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ISOPERIMETRIC CONSTANTS FOR CONSERVATIVE FUCHSIAN GROUPS

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In celebration of Professor Nobuyuki Suita's life

Abstract

The critical exponents of conservative Fuchsian groups are bounded from below by $1/2$. It is proved in this note that this result is sharp by giving a sequence of conservative Fuchsian groups whose critical exponents converge to $1/2$. The proof is carried out by estimating the isoperimetric constants of hyperbolic surfaces associated with the Fuchsian groups.

§1. Statement of results

A Fuchsian group Γ is a discrete group of orientation-preserving isometric automorphisms of the hyperbolic plane $(\mathbf{H}^2, \rho)$, which is further assumed to be torsion-free throughout this note. It acts on $\mathbf{H}^2$ properly discontinuously and freely. The unit disk $\mathbf{B}^2 \subset \mathbf{C}$ with a conformal metric $2|dz|/(1-|z|^2)$ is a model of the hyperbolic plane and Γ acts on $\mathbf{B}^2$ as a group of Möbius transformations. The boundary S^1 of the model $\mathbf{B}^2$ is located at infinity of the hyperbolic plane and the action of Γ extends to S^1.

The *critical exponent* of a Fuchsian group Γ is defined by

$$\delta(\Gamma) = \inf\left\{ s \geq 0 \,\middle|\, \sum_{\gamma \in \Gamma} \exp(-s\rho(0, \gamma(0))) < \infty \right\}.$$

This is an index which measures distribution of the orbit $\Gamma(0)$ and which is closely related to geometric structure of the associated hyperbolic surface $N_\Gamma := \mathbf{H}^2/\Gamma$. For details, see a survey article [6] and a textbook [8].

A limit point $\xi \in S^1$ of a Fuchsian group Γ is a point of accumulation of the orbit $\Gamma(0) \subset \mathbf{B}^2$. The set of all limit points is called the limit set of Γ and is denoted by $\Lambda(\Gamma)$. A limit point $\xi \in \Lambda(\Gamma)$ is called a conical limit point if $\Gamma(0)$ accumulates to ξ within a bounded distance of the geodesic ray from 0 towards ξ. The conical limit set $\Lambda_C(\Gamma)$ is the set of all conical limit points. A limit point $\xi \in \Lambda(\Gamma)$ is called a horocyclic limit point if $\Gamma(0)$ accumulates to ξ within *some*

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horoball tangent to S^1 at ξ. The *horocyclic limit set* $\Lambda_H(\Gamma)$ is the set of all horocyclic limit points.

The critical exponent can be estimated from below by the Hausdorff dimension (denoted by dim) of those limit sets. A fundamental result in this principle is an estimate by the conical limit set: $\delta(\Gamma) \geq \dim \Lambda_C(\Gamma)$. (Actually the equality holds for every non-elementary Fuchsian group Γ.) Another estimate by the horocyclic limit set is obtained similarly through elementary hyperbolic geometric observation [8, Theorem 2.1.1]. Indeed, ξ belongs to $\Lambda_H(\Gamma)$ if and only if there exists some constant $k > 0$ such that ξ is contained in

$$I_\gamma^{(k)} = \{x \in S^1 \mid |x - |\gamma(0)|^{-1}\gamma(0)| < k(1 - |\gamma(0)|)^{1/2}\}$$

for infinitely many elements $\gamma \in \Gamma$. Since the diameter of $I_\gamma^{(k)}$ is comparable with $\exp(-\rho(0, \gamma(0))/2)$, a standard argument on the Hausdorff measure yields the following claim.

Proposition 1. *If $\Lambda_H(\Gamma)$ has positive 1-dimensional Hausdorff measure, then $\delta(\Gamma) \geq 1/2$. More generally,*

$$\delta(\Gamma) \geq \frac{\dim \Lambda_H(\Gamma)}{2}$$

is satisfied.

The action of a Fuchsian group Γ on S^1 divides it into two parts up to null sets with respect to the 1-dimensional Hausdorff measure: the conservative part and the dissipative part. The latter part is characterized as the maximal measurable subset where a fundamental set for Γ can be chosen. According to Pommerenke [9] and Sullivan [13], the conservative part is coincident with the horocyclic limit set $\Lambda_H(\Gamma)$ up to null sets.

Definition. A Fuchsian group Γ is called *conservative* if the horospherical limit set $\Lambda_H(\Gamma)$ is equal to S^1 almost everywhere with respect to the 1-dimensional Hausdorff measure on S^1.

Proposition 1 in particular implies $\delta(\Gamma) \geq 1/2$ for every conservative Fuchsian group Γ. The purpose of this note is to show that this estimate is sharp in the following sense.

Theorem 2. *There exists a sequence of conservative Fuchsian groups $\{\Gamma_n\}_{n=1}^{\infty}$ such that their critical exponents $\delta(\Gamma_n) (\geq 1/2)$ converge to $1/2$ as $n \to \infty$.*

§2. Normal covering planer surfaces

A sequence of conservative Fuchsian groups as in Theorem 2 is given by normal subgroups Γ_n of Fuchsian groups $\hat{\Gamma}_n$ of cofinite area, which are defined as follows. For each positive integer $n \in \mathbf{N}$, choose a cocompact Fuchsian group G_n having a property that the injectivity radius at the origin $0 \in \mathbf{B}^2$ is greater than n, in other words, the hyperbolic ball $B(0,n) \subset \mathbf{B}^2$ with center 0 and radius n is mapped injectively under the covering projection $\mathbf{B}^2 \to N_{G_n}$. Remark that genera of such compact surfaces N_{G_n} should increase to infinity as $n \to \infty$ to meet this property. Let $R_n = \mathbf{B}^2 - G_n(0)$ be a planer surface obtained from $\mathbf{B}^2$ by removing the orbit of the origin. Let $\hat{R}_n = R_n/G_n$ be the quotient surface, which is coincident with a once-punctured Riemann surface obtained from N_{G_n}. Then, by uniformization theorem, R_n and $\hat{R}_n$ are represented by Fuchsian groups Γ_n and $\hat{\Gamma}_n$, respectively. Since the covering $R_n \to \hat{R}_n$ is normal, Γ_n is a normal subgroup of $\hat{\Gamma}_n$.

Conservativity of these particular Γ_n can be seen from [10, Example 1] and [12, Theorem 4.4]. More generally, the following claim is proved in [5, Theorem 5.1] and [7, Theorem 6].

Proposition 3. *Let Γ be a non-trivial normal subgroup of a Fuchsian group $\hat{\Gamma}$. Then $\Lambda_C(\hat{\Gamma}) \subset \Lambda_H(\Gamma)$. In particular, if $\hat{\Gamma}$ is of cofinite area, then Γ is conservative.*

In what follows, it is proved that the critical exponents $\delta(\Gamma_n)$ converge to $1/2$. Here we remark the following, though they are not directly related to the proof. By construction, the Fuchsian group Γ_n is non-elementary and contains a parabolic element. In this case, a strict inequality $\delta(\Gamma_n) > 1/2$ is always satisfied. See [6, Lemma 30] and [8, Lemma 3.5.4]. Another remark goes to an upper bound of $\delta(\Gamma_n)$. The above construction of a normal subgroup Γ_n appeared in [12], where Patterson proved a strict inequality $\delta(\Gamma_n) < 1$ by an argument using a spectral method as in the next section.

§3. Isoperimetric constants

The critical exponent $\delta(\Gamma)$ is related to the infimum of the *Rayleigh quotient*

$$\lambda_0(\Gamma) = \inf\left\{ \frac{\int_{N_\Gamma} |\nabla f|^2}{\int_{N_\Gamma} |f|^2} \,\middle|\, f \in C_0^\infty(N_\Gamma) \right\}.$$

This quantity is also called the *bottom of spectra* because $\lambda_0(\Gamma)$ is equal to the infimum of the set of eigenvalues with respect to the Laplace-Beltrami operator on the hyperbolic surface N_Γ. Then the following equation between $\delta(\Gamma)$ and $\lambda_0(\Gamma)$ is satisfied, which is due to Elstrodt, Patterson and Sullivan [14]. See also [6, Theorem 17].

Proposition 4. *Every Fuchsian group Γ satisfies*

$$\lambda_0(\Gamma) = \begin{cases} 1/4 & (\delta(\Gamma) \le 1/2) \\ \delta(\Gamma)(1-\delta(\Gamma)) & (\delta(\Gamma) \ge 1/2). \end{cases}$$

Proposition 4 implies that, in order that $\delta(\Gamma)$ is estimated from above, which is necessary for our main theorem, $\lambda_0(\Gamma)$ must be estimated from below. This will be done by using the following quantity.

Definition. The (linear) *isoperimetric constant* (or the Cheeger constant) of a hyperbolic surface N_Γ is defined by

$$h(\Gamma) = \sup_W \frac{A(W)}{\ell(\partial W)},$$

where the supremum is taken over all relatively compact domains $W \subset N_\Gamma$ with smooth boundary, $A(W)$ is the hyperbolic area of W and $\ell(\partial W)$ is the hyperbolic length of the boundary ∂W.

The concept of the isoperimetric constant has its origin in the isoperimetric problem on the Euclidean plane. The same problem can be formulated on the hyperbolic plane and the solution is also well-known. See [4, Section 10] for instance.

Proposition 5. *In the hyperbolic plane* $\mathbf{H}^2$, *consider any simple connected domain bounded by a simple closed curve with a given length* ℓ. *Among all such domains, the one having the greatest area is a hyperbolic ball* B *and its area is*

$$A(B) = \sqrt{\ell^2 + 4\pi^2} - 2\pi.$$

In particular, the isoperimetric constant $h(1)$ *of* $\mathbf{H}^2$ *is one.*

The following theorem due to Cheeger gives a relationship between $\lambda_0(\Gamma)$ and $h(\Gamma)$. See Chavel [2, Section IV.3].

Proposition 6. *Every Fuchsian group* Γ *satisfies*

$$\lambda_0(\Gamma) \geq \frac{1}{4h(\Gamma)^2}.$$

Note that the isoperimetric constant always satisfies $h(\Gamma) \geq 1$. Theorem 2 follows from Propositions 4 and 6 if $h(\Gamma_n) \to 1$ $(n \to \infty)$ for our Fuchsian groups Γ_n.

In the definition of the isoperimetric constant $h(\Gamma)$, the family of domains $W \subset N_\Gamma$ over which the supremum is taken can be slightly modified. If W has a boundary curve that bounds a topological disk in N_Γ, then by filling the disk, the area becomes larger but the boundary length becomes smaller. Hence W can be assumed to have no trivial boundary curves. Furthermore, relative compactness can be broken at each puncture. This is because if W has a boundary curve around a puncture then its length can be arbitrarily small keeping the area even larger by replacing it with a cuspidal curve arbitrary close to the puncture.

After all, W can be taken from a family of those which are topologically finite and whose boundary curves are homotopically non-trivial and non-cuspidal.

For each boundary curve c of such a domain W, there is a unique simple closed geodesic c^* freely homotopic to c. The domain W^* having the boundary curves c^* is homotopically equivalent to W in N_Γ.

Definition. A domain W^* in a hyperbolic surface R is called a *geodesic domain* if W^* is a topologically finite domain possibly with punctures and its boundary ∂W^* consists of a finite number of simple closed geodesics.

As the following theorem implies, the isoperimetric constant can be estimated by considering the supremum of $A(W^*)/\ell(\partial W^*)$ taken over all geodesic domains W^*.

Theorem 7. *Let W be a topologically finite domain with smooth boundary in a hyperbolic surface R and assume that boundary curves are non-trivial and non-cuspidal. Then the geodesic domain W^* homotopically equivalent to W in R satisfies*

$$\frac{A(W)}{\ell(\partial W)} \leq \frac{A(W^*)}{\ell(\partial W^*)} + 1.$$

Proof. The boundary ∂W consists of a finite number of simple closed curves $\{c_i\}_{i=1}^m$. For each c_i, there corresponds the simple closed geodesic c_i^*, which is a boundary component of the geodesic domain W^*. Let $\ell_i = \ell(c_i^*)$ be the length of c_i^* for each $i = 1, \dots, m$.

For the geodesic domain $W^* \subset R$, consider its Nielsen extension $\hat{W}$, which is a complete hyperbolic surface obtained by adding a geodesic annulus along each geodesic boundary component c_i^*. The original domain W can be embedded isometrically into $\hat{W}$. Consider the union $W \cup W^*$. It satisfies $A(W \cup W^*) \geq A(W)$ and $\ell(\partial(W \cup W^*)) \leq \ell(\partial W)$. Hence, in order to estimate $A(W)/\ell(\partial W)$ from above, we have only to examine such a domain $W \subset \hat{W}$ that contains W^*.

In each annular component H_i of $\hat{W} - W^*$ $(i = 1, \dots, m)$, consider a problem of finding a minimal-length curve c_i' that bounds an annulus of a prescribed area $A_i := A(W \cap H_i)$ together with the geodesic $c_i^* = \partial H_i \cap \partial W^*$. The existence of the minimizer c_i' is guaranteed by a compactness property (the Ascoli-Arzela theorem) of a family of homotopically non-trivial, equicontinuous embeddings of a circle into H_i. Moreover, we see that the c_i' is a curve of constant geodesic curvature. See for example [4] Section 5, Theorem 10.1 and its references about those arguments. Consequently, there exists a convex domain W' homotopically equivalent to W in $\hat{W}$ whose boundary components are c_i' $(i = 1, \dots, m)$ and that satisfies

$$\frac{A(W)}{\ell(\partial W)} \leq \frac{A(W')}{\ell(\partial W')}.$$

Each boundary component c_i' of the convex domain W' is equidistant from every point on c_i^*, where the distance can be represented as $\operatorname{arcsinh}(\tan\theta_i)$ by using an angle constant $\theta_i \in [0, \pi/2)$. In this representation, the length of c_i' is $\ell_i \sec\theta_i$ and the area of the annulus bounded by c_i^* and c_i' is $\ell_i \tan\theta_i$. Hence W' satisfies

$$\begin{aligned}\frac{A(W')}{\ell(\partial W')} &= \frac{A(W^*) + \sum_{i=1}^m \ell_i \tan\theta_i}{\sum_{i=1}^m \ell_i \sec\theta_i} \\ &= \frac{\dfrac{\sum \ell_i}{\sum \ell_i \sec\theta_i} A(W^*) + \dfrac{\sum \ell_i \tan\theta_i}{\sum \ell_i \sec\theta_i} \sum \ell_i}{\sum \ell_i} \leq \frac{A(W^*)}{\ell(\partial W^*)} + 1.\end{aligned}$$

Therefore, $A(W)/\ell(\partial W)$ is also bounded by the last term in the above inequality. $\square$

Remark that Fernández and Rodríguez [3, Lemma 1.2] gave an estimate of the same kind in the form

$$\frac{A(W)}{\ell(\partial W)} \leq \frac{A(W^*)}{\ell(\partial W^*)} + 2.$$

Our Theorem 7 succeeds in replacing their constant 2 with 1, which is the best possible constant. This improvement is crucial for proving Theorem 2, as is seen in the next section.

§4. Proof of Theorem 2

In this section, an estimate of the isoperimetric constant $h(\Gamma_n)$ is given from above, by which the proof of Theorem 2 will be complete.

Take a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ satisfying $\varepsilon_n \to 0$ as $n \to \infty$. By the definition of $h(\Gamma_n)$, there exists an admissible domain $W_n \subset R_n$ such that

$$h(\Gamma_n) < \frac{A(W_n)}{\ell(\partial W_n)} + \varepsilon_n.$$

Here W_n can be again replaced with a domain bounded by a *single* simple closed curve c_n. This is because filling a punctured disk in the *planer* surface R_n bounded by an inner boundary component of ∂W_n ($\neq c_n$) makes the area larger but the boundary length smaller. Hence W_n may be assumed to have a single boundary curve and at most a finite number of punctures.

Take the simple closed geodesic c_n^* freely homotopic to $c_n = \partial W_n$ in R_n and consider the geodesic domain W_n^* bounded by c_n^*. Set $A_n = A(W_n^*)$ and $\ell_n = \ell(c_n^*)$. Then Theorem 7 implies

$$h(\Gamma_n) < \frac{A_n}{\ell_n} + 1 + \varepsilon_n.$$

By the Gauss-Bonnet formula, if the number of punctures of W_n^* is $p(n)$, then
$$A_n = 2\pi(p(n) - 1) < 2\pi p(n).$$

An upper bound of $p(n)$ is obtained by using the hyperbolic metric on $\mathbf{B}^2$ instead of $R_n(\subset \mathbf{B}^2)$. Let ℓ_n' be the hyperbolic length of c_n^* in $\mathbf{B}^2$. Monotonicity of the hyperbolic density under the inclusion relation implies $\ell_n' \le \ell_n$. By Proposition 5, the area of W_n^* measured in $\mathbf{B}^2$ is bounded from above by
$$\sqrt{(\ell_n')^2 + 4\pi^2} - 2\pi < \ell_n' \le \ell_n.$$
Each puncture z of W_n^* has a mutually disjoint neighborhood $B(z,n) \subset \mathbf{B}^2$ of radius n, whose area is $4\pi \sinh^2(n/2)$. In these circumstances, $p(n)$ can be estimated as follows.

Lemma 8.
$$p(n) < \frac{\ell_n}{2\pi \sinh^2(n/2)} + \frac{\ell_n}{2n}$$

Proof. Consider the intersection of W_n^* and $B(z,n)$ for each puncture z of W_n^*. Then at least one of the following two conditions are satisfied:
(1) the area of the intersection $W_n^* \cap B(z,n)$ is greater than half the area of $B(z,n)$;
(2) the length of the intersection $\partial W_n^* \cap B(z,n)$ is greater than the diameter of $B(z,n)$.
Indeed, when $\partial W_n^* \cap B(z,n) \ne \emptyset$, there are subarcs α of ∂W_n^* and β of $\partial B(z,n)$ which together bound a domain containing z. If $\ell(\beta) \le \ell(\partial B(z,n))/2$, then $\ell(\alpha) > 2n$ and hence condition (2) is satisfied. If $\ell(\beta) > \ell(\partial B(z,n))/2$ and $\ell(\alpha) \le 2n$, then a half-disk of $B(z,n)$ divided by a diameter connecting two antipodal points on β is entirely contained in W_n^* and hence condition (1) is satisfied.

The area of $W_n^* \subset \mathbf{B}^2$ is bounded by ℓ_n and half the area of $B(z,n)$ is precisely $2\pi \sinh^2(n/2)$. Therefore, the number of punctures z in W_n^* satisfying condition (1) is at most $\ell_n/\{2\pi \sinh^2(n/2)\}$. Similarly, the number of punctures satisfying condition (2) is at most $\ell_n/(2n)$. $\square$

The combination of all estimates above yields
$$\begin{aligned} h(\Gamma_n) &< \frac{2\pi p(n)}{\ell_n} + 1 + \varepsilon_n \\ &< \frac{1}{\sinh^2(n/2)} + \frac{\pi}{n} + 1 + \varepsilon_n \to 1 \quad (n \to \infty). \end{aligned}$$
This implies that $\delta(\Gamma_n) \to 1/2$ as $n \to \infty$.

Remark that Álvarez, Pestana and Rodríguez [1, Theorem 1] gave an estimate of the isoperimetric constant by a similar method, however, our estimate has an advantage when the injectivity radius n grows to infinity.

§5. Questions and comments

There still remains a question whether the lower bound $1/2$ of the critical exponents is attained by a conservative Fuchsian group. As is mentioned in Section 2, such a Fuchsian group, if exists, contains no parabolic elements.

PROBLEM. *Does there exist a conservative Fuchsian group* Γ *with* $\delta(\Gamma) = 1/2$?

Note that if $\delta(\Gamma) < 1/2$ then the 1-dimensional Hausdorff measure of $\Lambda_H(\Gamma)$ is zero as in [11, Theorem 3]. This fact also follows from Proposition 1. Hence the exponent $1/2$ is critical for conservativity in this sense.

The sequence of Fuchsian groups $\{\Gamma_n\}$ in our construction (modulo conjugation) converges geometrically to a parabolic cyclic group G, which satisfied $\delta(G) = 1/2$. However, concerning the continuity of the critical exponent, only the lower semi-continuity holds in general: $\liminf_{n\to\infty} \delta(\Gamma_n) \geq \delta(G)$. See [6, Lemma 21]. This does not help us to prove Theorem 2.

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