(z)\frac{i}{2}\,dz \wedge d\bar{z}, \quad z \in \mathbf{C}.$$

Since the length $\|\eta\|_h$ of η with respect to h is bounded, there is a constant $C > 0$ such that

$$|\eta_f(z)|^2 \leqq 2\pi C s(z).$$

Let $0 < \delta < 1$. Using the concavity of the logarithmic function and Lemma 2.2, we have

$$\begin{aligned}
m_f(r;\eta) &= \int_{|z|=r} \log^+|\eta_f| \frac{d\theta}{2\pi} \leqq \frac{1}{2}\int_{|z|=r} \log(1+|\eta_f|^2)\frac{d\theta}{2\pi} \\
&\leqq \frac{1}{2}\log\left(1+\int_{|z|=r}|\eta_f|^2\frac{d\theta}{2\pi}\right) \leqq \frac{1}{2}\log\left(1+C\int_{|z|=r} s(z)\,d\theta\right) \\
&\leqq \frac{1}{2}\log\left(1+\frac{C}{r}\frac{d}{dr}\int_{\Delta(r)} s(z)t\,dt\wedge d\theta\right) \\
&\leqq \frac{1}{2}\log\left(1+\frac{C}{r}\left(\int_{\Delta(r)} s(z)\frac{i}{2}\,dz\wedge d\bar{z}\right)^{1+\delta}\right)\|_\delta \\
&\leqq \frac{1}{2}\log\left(1+Cr^\delta\left(\frac{d}{dr}\int_1^r\frac{dt}{t}\int_{\Delta(t)} f^*\omega\right)^{1+\delta}\right)\|_\delta \\
&\leqq \frac{1}{2}\log\left(1+Cr^\delta\left(\int_1^r\frac{dt}{t}\int_{\Delta(t)} f^*\omega\right)^{(1+\delta)^2}\right)\|_\delta \\
&\leqq \frac{1}{2}\log(1+Cr^\delta(T_f(r;h))^{(1+\delta)^2})\|_\delta \\
&\leqq \delta\log^+ r + 2\log^+ T_f(r;h)\|_\delta.
\end{aligned}$$

Remark. It is interesting to observe that the complex analyticity of η is completely irrelevant to the above Lemma 2.1. As in Nevanlinna theory in complex analysis Lemma 2.1 is expected to apply to the Kobayashi hyperbolicity problem.

3. Cartan-Nochka's Theorem

Let H_j, $1 \leqq j \leqq q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ defined by

$$H_j\colon \quad \sum_{k=0}^{n} h_{jk}w^k = 0, \quad 1 \leqq j \leqq q,$$

where $[w^0, \dots, w^n]$ is a homogeneous coordinate system of $\mathbf{P}^n(\mathbf{C})$. Set the index set $Q = \{1, \dots, q\}$. For a subset $R \subset Q$, $|R|$ denotes its cardinality.

DEFINITION. Let $N \geqq n$ and $q \geqq N+1$. We say that H_j, $j \in Q$ are in N-*subgeneral position* if for every subset $R \subset Q$ with $|R| = N+1$

$$\bigcap_{j \in R} H_j = \emptyset.$$

If they are in n-subgeneral position, we simply say that they are in *general position*.

Being in N-subgeneral position is equivalent to that for an arbitrary $(N+1, n+1)$-matrix $(h_{jk})_{j \in R, 0 \leqq k \leqq n}$

$$\operatorname{rank}(h_{jk})_{j \in R, 0 \leqq k \leqq n} = n+1.$$

Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. Let $z = (z_j)$ be the natural coordinate system of $\mathbf{C}^m$, $\|z\| = \sqrt{\sum_j |z_j|^2}$ and let ω be the Fubini-Study metric form on $\mathbf{P}^n(\mathbf{C})$. We define the order function $T_f(r)$ with respect to ω by

$$T_f(r) = \int_0^r \frac{dt}{t^{2m-1}} \int_{\|z\|<t} \left(\frac{i}{2\pi} \partial\bar{\partial}\|z\|^2\right)^{2m-2} \wedge f^*\omega.$$

Cf. [NO$\frac{84}{90}$] for general notation in Nevanlinna theory. We denote a such small term by $S_f(r)$ that for an arbitrarily small positive number δ

$$S_f(r) \leqq \delta \log r + O(\log T_f(r))\|_\delta.$$

For a hyperplane $H \subset \mathbf{P}^n(\mathbf{C})$ such that $H \not\supset f(\mathbf{C}^m)$ we have the pull-backed divisor f^*H on $\mathbf{C}^m$ and the irreducible decomposition $f^*H = \sum_j v_j Z_j$. We define the truncated divisor $(f^*H)_{[k]}$ to the level $k \in \mathbf{N} \cup \{\infty\}$ by

$$(f^*H)_{[k]} = \sum_j \min\{v_j, k\} Z_j.$$

We define the counting function $N_k(r, f^*H)$ of the divisor $(f^*H)_{[k]}$ by

$$N_k(r, f^*H) = \int_1^r \frac{dt}{t^{2m-1}} \int_{(f^*H)_{[k]} \cap \{\|z\|<t\}} \left(\frac{i}{2\pi} \partial\bar{\partial}\|z\|^2\right)^{2m-2}.$$

and set $N(r, f^*H) = N_\infty(r, f^*H)$ (cf. [NO$\frac{84}{90}$], [Fu93]).

THEOREM 3.1 ([Nc83] for $m = 1$). *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate meromorphic mapping. Let H_j, $1 \leqq j \leqq q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in N-subgeneral position. Then we have*

$$(q - 2N + n - 1) T_f(r) \leqq \sum_{j=1}^q N_n(r, f^*H_j) + S_f(r). \tag{3.2}$$

Remark. i) ($m = 1$) The case of $m = 1$ is essential. H. Cartan [Ca33] proved this when H_j, $1 \leqq j \leqq q$ are in general position.

ii) ($m \geqq 1$) By Weyl-Ahlfors' method Chen [Ch90] proved

$$(q - 2N + n - 1)T_f(r, L) + \frac{N+1}{n+1} N(r, (W(f))) \leqq \sum_{j=1}^{q} N(r, f^*H_j) + S_f(r),$$

where $(W(f))$ denotes the divisor defined by the Wronskian of f (see (3.6)). After this formulation it is unable to deduce (3.2).

iii) ($m = 1$) By Weyl-Ahlfors' method combined with his own technique H. Fujimoto [Fu93] proved that for an arbitrary $\varepsilon > 0$

$$(q - 2N + n - 1)T_f(r) \leqq \sum_{j=1}^{q} N_n(r, f^*H_j) + \varepsilon T_f(r)\|_\varepsilon.$$

Here, the estimate of the small error term is not as good as in (3.2); it is noticed that the type of error term is in general deeply related to the possible truncation level of counting functions in the right-hand side of (3.2) (see [NWY02] Example (5.36)).

Let H_j, $j \in Q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in N-subgeneral position. For $R \subset Q$ we set

$$V(R) = \text{the vector subspace spanned by } (h_{jk})_{0 \leqq k \leqq n},\ j \in R \text{ in } \mathbf{C}^{n+1},$$

$$\mathrm{rk}(R) = \dim V(R), \quad \mathrm{rk}(\emptyset) = 0.$$

We recall now lemmas due to Nochka (see [Nc83], [Ch90], [Fu93]).

LEMMA 3.3 ([Nc83], [Ch90], [Fu93]). *Let H_j, $j \in Q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in N-subgeneral position, and assume that $q > 2N - n + 1$. Then there are positive rational constants $\omega(j)$, $j \in Q$ satisfying the following*:

(i) $0 < \omega(j) \leqq 1$, $\forall j \in Q$.

(ii) *Setting* $\tilde{\omega} = \max_{j \in Q} \omega(j)$, *one gets*

$$\sum_{j=1}^{q} \omega(j) = \tilde{\omega}(q - 2N + n - 1) + n + 1.$$

(iii) $\dfrac{n+1}{2N - n + 1} \leqq \tilde{\omega} \leqq \dfrac{n}{N}$.[1]

(iv) *For* $R \subset Q$ *with* $0 < |R| \leqq N + 1$, $\sum_{j \in R} \omega(j) \leqq \mathrm{rk}(R)$.

[1] The bound $\frac{n}{N}$ which is better than the original one $\frac{n+1}{N+1}$, was suggested by N. Toda by a careful check of the proof.

The above $\omega(j)$ are called the *Nochka weights*, and $\tilde{\omega}$ the *Nochka constant*.

Lemma 3.4 ([Nc83], [Ch90], [Fu93]). *Let $q > 2N - n + 1$, and let $\{H_j\}_{j \in Q}$ be a family of hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in N-subgeneral position. Let $\{\omega(j)\}_{j \in Q}$ be its Nochka weights.*

Let $E_j \geqq 1$, $j \in Q$ be arbitrarily given numbers. Then for every subset $R \subset Q$ with $0 < |R| \leqq N + 1$, there are distinct indices $j_1, \dots, j_{\mathrm{rk}(R)} \in R$ such that $\mathrm{rk}(\{j_l\}_{l=1}^{\mathrm{rk}(R)}) = \mathrm{rk}(R)$ and

$$\prod_{j \in R} E_j^{\omega(j)} \leqq \prod_{l=1}^{\mathrm{rk}(R)} E_{j_l}.$$

Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping. Fix a homogeneous coordinate system $w = [w^0, \dots, w^n]$ of $\mathbf{P}^n(\mathbf{C})$ and let $f(z) = [f^0(z), \dots, f^n(z)]$ be a reduced representation.

Assume that H_j are defined by

$$\text{(3.5)} \qquad H_j\colon \quad \hat{H}_j(w) = \sum_{k=0}^{n} h_{jk} w^k = 0, \quad 1 \leqq j \leqq q,$$

$$\|\hat{H}_j\| = \left(\sum_k |h_{jk}|^2 \right)^{1/2} = 1, \quad \frac{|\hat{H}_j(w)|}{\|w\|} \leqq 1.$$

After [Fu85] and [Ng97] §2 (b), we define the Wronskian $W(f) = W(f^0, \dots, f^n) \not\equiv 0$, and the logarithmic Wronskian $\Delta(f^0, \dots, f^n)$ as follows:

$$\text{(3.6)} \qquad W(f^0, \dots, f^n) = \begin{vmatrix} f^0 & \cdots & f^n \\ D^{(1)} f^0 & \cdots & D^{(1)} f^n \\ \vdots & \vdots & \vdots \\ D^{(n)} f^0 & \cdots & D^{(n)} f^n \end{vmatrix},$$

$$\Delta(f^0, \dots, f^n) = \begin{vmatrix} 1 & \cdots & 1 \\ \dfrac{D^{(1)} f^0}{f^0} & \cdots & \dfrac{D^{(1)} f^n}{f^n} \\ \vdots & \vdots & \vdots \\ \dfrac{D^{(n)} f^0}{f^0} & \cdots & \dfrac{D^{(n)} f^n}{f^n} \end{vmatrix}.$$

Here $D^{(j)} = \left(\dfrac{\partial}{\partial z^1} \right)^{\alpha_1(j)} \cdots \left(\dfrac{\partial}{\partial z^m} \right)^{\alpha_m(j)}$ are some partial differentiations of order at most j. Because of the choice of $D^{(j)}$ we have the following functional equations for a meromorphic function g on $\mathbf{C}^m$ and $A \in \mathrm{GL}(n+1, \mathbf{C})$:

$$
\begin{aligned}
(3.7)\qquad W(gf^0,\dots,gf^n) &= g^{n+1}W(f^0,\dots,f^n),\\
W((f^0,\dots,f^n)A) &= W(f^0,\dots,f^n)\times(\det A),\\
\Delta(gf^0,\dots,gf^n) &= \Delta(f^0,\dots,f^n),\\
\Delta\left(1,\frac{f^1}{f^0},\dots,\frac{f^n}{f^0}\right) &= \Delta(f^0,\dots,f^n).
\end{aligned}
$$

The following lemma is a key to get the correct truncation level of counting functions:

LEMMA 3.8 ([Fu93] Lemma 3.2.13). *Let the notation be as above. Then the following inequality holds as divisors on* $\mathbf{C}^m$ *with rational coefficients*:

$$\sum_{j\in Q}\omega(j)(\hat{H}_j\circ f)-(W(f^0,\dots,f^n))\leqq\sum_{j\in Q}\omega(j)(f^*H_j)_{[n]}.$$

Remark. H. Fujimoto [Fu93] gave a detailed proof of this lemma for $m=1$, and the same proof works for general $m\geqq 1$.

For a subset $R\subset Q$, $|R|=n+1$ we define $W((\hat{H}_j\circ f, j\in R))$ and $\Delta((\hat{H}_j\circ f, j\in R))$ as Wronskian and logarithmic Wronskian of $\hat{H}_j\circ f$, $j\in R$, respectively.

Now we prove a key lemma of the proof of Theorem 3.1:

LEMMA 3.9. *Let* $q>2N-n+1$ *and let* $\omega(j)$, $\tilde{\omega}$ *be the Nochka weights and constant of* $\{H_j\}_{j\in Q}$, *respectively. Then there is a positive constant C dependent on* $\{\hat{H}_j\}_{j\in Q}$ *such that for an arbitrary* $z\in\mathbf{C}^m\backslash\{\prod_{j\in Q}\hat{H}_j\circ f=0\}$

$$\|f(z)\|^{\tilde{\omega}(q-2N+n-1)}\leqq C\frac{\prod_{j\in Q}|\hat{H}_j(f(z))|^{\omega(j)}}{|W(f^0(z),\dots,f^n(z))|}\times\left\{\sum_{R\subset Q,\,|R|=n+1}|\Delta((\hat{H}_j\circ f(z), j\in R))|\right\}.$$

Proof. By the definition of N-subgeneral position, for an arbitrary point $w\in\mathbf{P}^n(\mathbf{C})$, there exists $S\subset Q$, $|S|=q-N-1$ such that $\prod_{j\in S}\hat{H}_j(w)\neq 0$. Therefore, there is a constant $C_1>0$ such that

$$C_1^{-1}<\sum_{|S|=q-N-1}\prod_{j\in S}\left(\frac{|\hat{H}_j(w)|}{\|w\|}\right)^{\omega(j)}<C_1,\quad\forall w\in\mathbf{P}^n(\mathbf{C}).\tag{3.10}$$

We consider those $w\in\mathbf{P}^n(\mathbf{C})$ such that $\prod_{j\in Q}\hat{H}_j(w)\neq 0$. Setting $R=Q\backslash S$, we have

$$\prod_{j\in S}\left(\frac{|\hat{H}_j(w)|}{\|w\|}\right)^{\omega(j)}=\prod_{j\in R}\left(\frac{\|w\|}{|\hat{H}_j(w)|}\right)^{\omega(j)}\cdot\frac{\prod_{j\in Q}|\hat{H}_j(w)|^{\omega(j)}}{\|w\|^{\sum_{j\in Q}\omega(j)}}.$$

By making use of Lemma 3.3 (ii) and $\mathrm{rk}(R) = n+1$ for R, we obtain a subset $\{j_1, \dots, j_{n+1}\} = R^\circ \subset R$ given by Lemma 3.4, so that

$$\prod_{j\in S}\left(\frac{|\hat{H}_j(w)|}{\|w\|}\right)^{\omega(j)} \leqq \left(\prod_{j\in R^\circ}\frac{\|w\|}{|\hat{H}_j(w)|}\right)\cdot\frac{\prod_{j\in Q}|\hat{H}_j(w)|^{\omega(j)}}{\|w\|^{\tilde{\omega}(q-2N+n-1)+n+1}} \tag{3.11}$$

$$= \frac{1}{\prod_{j\in R^\circ}|\hat{H}_j(w)|}\cdot\frac{\prod_{j\in Q}|\hat{H}_j(w)|^{\omega(j)}}{\|w\|^{\tilde{\omega}(q-2N+n-1)}}.$$

Because of Wronskian's property (3.7), there is a constant $c(R^\circ) > 0$ such that

$$c(R^\circ)\frac{|W((\hat{H}_j\circ f, j\in R^\circ))|}{|W(f^0,\dots,f^n)|} = 1.$$

For $z \in \mathbf{C}^m\backslash\{\prod_{j\in Q}\hat{H}_j\circ f = 0\}$ this with (3.11) implies

$$\prod_{j\in S}\left(\frac{|\hat{H}_j\circ f(z)|}{\|f(z)\|}\right)^{\omega(j)} \leqq c(R^\circ)\frac{1}{\|f(z)\|^{\tilde{\omega}(q-2N+n-1)}}$$

$$\cdot\frac{\prod_{j\in Q}|\hat{H}_j\circ f(z)|^{\omega(j)}}{|W(f^0(z),\dots,f^n(z))|}\cdot\frac{|W((\hat{H}_j\circ f(z), j\in R^\circ))|}{\prod_{j\in R^\circ}|\hat{H}_j\circ f(z)|}$$

$$= c(R^\circ)\frac{1}{\|f(z)\|^{\tilde{\omega}(q-2N+n-1)}}$$

$$\cdot\frac{\prod_{j\in Q}|\hat{H}_j\circ f(z)|^{\omega(j)}}{|W(f^0(z),\dots,f^n(z))|}\cdot|\Delta((\hat{H}_j\circ f(z), j\in R^\circ))|.$$

Hence, setting $C = C_1\max_{R^\circ}\{c(R^\circ)\}$, we obtain the desired inequality. *Q.E.D.*

Proof of Theorem 3.1. We may assume that $q-2N+n-1>0$. By Lemmas 3.9, 3.8, and Jensen's formula we have

$$\begin{aligned}\tilde{\omega}(q-2N+n-1)T_f(r) &\\ \leqq \sum_{j=1}^{q}\omega(j)N_n(r, f^*H_j)&\\ + \frac{1}{2\pi}\int_{|z|=r}\log\left(\sum_{R\subset Q,|R|=n+1}|\Delta((\hat{H}_j\circ f(z), j\in R))|\right)d\theta + O(1)&\end{aligned} \tag{3.12}$$

$$\leqq \tilde{\omega} \sum_{j=1}^{q} N_n(r, f^* H_j)$$
$$+ \frac{1}{2\pi} \int_{|z|=r} \log \left(\sum_{R \subset Q, |R|=n+1} |\Delta((\hat{H}_j \circ f(z), j \in R))| \right) d\theta + O(1).$$

It follows that

$$(3.13) \quad (q - 2N + n - 1) T_f(r)$$
$$\leqq \sum_{j=1}^{q} N_n(r, f^* H_j)$$
$$+ \frac{1}{2\pi\tilde{\omega}} \int_{|z|=r} \log \left(\sum_{R \subset Q, |R|=n+1} |\Delta((\hat{H}_j \circ f(z), j \in R))| \right) d\theta + O(1).$$

By making use of Nevanlinna's lemma on logarithmic derivative generalized over $\mathbf{C}^m$ by A. L. Vitter [Vi77], we deduce

$$\frac{1}{2\pi\tilde{\omega}} \int_{|z|=r} \log \left(\sum_{R \subset Q, |R|=n+1} |\Delta((\hat{H}_j \circ f(z), j \in R))| \right) d\theta$$
$$\leqq \frac{1}{\tilde{\omega}} \left(\sum_{R \subset Q, |R|=n+1} \frac{1}{2\pi} \int_{|z|=r} \log^+ |\Delta((\hat{H}_j \circ f(z), j \in R))| \, d\theta \right) + O(1)$$
$$= S_f(r).$$

From this and (3.13) the desired inequality follows. *Q.E.D.*

Remark on a generalization. We give a generalization of Theorem 3.1 by combining the method in §2 with that of [Ng76] (cf. [Ng03] Chap. 4). Let $\pi : X \to \mathbf{C}^m$ be a finite analytic covering space, that is, X is a normal irreducible complex space and π is a finite mapping. Let R be the ramification divisor of π, and let p be the sheet number. Let $f : X \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping and take a representation $f(z) = [f^0(z), \ldots, f^n(z)]$, which is not necessarily reduced. The Wronskian $W(f^0, \ldots, f^n)$ is defined outside R (cf. (3.6)) and then is extended meromorphically on X (see [Fu85], [Ng97] §2 (b)).

If f separates the fibers of π, we have ([Ng76])

$$(3.14) \qquad N(r, R) \leqq (2p - 2) T_f(r) + O(1).$$

Taking account of the order of partial differentiations, one gets

$$(3.15) \qquad (W(f^0, \ldots, f^n)) + \frac{n(n+1)}{2} R \geqq 0.$$

As deduced (3.12) by making use of Lemmas 3.9 and 3.8, we count the divisor $(W(f^0, \dots, f^n))$ so that for hyperplanes $\{H_j\}_{j=1}^q$ in N-subgeneral position, we obtain

$$(3.16) \quad \tilde{\omega}(q - 2N + n - 1)T_f(r) \leqq \tilde{\omega} \sum_{j=1}^{q} N_n(r, f^* H_j) + \frac{n(n+1)}{2} N(r, R) + S_f(r).$$

It follows from (3.14) that

$$(3.17) \qquad \frac{n(n+1)}{2} N(r, R) \leqq n(n+1)(p-1)T_f(r) + O(1).$$

By (3.16), (3.17) and Lemma 3.3 (iii) we have

THEOREM 3.18 (cf. [Ng03] Chap. 4 §3). *Let $f : X \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate meromorphic mapping. Let $\{H_j\}_{j=1}^q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in N-subgeneral position. Then*

$$(q - 2N + n - 1 - (p-1)n(2N - n + 1))T_f(r) \leqq \sum_{j=1}^{q} N_n(r, f^* H_j) + S_f(r).$$

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