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A REMARK ON UNIVERSAL COVERINGS OF HOLOMORPHIC FAMILIES OF RIEMANN SURFACES

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Dedicated to the memory of Professor Nobuyuki Suita

Abstract

We study the universal covering space $\tilde{M}$ of a holomorphic family (M, π, R) of Riemann surfaces over a Riemann surface R. The main result is that (1) $\tilde{M}$ is topologically equivalent to a two-dimensional cell, (2) $\tilde{M}$ is analytically equivalent to a bounded domain in $\mathbf{C}^2$, (3) $\tilde{M}$ is not analytically equivalent to the two-dimensional unit ball $\mathbf{B}_2$ under a certain condition, and (4) $\tilde{M}$ is analytically equivalent to the two-dimensional polydisc Δ^2 if and only if the homotopic monodoromy group of (M, π, R) is finite.

1. Introduction

1.1. It is well-known as Koebe's uniformization theorem for a Riemann surface that the universal covering space $\tilde{R}$ of a complex manifold R of dimension one is given as follows (cf. Bers [4] and Shafarevich [22], pp. 380–401).

(1) $\tilde{R}$ is biholomorphically equivalent to the Riemann sphere $\hat{\mathbf{C}}$ if and only if R is also biholomorphically equivalent to $\hat{\mathbf{C}}$.
(2) $\tilde{R}$ is biholomorphically equivalent to the complex plane $\mathbf{C}$ if and only if R is biholomorphically equivalent to $\mathbf{C}$, $\mathbf{C}\backslash\{0\}$ or a torus.
(3) $\tilde{R}$ is biholomorphically equivalent to the unit disc Δ if and only if R is not biholomorphically equivalent to $\hat{\mathbf{C}}$, $\mathbf{C}$, $\mathbf{C}\backslash\{0\}$ or a torus.

1.2. However, universal coverings and fundamental groups of complex manifolds of higher dimension are very complicated. We give some examples (cf. Shafarevich [22], pp. 401–408).

(1) There are infinitely many different simply-connected compact complex manifolds of dimension $n \geq 2$.

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(2) For a given finite group Γ, there exists a compact complex manifold of dimension $n \geq 2$ whose fundamental group is isomorphic to Γ.
(3) The polydisc Δ^n of dimension $n \geq 2$ is not biholomorphically equivalent to the unit ball $\mathbf{B}_n$ (Poincaré's theorem, cf. Narashimhan [17], p. 70).

1.3. P. A. Griffiths [8] got the following uniformization theorem of quasi-projective varieties. Here we describe the case of dimension two. Let $\hat{M}$ be a two-dimensional, irreducible, smooth quasi-projective algebraic variety over the complex number field. For every point p in $\hat{M}$, there exists a Zariski neighborhood M of p such that M has a holomorphic fibration (M, π, R) of Riemann surfaces of type (g, n) with $2g - 2 + n > 0$ over a hyperbolic Riemann surface R of analytically finite type. (We give a definition of a holomorphic fibration in the next section.) Then Griffiths proved that the universal covering space $\tilde{M}$ is topologically equivalent to a two-dimensional cell and biholomorphically equivalent to a bounded domain of holomorphy in $\mathbf{C}^2$ by using the theory of simultaneous uniformization of Riemann surfaces due to Bers.

1.4. In this paper we study some function-theoretic properties of the universal covering space $\tilde{M}$ of a holomorphic family of Riemann surfaces (M, π, R). Our Main results are follows:

Theorem 1. *The universal covering space $\tilde{M}$ of a holomorphic family of Riemann surfaces (M, π, R) of type (g, n) is not biholomorphically equivalent to the two-dimensional unit ball $\mathbf{B}_2$ provided that (M, π, R) is locally trivial, $n > 0$, or R is not compact.*

By Rosay's theorem [19] we have a corollary.

Corollary 1. *The universal covering space $\tilde{M}$ of a holomorphic family of Riemann surfaces (M, π, R) of type (g, n) is not biholomorphically equivalent to any two-dimensional strongly pseudoconvex domains provided that (M, π, R) is locally trivial, $n > 0$, or R is not compact.*

Theorem 2. *The universal covering space $\tilde{M}$ of a holomorphic family of Riemann surfaces (M, π, R) is biholomorphically equivalent to the two-dimensional polydisc Δ^2 if and only if all the fibers $S_t = \pi^{-1}(t)$ are biholomorphically equivalent.*

As a corollary we have the following (see Imayoshi [9]).

Corollary 2. *The universal covering space $\tilde{M}$ of a holomorphic family of Riemann surfaces (M, π, R) is biholomorphically equivalent to the two-dimensional polydisc Δ^2 if and only if the homotopic monodromy group $\mathscr{M}$ of (M, π, R) is finite.*

In the case where R has punctures, i.e., it is not compact, these results were obtained in Imayoshi [9]. In this paper we do not assume that R has punctures.

However, in Theorem 1, if R is compact, we assume that (M, π, R) is locally trivial, or $n > 0$, i.e., every fiber S_t has punctures. It is known that a Kodaira surface M has a locally non-trivial fibration (M, π, R) of type $(g, 0)$ over a compact Riemann surface R (Kas [12], Kodaira [14]), and its universal covering $\tilde{M}$ is not biholomorphically equivalent to $\mathbf{B}_2$ (Atiyah [1], Shabat [20], [21]). It is not known whether except for a kind of Kodaira surfaces there exits a locally non-trivial holomorphic family of Riemann surfaces of type $(g, 0)$ over a compact Riemann.

1.5. This paper is organized as follows: In §2 we give a definition of holomorphic families (M, π, R) of Riemann surfaces and some examples of these families. In §3 we explain briefly Teichmüller theory used in this paper. In §4, using Teichmüller theory we construct canonically a universal covering space $\tilde{M}$ and its universal covering transformation group $\mathcal{G}$. Theorem 1 is proved in §5, and Theorem 2 is proved in §6 and §7.

2. Holomorphic families of Riemann surfaces

2.1. A holomorphic family (M, π, R) of Riemann surfaces over a Riemann surface R is defined as follows. Let $\hat{M}$ be a two-dimensional complex manifold, C a one-dimensional analytic subset of $\hat{M}$ or an empty set, and R be a Riemann surface. Assume that a proper holomorphic map $\hat{\pi} : \hat{M} \to R$ satisfies two conditions:

(i) by setting $M = \hat{M} \backslash C$ and $\pi = \hat{\pi}|M$, the holomorphic map π is of maximal rank at every point of M, and

(ii) the fiber $S_t = \pi^{-1}(t)$ over each $t \in R$ is a Riemann surface of fixed analytically finite type (g, n), where g is the genus of S_t and n is the number of punctures of S_t, i.e., it is obtained by removing n distinct points from a compact Riemann surface of genus g.

We call such a triple (M, π, R) a *holomorphic family of Riemann surfaces of type* (g, n) over R. We assume throughout this paper that $2g - 2 + n > 0$, and R is a hyperbolic Riemann surface of analytically finite type.

2.2. We give some examples of holomorphic families of Riemann surfaces.

Example 1. Take two hyperbolic Riemann surfaces R_0, S_0 of analytically finite type. Let $M_0 = R_0 \times S_0$ and $\pi_0 : M_0 = R_0 \times S_0 \to R_0$ be the canonical projection. Then (M_0, π_0, R_0) is a holomorphic family of Riemann surfaces of type (g_0, n_0), where (g_0, n_0) is the type of S_0.

A holomorphic family (M, π, R) is said to be *globally trivial* if there exist biholomorphic maps $F : M \to M_0 = R_0 \times S_0$ and $f : R \to R_0$ with $\pi_0 \circ F = f \circ \pi$. A holomorphic family is said to be *locally trivial* if it is analytically a local tirivial fiber bundle.

The universal covering $\tilde{M}_0$ of M_0 is biholomorphically equivalent to $\tilde{R}_0 \times \tilde{S}_0 \cong \Delta^2$. Poincaré's Theorem shows that $\tilde{M}_0$ is not biholomorphically equivalent to the unit ball $\mathbf{B}_2$. This is a trivial example of Theorems 1 and 2.

Example 2. Let R be a hyperbolic Riemann surface of analytically finite type (g, n). Let $M = \{(p, q) \in R \times R \mid p \neq q\}$ and $\pi : M \to R$ be the canonical projection. Then (M, π, R) is a locally non-tirivial holomorphic family of Riemann surfaces of type $(g, n+1)$. Theorems 1 and 2 imply that the universal covering $\tilde{M}$ of M is biholomorphically equivalent to neither Δ^2 nor $\mathbf{B}_2$.

Example 3. Set $R = \mathbf{C} \backslash \{0\}$ and $M = \{(x, y, t) \in \mathbf{C}^2 \times R \mid y^2 = x^3 - t\}$. Let $\pi : M \to R$ be the canonical projection. Then (M, π, R) is a holomorphic family of Riemann surfaces of type $(1, 1)$, which is locally trivial, but not globally trivial. In this case $\tilde{M}$ is biholomorphically equivalent to Δ^2.

Example 4. Set $R = \mathbf{C} \backslash \{0, 1\}$ and $M = \{(x, y, t) \in \mathbf{C}^2 \times R \mid y^2 = x(x-1)(x-t)\}$. Let $\pi : M \to R$ be the canonical projection. Then (M, π, R) is a holomorphic family of Riemann surfaces of type $(1, 1)$, which is not locally trivial. Hence Theorems 1 and 2 show that $\tilde{M}$ of M is biholomorphically equivalent to neither Δ^2 nor $\mathbf{B}_2$.

Example 5. Kodaira [14] constructed a locally non-trivial holomorphic family (M, π, R) of Riemann surfaces of type $(g, 0)$ over a closed Riemann surface R. See also Atiyah [1], Barth, Peters and Van de Ven [2], Kas [12], and Riera [18]. We call such a complex surface M a *Kodaira surface*.

Since this family is not locally trivial, Theorem 2 implies that $\tilde{M}$ is not biholomorphically equivalent to Δ^2 (cf. Atiyah [1], p. 79). It is also known that $\tilde{M}$ is not biholomorphically equivalent to $\mathbf{B}_2$ (see Atiyah [1], p. 79).

Example 6. As stated in §1, for a two-dimensional, irreducible, smooth quasi-projective algebraic surface $\hat{M}$ over the complex number field and for every point $p \in \hat{M}$, there exists a Zariski neighborhood M of p such that M has a holomorphic fibration (M, π, R) of Riemann surfaces over a Riemann surface R.

3. Teichmüller theory

3.1. In order to construct canonically a universal covering space $\tilde{M}$ of a holomorphic family (M, π, R) of Riemann surfaces of type (g, n), we use Teichmüller theory. We shall explain it in brief (refer to Bers [5], and Imayoshi and Taniguchi [10]).

Let S be a fixed Riemann surface of analytically finite type (g, n) with $2g - 2 + n > 0$. A *marked Riemann surface* (S, f, S') is a Riemann surface S' of analytically finite type (g, n) with a quasiconformal map $f : S \to S'$. We define an equivalence relation between marked surfaces (S, f_1, S_1) and (S, f_2, S_2) if there exists a conformal map $h : S_1 \to S_2$ such that the self-map $f_2^{-1} \circ h \circ f_1 : S \to S$ is homotopic to the identity. We denote by $[S, f, S']$ the equivalence class of a representative (S, f, S'). The *Teichmüller space* $T(S)$ of a Riemann surface S is the set of all these equivalence classes $[S, f, S']$. Let $\mathrm{Mod}(S)$ be the set of all homotopy classes $[f_0]$ of quasiconformal self-maps $f_0 : S \to S$. We call $\mathrm{Mod}(S)$ the *Teichmüller modular group* of R. Every element $[f_0]$ acts on $T(R)$ by

$$[f_0]_*([S, f, S']) = [S, f \circ f_0^{-1}, S'].$$

3.2. Let G be a finitely generated Fuchsian group of the first kind with no elliptic elements acting on the upper half-plane U such that the quotient space $S \cong U/G$ is of type (g, n). Let $Q_{norm}(G)$ be the set of all quasiconformal automorphisms w of U leaving 0, 1, ∞ fixed and satisfying $wGw^{-1} \subset PSL(2, \mathbf{R})$, where $PSL(2, \mathbf{R})$ is the set of all real Möbius transformations. Two elements w_1 and w_2 of $Q_{norm}(G)$ are equivalent if $w_1 = w_2$ on the real axis $\mathbf{R}$. The *Teichmüller space* $T(G)$ of G is the set of all equivalence classes $[w]$ obtained by classifying $Q_{norm}(G)$ by the above equivalence relation.

Let $L^\infty(U, G)_1$ be the complex Banach space of (equivalence classes of) bounded complex-valued measurable functions μ on U satisfying

$$\mu \circ g \frac{\overline{g'}}{g'} = \mu, \quad {}^\forall g \in G, \quad \text{and} \quad \|\mu\|_\infty < 1.$$

For an element $\mu \in L^\infty(U, G)_1$ denote by w_μ the element in $Q_{norm}(G)$ with Beltrami coefficient μ. Let W^μ be the quasiconformal automorphism of the Riemann sphere $\hat{\mathbf{C}}$ such that W^μ has the Beltrami coefficient μ on the upper half-plane U, and comformal on the lower half-plane L, and

$$W^\mu(z) = \frac{1}{z + i} + O(|z + i|) \tag{3.1}$$

as $z \to -i$. This map W^μ is uniquely determined by $[w_\mu]$ up to the equivalence relation, i.e., $w_\mu = w_\nu$ on $\mathbf{R}$ if and only if $W^\mu = W^\nu$ on L. We set $T_\beta(G) = \{[W^\mu] \mid \mu \in L^\infty(U, G)_1\}$, which is called the *Bers Teichmüller space* of G.

Let ϕ_μ be the Schwarzian derivative of W^μ on L. Then ϕ_μ is an element of the space $B_2(L, G)$ of bounded holomorphic quadratic differentials for G on L. The space $B_2(L, G)$ is a $(3g - 3 + n)$-dimensional complex vector space. Bers proved that the map sending $[W^\mu]$ into ϕ_μ is a biholomorphic map of $T_\beta(G)$ onto a holomorphically convex bounded domain of $B_2(L, G)$, which is denoted the same notation $T_\beta(G)$.

Denote by $N(G)$ the set of all quasiconformal automorphisms ω of U with $\omega G \omega^{-1} = G$. Two elements $\omega_1, \omega_2 \in N(G)$ are equivalent if $\omega_1 = \omega_2 \circ g_0$ on the real axis $\mathbf{R}$ for some $g_0 \in G$. Denote by $[\omega]$ the equivalence class of a representative ω. Let $\mathrm{Mod}(G)$ be the set of all equivalence classes $[\omega]$ in $N(G)$. We call $\mathrm{Mod}(G)$ the *Teichmüller modular group* of G. Every element $[\omega]$ acts on $T(G)$ by

$$[\omega]_*([w]) = [\lambda \circ w \circ \omega^{-1}],$$

where $[w] \in T(G)$ and $\lambda \in PSL(2, \mathbf{R})$ with $\lambda \circ w \circ \omega^{-1} \in Q_{norm}(G)$.

4. Construction of the universal covering space $\tilde{M}$ of a holomorphic family (M, π, R) of Riemann surfaces

4.1. We shall describe a way to construct a universal covering space $\tilde{M}$ of a given holomorphic family (M, π, R) of Riemann surfaces of type (g, n) by using Teichmüller theory. This is due to Griffiths [8].

Let (M, π, R) be a holomorphic family of Riemann surfaces of type (g, n) over R. Take a universal covering $\rho : \Delta \to R$ with covering transformation group Γ. Then there exists a holomorphic map $\Phi : \Delta \to T(S)$ sending $\tau \in \Delta$ into $[S, f_\tau, S_{\rho(\tau)}]$, where $f_\tau : S \to S_{\rho(\tau)}$ is a quasiconformal map moving continuously with resptect to the parameter τ. We call this holomorphic map $\Phi : \Delta \to T(S)$ a *representation* of (M, π, R) into a Teichmüller space $T(S)$. The representation Φ induces a group homomorphism $\Phi_* : \Gamma \to \mathrm{Mod}(S)$ satisfying $\Phi \circ \gamma = \Phi_*(\gamma) \circ \Phi$ for all $\gamma \in \Gamma$.

4.2. Identify $T(S)$ with $T_\beta(G)$. Then we obtain a *representation* $\Psi : \Delta \to T_\beta(G)$ of (M, π, R) into $T(G)$ and a biholomorphic map $F_\tau : D_\tau / G_\tau \to S_{\rho(\tau)}$ for each $\tau \in \Delta$, where $\Psi(\tau) = [W^{\mu(\tau)}]$, $D_\tau = W^{\mu(\tau)}(U)$, and $G_\tau = W^{\mu(\tau)} G (W^{\mu(\tau)})^{-1} \subset PSL(2, \mathbf{C})$.

We set

$$\tilde{M} = \{(\tau, w) \mid \tau \in \Delta, w \in D_\tau\}.$$

This set $\tilde{M}$ is topologically equivalent to a two-dimensional cell. From (3.1) Koebe's one-quarter theorem shows that $D_\tau \subset \{|w| < 2\}$ for all $\tau \in \Delta$, and so $\tilde{M}$ is a bounded domain in $\mathbf{C}^2$. It is also shown that $\tilde{M}$ is a domain of holomorphy. Let $\tilde{\pi} : \tilde{M} \to \Delta$ be the holomorphic map sending (τ, w) into τ. Then the fiber $\tilde{\pi}^{-1}(\tau)$ of $(\tilde{M}, \tilde{\pi}, \Delta)$ over τ is biholomorphically equivalent D_τ.

Let $\Pi : \tilde{M} \to M$ be the holomorphic map sending (τ, w) into $F_\tau(w)$. Then $\Pi : \tilde{M} \to M$ is the universal covering of M constructed by Griffiths [8].

4.3. We shall explicitly express the elements of the covering transformation group $\mathscr{G}$ of the the universal covering $\Pi : \tilde{M} \to M$. For each element $\gamma \in \Gamma$, the *homotopic monodromy* $\mathscr{M}_\gamma$ of γ for (M, π, R) is the element of the Teichmüller modular group $\mathrm{Mod}(G)$ with $\Phi \circ \gamma = \mathscr{M}_\gamma \circ \Phi$. The subgroup $\mathscr{M} = \{\mathscr{M}_\gamma \mid \gamma \in \Gamma\}$ of $\mathrm{Mod}(G)$ is called the *homotopic monodromy group* of (M, π, R) with respect to the representation Φ.

Denote by $N(G)$ the set of all quasiconformal automorphisms ω of U with $\omega G \omega^{-1} = G$. Take an element $\omega_\gamma \in N(G)$ inducing $\mathscr{M}_\gamma$, i.e., $[\omega_\gamma] = \mathscr{M}_\gamma$. We may assume that $\omega_{\gamma \circ \delta} = \omega_\gamma \circ \omega_\delta$ for all $\gamma, \delta \in \Gamma$.

For each $\tau \in \Delta$, let $[w_{\mu(\tau)}]$ be the point of $T(G)$ with Beltrami coefficient $\mu(\tau) \in L^\infty(U, G)_1$ corresponding to the $\Psi(\tau) \in T_\beta(G)$. For every $g \in G$, we set

$w_{\nu(\tau)} = \lambda \circ w_{\mu(\tau)} \circ (\omega_\gamma \circ g)^{-1} \in Q_{norm}(G)$, where λ is a real Möbius transformation. Note that $w_{\nu(\tau)} = w_{\mu(\gamma(\tau))}$.

If we set

$$(\gamma, g)(\tau, w) = (\gamma(\tau), W^{\mu(\gamma(\tau))} \circ (\omega_\gamma \circ g) \circ (W^{\mu(\tau)})^{-1}(w)),$$

then the map (γ, g) is an analytic automorphism of $\tilde{M}$ (see Bers [3], Theorem 2, p. 95). We set

$$H(\gamma, g)(\tau, w) = W^{\mu(\gamma(\tau))} \circ (\omega_\gamma \circ g) \circ (W^{\mu(\tau)})^{-1}(w).$$

Then $H(\gamma, g)(\tau, \cdot) : D_\tau \to D_{\gamma(\tau)}$ is a conformal map such that $G_{\gamma(\tau)} = H(\gamma, g)(\tau, \cdot) G_\tau (H(\gamma, g)(\tau, \cdot))^{-1}$ and $H(\gamma, g)(\tau, \cdot)$ induces a conformal map of D_τ / G_τ onto $D_{\gamma(\tau)} / G_{\gamma(\tau)}$.

Now the covering transformation group $\mathscr{G}$ of the universal covering $\Pi : \tilde{M} \to M$ is identified with the set $\Gamma \times G$. By definition, we have the relation

$$(\gamma, g) \circ (\delta, h) = (\gamma \circ \delta, \omega_\delta^{-1} \circ g \circ \omega_\delta \circ h)$$

for all $\gamma, \delta \in \Gamma$ and $g, h \in G$, which implies that $\mathscr{G}$ is a semi-direct product of Γ by G. Note that $(\gamma, g) = (\delta, h)$ if and only if $\gamma = \delta$ and $g = h$.

5. Proof of Theorem 1

5.1. In this section we shall give a proof of Theorem 1. We use the notation in §3 and §4.

If (M, π, R) is locally trivial, then the representation Ψ of (M, π, R) into a Teichmüller space $T(G)$ is constant. Hence $\tilde{M} = \Delta \times D_0 \cong \Delta \times \Delta$, which implies that $\tilde{M}$ is not biholomorphically equivalent to the unit ball $\mathbf{B}_2$ by Poincaré's Theorem.

If the base surface R is not compact, the assertion of Theorem 1 is shown in Imayoshi [9], pp. 584–586.

5.2. Let us consider the case $n > 0$, i.e., every fiber $S_t = \pi^{-1}(t)$ is not compact. Assume that there exists a biholomorphic map $F = (F_1, F_2) : \tilde{M} \to \mathbf{B}_2$.

We may assume that for every $\Phi_*(\gamma) = [f_\gamma] \in \mathrm{Mod}(S)$, $\gamma \in \Gamma$, the quasiconformal self-map $f_\gamma : S \to S$ fixes each puncture of S. In fact, the subgroup $\mathscr{M}' = \{[f_\gamma] \in \Phi_*(\Gamma) \mid f_\gamma$ fixes every puncture of $S\}$ of $\Phi_*(\Gamma)$ is a normal subgroup $\mathscr{M}$ of finite index. Let $\Gamma' = \{\gamma \in \Gamma \mid [f_\gamma] \in \mathscr{M}'\}$. Then Γ' is a normal subgroup of Γ and Γ/Γ' is canonically isomorphic to $\mathscr{M}/\mathscr{M}'$. Hence Γ' is a normal subgroup of Γ of finite index. Then there exists a unramified finite-sheeted covering $\rho_0 : R' \to R$ such that the fundamental group of R' is isomorphic to Γ' and the covering transformation group of $\rho_0 : R' \to R$ is isomorphic to Γ/Γ'. Let $\pi' : M' \to R'$ be the fiber product of $\pi : M \to R$ by $\rho_0 : R' \to R$, i.e., $M' = \{(p, t') \in M \times R' \mid \pi(p) = \rho_0(t')\}$ and $\pi'(p, t') = t'$. Then the fiber $\pi'^{-1}(t')$ of M' over t' is biholomorphic to the fiber $\pi^{-1}(\rho_0(t'))$ of M over $\rho_0(t')$, and the monodromy of (M', π', R') with respect to arbitrary $\gamma' \in \Gamma'$ is $[f_{(\rho_0)_*(\gamma')}] \in \mathscr{M}'$. Since $\tilde{M}$ is biholomorphically equivalent to $\mathbf{B}_2$, we see that the universal covering space $\tilde{M}'$ of M' is also biholomorphically equivalent to $\mathbf{B}_2$. Therefore we may consider (M', π', R') in place of (M, π, R).

5.3. Now suppose that there exists a biholomorphic map $F = (F_1, F_2) : \tilde{M} \to \mathbf{B}_2$, and that for every $\Phi_*(\gamma) = [f_\gamma] \in \mathrm{Mod}(S)$, $\gamma \in \Gamma$, the quasiconformal self-map $f_\gamma : S \to S$ fixes each puncture of S. We may also assume that for every puncture p_0 of S there exists a neighborhood U_{p_0} of p_0 such that $f_\gamma(p) = p$ for all $p \in U_{p_0}$.

We set $t_0 = \rho(0)$, and $S = S_{t_0} = \pi^{-1}(t_0) \cong U/G$. Take a cusp point $\zeta_0^* \in \partial U$ for G. From the assumption that the quasiconformal self-map $f_\gamma : S \to S$ inducing $\Phi_*(\gamma)$ fixes each puncture of S it follows that for $\Psi_*(\gamma) = [\omega_\gamma] \in \mathrm{Mod}(G)$ there exists an element $g_\gamma \in G$ such that

$$g_\gamma \circ \omega_\gamma(w) = w \tag{5.1}$$

for any point w in a cusped region belonging to ζ_0^* for G.

We set

$$\begin{aligned} W^0(z) &= \frac{1}{z+1}, \\ G_0 &= W^0 G (W^0)^{-1}, \\ \zeta_0 &= W^0(\zeta_0^*) \in \partial D_0 = \partial W^0(U). \end{aligned}$$

5.4. Consider the holomorphic motion V^τ of ∂D_0 given by

$$V^\tau(\zeta) = W^{\mu(\tau)} \circ (W^0)^{-1}(\zeta), \quad (\tau, \zeta) \in \Delta \times \partial D_0.$$

Note that V is G_0-*equivariant*, that is, it satisfies the relation

$$V^\tau(g(\zeta)) = g^\tau(V^\tau(\zeta)) \quad \text{on } \Delta \times \partial D_0 \tag{5.2}$$

for all $g \in G_0$, where $g^\tau = W^{\mu(\tau)} \circ g \circ (W^{\mu(\tau)})^{-1}$. Then an equivariant version of Slodkowski's extension theorem implies that the G_0-equivariant holomorphic motion V of ∂D_0 can be extended to a holomorphic motion of $\hat{\mathbf{C}}$ (still called V^τ) in such a way that (5.2) holds for all $g_0 \in G_0$, $\tau \in \Delta$, and $w \in \hat{\mathbf{C}}$ (see Earle, Kra and Krushkal' [7], p. 928).

Take a sequence $\{w_n\}_{n=1}^\infty$ in a cusped region belonging to ζ_0 for G_0 with $\lim_{n\to\infty} w_n = \zeta_0$. We define a holomorphic map $\Delta \to \tilde{M}$ by

$$s_n(\tau) = (\tau, V^\tau(w_n)),$$

which is a holomorphic section of $(\tilde{M}, \tilde{\Pi}, \Delta)$. Here $\tilde{\Pi} : \tilde{M} \to \Delta$ is the holomorphic map given by $\tilde{\Pi}(\tau, w) = \tau$.

We put $h_\gamma = (\omega_\gamma)^{-1} \circ g_\gamma \circ \omega_\gamma$ and

$$\begin{aligned} H_\gamma(\tau, w) &= H_{(\gamma, h_\gamma)}(\tau, w) \\ &= W^{\mu(\gamma(\tau))} \circ \omega_\gamma \circ h_\gamma \circ (W^{\mu(\tau)})^{-1}(w). \end{aligned}$$

From (5.1) we get

$$H_\gamma(\tau, W^{\mu(\tau)}(w_n)) = W^{\mu(\gamma(\tau))} \circ \omega_\gamma \circ h_\gamma \circ (w_n) \tag{5.3}$$
$$= W^{\mu(\gamma(\tau))} \circ g_\gamma \circ \omega_\gamma(w_n)$$
$$= W^{\mu(\gamma(\tau))}(w_n).$$

Let d_{D_τ} be the Poincaré distance on D_τ. Then we obtain the following lemma:

Lemma 1. *There exists a positive constant K depending on γ and τ such that*

$$d_{D_{\gamma(\tau)}}(H_\gamma(\tau, V^\tau(w_n)), V^{\gamma(\tau)}(w_n)) \le K. \tag{5.4}$$

Proof. Noting $H_\gamma : D_\tau \to D_{\gamma(\tau)}$ is conformal and (5.3), we get

$$\begin{aligned} &d_{D_{\gamma(\tau)}}(H_\gamma(\tau, V^\tau(w_n)), V^{\gamma(\tau)}(w_n)) \\ &\quad \le d_{D_{\gamma(\tau)}}(H_\gamma(\tau, V^\tau(w_n)), H_\gamma(\tau, W^{\mu(\tau)}(w_n))) \\ &\qquad + d_{D_{\gamma(\tau)}}(H_\gamma(\tau, W^{\mu(\tau)}(w_n)), V^{\gamma(\tau)}(w_n)) \\ &\quad = d_{D_\tau}(W^{\mu(\tau)}(w_n), V^\tau(w_n)) + d_{D_{\gamma(\tau)}}(W^{\mu(\gamma(\tau))}(w_n), V^{\gamma(\tau)}(w_n)). \end{aligned} \tag{5.5}$$

Since V^τ is quasiconformal on $\hat{\mathbf{C}}$ by a theorem due to Mañé, Sud and Sullivan (cf. Bers and Royden [6], Theorem 1, p. 492), and V^τ and $W^{\mu(\tau)}$ have the same boundary values on ∂D_0, Theichmüller's theorem implies that there exists a positive constant K_1 such that

$$d_{D_\tau}(W^{\mu(\tau)}(w_n), V^\tau(w_n)) \le K_1 \tag{5.6}$$

for any n (see Theichmüller [24], and Kra [15], Lemma 1, p. 234). Similarly, we find a positive constant K_2 so that

$$d_{D_{\gamma(\tau)}}(W^{\mu(\gamma(\tau))}(w_n), V^{\gamma(\tau)}(w_n)) \le K_2 \tag{5.7}$$

for any n. Hence from (5.5), (5.6) and (5.7) we have

$$d_{D_{\gamma(\tau)}}(H_\gamma(\tau, V^\tau(w_n)), V^{\gamma(\tau)}(w_n)) \le K_1 + K_2 \tag{5.8}$$

for any n. □

5.5. Since $\mathbf{B}_2$ is a bounded domain, we may assume that $\{F \circ s_n\}_{n=0}^\infty$ converges uniformly on compact subsets of Δ. We may also assume that

$$\lim_{n\to\infty} F \circ s_n(\tau) = \lim_{n\to\infty} F(\tau, V^\tau(w_n)) = (1, 0) \in \partial\mathbf{B}_2 \tag{5.9}$$

for every $\tau \in \Delta$ (see Imayoshi [9], pp. 584–585).

Let $F_* : \mathscr{G} \to \mathrm{Aut}(\mathbf{B}_2)$ be the group homomorphism defined by

$$F \circ (\gamma, g) = F_*(\gamma, g) \circ F$$

for every $(\gamma, g) \in \mathscr{G} = \Gamma \ltimes G$.

Setting $\chi_\gamma = (\gamma, h_\gamma)$, we show that

$$F_*(\chi_\gamma)(1,0) = (1,0)$$

for all $\gamma \in \Gamma$ as follows. Consider

$$F \circ \chi_\gamma(\tau, V^\tau(w_n)) = F_*(\chi_\gamma) \circ F(\tau, V^\tau(w_n)). \tag{5.10}$$

From (5.9) we have

$$\lim_{n\to\infty} F_*(\chi_\gamma) \circ F(\tau, V^\tau(w_n)) = F_*(\chi_\gamma)(1,0). \tag{5.11}$$

Let $d_{\mathbf{B}_2}$ be the Kobayashi distance on $\mathbf{B}_2$. (For the Kobayashi distance refer to Jarnicki and Pflug [11], and Kobayashi [13].) The distance decreasing property for holomorphic maps with respect to Kobayashi distances guarantees that

$$\begin{aligned} &d_{\mathbf{B}_2}(F(\gamma(\tau), H_\gamma(\tau, V^\tau(w_n))), F(\gamma(\tau), V^{\gamma(\tau)}(w_n))) \\ &\qquad \le d_{D_{\gamma(\tau)}}(H_\gamma(\tau, V^\tau(w_n)), V^{\gamma(\tau)}(w_n)) \le K. \end{aligned} \tag{5.12}$$

From (5.9) and (5.12) we conclude that

$$\begin{aligned} \lim_{n\to\infty} F \circ \chi_\gamma(\tau, V^\tau(w_n)) &= \lim_{n\to\infty} F(\gamma(\tau), H_\gamma(\tau, V^\tau(w_n))) \\ &= (1,0). \end{aligned} \tag{5.13}$$

Therefore form (5.10), (5.11) and (5.13) we have

$$F_*(\chi_\gamma)(1,0) = (1,0)$$

for any $\gamma \in \Gamma$.

By the same way as Imayoshi [9], pp. 585–587 we can prove Theorem 1 and Corollary 1. This completes the proof of Theorem 1 and Corollary 1.

6. Proof of Theorem 2 for $n > 0$

6.1. We recall the following three lemmas:

Lemma 2. *Any analytic automorphism of* $\Delta^2 = (|z| < 1) \times (|w| < 1)$ *is either one of following two types*:
(I) $(A, B)(z, w) = (A(z), B(w))$,
(II) $(A, B)(z, w) = (A(w), B(z))$,
where $A, B \in \mathrm{Aut}(\Delta)$.

(See Narashimhan [17], Proposition 3, p. 68.)

Lemma 3. *Two Möbius transformations A and B are commutative if and only if* $\mathrm{Fix}(A) = \mathrm{Fix}(B)$, *i.e, they have the same set of fixed points provided that neither is the identity and provided that A or B is not a transformation of order two.*

(See Lehner [16], Theorems 1 and 2, p. 72.)

LEMMA 4. *Let A be a hyperbolic or loxodromic transformation and let B be a Möbius transformation which has one and only one fixed point in common with A. Then the sequence $\{B \circ A^n \circ B^{-1} \circ A^{-n}\}_{n=1}^{\infty}$ of Möbius transformations converges to a Möbius transformation as $n \to \infty$ or $-\infty$, so the group $\langle A, B \rangle$ generated by A and B is not discrete.*

(See Lehner [16], Theorem 2E, p. 94.)

6.2. Assume that (M, π, R) is a holomorphic family of Riemann surfaces of type (g, n) with $n > 0$ and there exists a biholomorphic map $F = (F_1, F_2) : \tilde{M} \to \Delta^2$.

First assume that for every $\Phi_*(\gamma) = [f_\gamma] \in \mathrm{Mod}(S)$, $\gamma \in \Gamma$, the quasiconformal self-map $f_\gamma : S \to S$ fixes every puncture of S, and that $F_*(\gamma, g)$ is of type (I) for all $(\gamma, g) \in \mathscr{G} = \Gamma \ltimes G$.

We use the notation in §3, §4 and §5. Let g_0 be a parabolic element of G with fixed point ζ_0^*. Set $\zeta_\tau = W^{\mu(\tau)}(\zeta_0^*) \in \partial D_\tau$, $\tau \in \Delta$. For any $\gamma \in \Gamma$ there exists an element $g_\gamma \in G$ satisfying (5.1). We put

$$h_\gamma = \omega_\gamma^{-1} \circ g_\gamma \circ \omega_\gamma,$$

$$H_\gamma(\tau, \zeta_\tau) = H_{(\gamma, h_\gamma)}(\tau, \zeta_\tau) = W^{\mu(\gamma(\tau))} \circ (\omega_\gamma \circ h_\gamma) \circ (W^{\mu(\tau)})^{-1}(\zeta_\tau).$$

Then we obtain

$$\zeta_{\gamma(\tau)} = H_\gamma(\tau, \zeta_\tau). \tag{6.1}$$

We put

$$(A_\gamma, B_\gamma) \circ F = F \circ (\gamma, h_\gamma), \tag{6.2}$$

where $(A_\gamma, B_\gamma) \in \mathrm{Aut}(\Delta^2)$.

Using the holomorphic motion V^τ in §5.4, we define a sequence $\{(\varphi_n, \psi_n)\}_{n=1}^{\infty}$ of holomorphic maps from Δ into Δ^2 by

$$(\varphi_n(\tau), \psi_n(\tau)) = (F_1(\tau, V^\tau(w_n)), F_2(\tau, V^\tau(w_n))). \tag{6.3}$$

We may assume that $\{(\varphi_n, \psi_n)\}_{n=1}^{\infty}$ converges uniformly to a holomorphic map $(\varphi_0, \psi_0) : \Delta \to \bar{\Delta}^2$ on compact subsets of Δ. Then the maximum theorem for holomorphic functions yields one of the following four cases:

(1) $(\varphi_0, \psi_0)(\Delta) \subset \Delta^2$.

(2) (φ_0, ψ_0) is constant on Δ with value $(c_1, c_2) \in (\partial\Delta)^2$.

(3) φ_0 is constant on Δ with value $c_1 \in \partial\Delta$, and $\psi_0(\Delta) \subset \Delta$.

(4) $\varphi_0(\Delta) \subset \Delta$, and ψ_0 is constant on Δ with value $c_2 \in \partial\Delta$.

Since F is a proper map, case (1) does not occur. We show that case (2) neither occurs as follows. Assume that (φ_0, ψ_0) is constant on Δ with value $(c_1, c_2) \in (\partial\Delta)^2$. From (5.8), (6.1), (6.2) and (6.3), for any $\gamma \in \Gamma$ we obtain

$$A_\gamma(c_1) = c_1, \quad \text{and} \quad B_\gamma(c_2) = c_2. \tag{6.4}$$

Take two elements $\gamma, \delta \in \Gamma$ with $\gamma\delta \neq \delta\gamma$. Then we have $(\gamma, h_\gamma) \circ (\delta, h_\delta) \neq (\delta, h_\delta) \circ (\gamma, h_\gamma)$, and so $(A_\gamma, B_\gamma) \circ (A_\delta, B_\delta) \neq (A_\delta, B_\delta) \circ (A_\gamma, B_\gamma)$. Hence we get

$$A_\gamma A_\delta \neq A_\delta A_\gamma, \quad \text{or} \quad B_\gamma B_\delta \neq B_\delta B_\gamma. \tag{6.5}$$

Therefore Lemmas 3, 4, (6.4) and (6.5) imply that $F_*(\mathscr{G})$ is not discrete. This is a contradiction.

In case (3) we see that ψ_0 is not constant as follows. Suppose that ψ_0 is constant with value $c_2 \in \Delta$. From (5.8), (6.1), (6.2) and (6.3), for any $\gamma \in \Gamma$ we have

$$A_\gamma(c_1) = c_1, \quad \text{and} \quad B_\gamma(c_2) = c_2. \tag{6.6}$$

Let c_1^* be the reflection of c_1 with respect to the unit circle $\partial\Delta$. Since $A_\gamma \in \mathrm{Aut}(\Delta)$, we see that

$$A_\gamma(c_1^*) = c_1^* \tag{6.7}$$

for any $\gamma \in \Gamma$. Hence Lemma 3, (6.6) and (6.7) imply that

$$A_\gamma A_\delta = A_\delta A_\gamma \tag{6.8}$$

for all $\gamma, \delta \in \Gamma$.

Take two elements $\gamma, \delta \in \Gamma$ with $\gamma\delta \neq \delta\gamma$. Then we have $(\gamma, h_\gamma) \circ (\delta, h_\delta) \neq (\delta, h_\delta) \circ (\gamma, h_\gamma)$, and so $(A_\gamma, B_\gamma) \circ (A_\delta, B_\delta) \neq (A_\delta, B_\delta) \circ (A_\gamma, B_\gamma)$. Noting (6.8) we get

$$B_\gamma B_\delta \neq B_\delta B_\gamma. \tag{6.9}$$

Therefore Lemmas 3, 4, (6.8) and (6.9) imply that $F_*(\mathscr{G})$ is not discrete. This is a contradiction.

Now assume that φ_0 is constant on Δ with value $c_1 \in \partial\Delta$ and $\psi_0 : \Delta \to \Delta$ is a non-constant holomorphic map. Let $F \circ (1, g_0) = (A_0, B_0) \circ F$. Then from (6.1), (6.2) and (6.3) we obtain

$$\varphi_0(\tau) = A_0 \circ \varphi_0(\tau) = A_0(c_1) = c_1, \quad \text{and} \quad \psi_0(\tau) = B_0 \circ \psi_0(\tau).$$

Since ψ_0 is not constant, we see that $A_0(c_1) = c_1$, and $B_0 = 1$, and so $F_*(1, g_0) = (A_0, 1)$, where A_0 is of infinite order and has a fixed point $c_1 \in \partial\Delta$. By a theorem due to Shimizu [23] (Theorem 2, p. 39), we see that

$$\mathscr{G}_1^* = \{A_g \in \mathrm{Aut}(\Delta) \mid (A_g, B_g) = F_*(1, g),\ g \in G\},$$
$$\mathscr{G}_2^* = \{B_g \in \mathrm{Aut}(\Delta) \mid (A_g, B_g) = F_*(1, g),\ g \in G\}$$

are discrete.

If $F_2|_{D_\tau} : D_\tau \to \Delta$ is not constant, then $F_2|_{D_\tau}$ induces a non-constant holomorphic map $[F_2]_\tau : D_\tau/G_\tau \to \Delta/\mathscr{G}_2^*$. Since the Riemann surface D_τ/G_τ is of analytically finite type, we see that $[F_2]_\tau$ has a holomorphic extension to the compactification of D_τ/G_τ. Hence we have $\psi_0(\tau) \in \partial\Delta$, and so by the maximum principle we conclude that ψ_0 is constant on Δ, which is a contradiction. Therefore, F_2 is constant on D_τ for all $\tau \in \Delta$, and $F_*(1, g)$ is of form $(A_g, 1)$ for any $g \in G$, i.e.,

$$\mathscr{G}_1^* = \{A_g \in \mathrm{Aut}(\Delta) \mid (A_g, 1) = F_*(1, g), g \in G\}.$$

Therefore $S_{\rho(\tau)} \cong D_\tau/G_\tau \cong \Delta/\mathscr{G}_1^*$ for every $\tau \in \Delta$. Similarly, in case (4) we can show that all fibers S_t are biholomorphically equivalent.

6.3. Next we prove that $F_*(\gamma, g)$ is of type (I) for all $(\gamma, g) \in \mathscr{G} = \Gamma \ltimes G$ provided that for every $\Phi_*(\gamma) = [f_\gamma] \in \mathrm{Mod}(S)$, $\gamma \in \Gamma$, the quasiconformal self-map $f_\gamma : S \to S$ fixes every puncture of S. Assume that $\mathscr{G}_0^* = \{(A, B) \mid (A, B) = F \circ (\gamma, g) \circ F^{-1}$ is of type (I), $(\gamma, g) \in \Gamma \ltimes G\}$ is a subgroup of $\mathscr{G}^*$ of index two. By the same argument as in §6.2 we see that one of the following two cases holds:

(1) F_1 is constant and F_2 is non-constant on D_τ for all $\tau \in \Delta$.

(2) F_2 is constant and F_1 is non-constant on D_τ for all $\tau \in \Delta$.

In case (1), if $(A, B) = (A, B) = F \circ (\gamma, g) \circ F^{-1}$ is of type (II) for some (γ, g), then we have $F_1(\gamma(\tau), H_{(\gamma, g)}(\tau, w)) = A \circ F_2(\tau, w)$. Since F_1 is constant and $A \circ F_2$ is non-constant on D_τ, we have a contradiction. Hence every $F \circ (\gamma, g) \circ F^{-1}$ is of type (I). Similarly it follows that in case (2), every $F \circ (\gamma, g) \circ F^{-1}$ is of type (I).

6.4. If Γ has an element γ such that $\Phi_*(\gamma) = [f_\gamma] \in \mathrm{Mod}(S)$ does not fix a puncture of S, then the same reasoning as one in §5.2 implies that all fibers S_t are biholomorphically equivalent.

7. Proof of Theorem 2 for a compact complex surface M

7.1. We shall give a proof of Theorem 2 in the case where M is compact, that is, the base surface R is compact and $n = 0$, i.e., every fiber $S_t = \pi^{-1}(t)$ is also compact.

Assume that there exists a biholomorphic map $F = (F_1, F_2) : \Delta^2 \to \tilde{M}$. We also assume that every element of $\mathscr{G}^* = F^{-1}\mathscr{G}F$ is of type (I).

We shall show that $F = (F_1, F_2)$ satisfies the following:

$$\frac{\partial F_1}{\partial z} = 0 \quad \text{on } \Delta^2, \quad \text{or} \quad \frac{\partial F_1}{\partial w} = 0 \quad \text{on } \Delta^2. \tag{7.1}$$

In order to obtain (7.1) we show that

$$\lim_{n\to\infty} \frac{\partial F_1}{\partial w}(z_n, w_n) \times \frac{\partial F_2}{\partial w}(z_n, w_n) = 0 \tag{7.2}$$

for any point $(\zeta_0, w_0) \in \partial\Delta \times \Delta$ and any sequence $\{(z_n, w_n)\}_{n=1}^\infty$ of points in Δ^2 with $\lim_{n\to\infty}(z_n, w_n) = (\zeta_0, w_0)$.

Suppose that (7.2) does not hold for some (ζ_0, w_0) and $\{(z_n, w_n)\}_{n=1}^\infty$. Then there exists a positive constant ε_0 and a subsequence $\{(z_{n_j}, w_{n_j})\}_{j=1}^\infty$ such that

$$\left|\frac{\partial F_1}{\partial w}(z_{n_j}, w_{n_j}) \times \frac{\partial F_2}{\partial w}(z_{n_j}, w_{n_j})\right| \geq \varepsilon_0 \tag{7.3}$$

for all j.

Since $\tilde{M}$ is a bounded domain, we may assume that the sequence $\{F(z_{n_j},\cdot)\}_{j=1}^{\infty}$ of holomorphic maps $F(z_{n_j},\cdot):\Delta=(|w|<1)\to\tilde{M}$ converges to a holomorphic map $\varphi=(\varphi_1,\varphi_2):\Delta\to\overline{\tilde{M}}$ uniformly on compact subsets of Δ, where $\overline{\tilde{M}}$ is the closure of $\tilde{M}$.

Let $\mathscr{F}$ be a fundamental set for $\mathscr{G}^*$. Note that $\mathscr{F}\Subset\Delta^2$, for M is compact. Then we can find a sequence $\{(a_j,b_j)\}_{j=1}^{\infty}$ of points in $\mathscr{F}$ and a sequence $\{A_j,B_j\}_{j=1}^{\infty}$ of elements in $\mathscr{G}^*$ such that

$$(A_j,B_j)(a_j,b_j)=(A_j(a_j),B_j(b_j))=(z_{n_j},w_{n_j}).$$

We may assume that (a_j,b_j) converges to a point $(a_0,b_0)\in\Delta^2$. We may also assume that (A_j,B_j) converges to (ζ_0,B_0) uniformly on compact subsets of Δ^2, where ζ_0 is the constant map with value ζ_0 and $B_0\in\mathrm{Aut}(\Delta)$. Because conditions $\lim_{j\to\infty}a_j=a_0\in\Delta$ and $\lim_{j\to\infty}A_j(a_j)=\zeta_0\in\partial\Delta$ imply $\lim_{j\to\infty}A_j=\zeta_0$, and conditions $\lim_{j\to\infty}b_j=b_0\in\Delta$ and $\lim_{j\to\infty}B_j(b_j)=w_0\in\Delta$ imply $\lim_{j\to\infty}B_j=B_0\in\mathrm{Aut}(\Delta)$.

We put $F_*(A_j,B_j)=F\circ(A_j,B_j)\circ F^{-1}=(\gamma_j,g_j)\in\mathscr{G}=\Gamma\ltimes G$. Then we have

$$F_1(A_j(a_j),B_j(b_j))=\gamma_j\circ F_1(a_j,b_j), \tag{7.4}$$

$$F_2(A_j(a_j),B_j(b_j))=H_j(F_1(a_j,b_j),F_2(a_j,b_j)), \tag{7.5}$$

where $H_j=H_{(\gamma_j,g_j)}$.

Since $F:\Delta^2\to\tilde{M}$ is biholomorphic, we see that $(\gamma_j,g_j)\circ F(a_j,b_j)=F(z_{n_j},w_{n_j})$ converges to a boundary point $\varphi(w_0)=(\varphi_1(w_0),\varphi_2(w_0))$ of $\tilde{M}$.

If $\varphi_1(w_0)\in\partial\Delta=(|\tau|<1)$, then we may assume that $\{\gamma_j\}_{j=1}^{\infty}$ converges to a constant map $\varphi_1(w_0)$ uniformly on compact subsets of Δ, because $\gamma_j\in\mathrm{Aut}(\Delta)$, $\lim_{j\to\infty}F_1(a_j,b_j)=F_1(a_0,b_0)\in\Delta$, and $\lim_{j\to\infty}\gamma_j\circ F_1(a_j,b_j)=\lim_{j\to\infty}F_1(z_{n_j},b_{n_j})=\varphi_1(w_0)\in\partial\Delta$. Hence from

$$\begin{aligned}\frac{d\gamma_j\circ F_1(a_j,w)}{dw}&=\frac{dF_1(A_j(a_j),B_j(w))}{dw}\\&=\frac{dF_1(z_{n_j},B_j(w))}{dw}\\&=\frac{\partial F_1}{\partial w}(z_{n_j},B_j(w))\times B_j'(w)\end{aligned}$$

we obtain

$$\lim_{j\to\infty}\frac{\partial F_1}{\partial w}(z_{n_j},B_j(w))\times B_0'(w)=0.$$

Since $B_j(b_j)=w_{n_j}$ and $B_0'(w)\neq 0$, we conclude that

$$\lim_{j\to\infty}\frac{\partial F_1}{\partial w}(z_{n_j},w_{n_j})=0.$$

Since $\lim_{j\to\infty} \partial F_2/\partial w(z_{n_j}, w_{n_j})$ exists, we have

$$\lim_{j\to\infty} \frac{\partial F_1}{\partial w}(z_{n_j}, w_{n_j}) \times \frac{\partial F_2}{\partial w}(z_{n_j}, w_{n_j}) = 0,$$

which is a contradiction to (7.3).

If $\varphi_1(w_0) \in \Delta = (|\tau| < 1)$, then we may assume that there exists an element $\gamma_0 \in \Gamma$ such that $\gamma_j = \gamma_0$ for any j. In fact, assuming γ_j converges to a holomorphic map $\gamma_0 : \Delta \to \bar{\Delta}$ uniformly on compact subsets of Δ, the assumptions $\lim_{j\to\infty} F_1(a_j, b_j) = F_1(a_0, b_0) \in \Delta$ and $\lim_{j\to\infty} \gamma_j \circ F_1(a_j, b_j) \to \varphi(w_0) \in \Delta$ imply that $\gamma_0 \in \mathrm{Aut}(\Delta)$, and the discreteness of Γ implies that $\gamma_j = \gamma_k$ for all sufficiently large j and k. Let $\tau_j = F_1(a_j, b_j)$ and $\tau_0 = F_1(a_0, b_0)$. Then $H_j(\tau_j, \cdot) : D_{\tau_j} \to D_{\gamma_0(\tau_j)}$ is conformal and we may assume that $\{H_j(\tau_j, \cdot)\}_{j=1}^{\infty}$ converges to a holomorphic map $H_0 : D_{\tau_0} \to \overline{D_{\gamma_0(\tau_0)}}$ uniformly on compact subsets of D_{τ_0}. Since $H_0(F(a_0, b_0)) = \varphi_2(w_0) \in \partial D_{\gamma_0(\tau_0)}$, we see that H_0 is constant on D_{τ_0}. Hence from

$$\begin{aligned}\frac{dH_j \circ F(a_j, w)}{dw} &= \frac{dF_2(A_j(a_j), B_j(w))}{dw} \\ &= \frac{dF_2(z_{n_j}, B_j(w))}{dw} \\ &= \frac{\partial F_2}{\partial w}(z_{n_j}, B_j(w)) \times B_j'(w)\end{aligned}$$

we obtain

$$\lim_{j\to\infty} \frac{\partial F_2}{\partial w}(z_{n_j}, B_j(w)) \times B_0'(w) = 0.$$

Since $B_j(b_j) = w_{n_j}$ and $B_0'(w) \neq 0$, we conclude that

$$\lim_{j\to\infty} \frac{\partial F_2}{\partial w}(z_{n_j}, w_{n_j}) = 0.$$

Since $\lim_{j\to\infty} \partial F_1/\partial w(z_{n_j}, w_{n_j})$ exists, we have

$$\lim_{j\to\infty} \frac{\partial F_1}{\partial w}(z_{n_j}, w_{n_j}) \times \frac{\partial F_2}{\partial w}(z_{n_j}, w_{n_j}) = 0,$$

which is a contradiction to (7.3).

Therefore we have (7.2) for any point $(\zeta_0, w_0) \in \partial\Delta \times \Delta$ and any sequence $\{(z_n, w_n)\}_{n=1}^{\infty}$ of points in Δ^2 with $\lim_{n\to\infty}(z_n, w_n) = (\zeta_0, w_0)$. Then Radó's theorem implies

$$\frac{\partial F_1}{\partial w} \times \frac{\partial F_2}{\partial w} = 0 \quad \text{on } \Delta^2. \tag{7.6}$$

(See Narashimhan [17], Theorem 1, p. 53). Hence we have

$$\frac{\partial F_1}{\partial w} = 0 \quad \text{on } \Delta^2, \quad \text{or} \quad \frac{\partial F_2}{\partial w} = 0 \quad \text{on } \Delta^2. \tag{7.7}$$

By a similar way as above we obtain

$$\frac{\partial F_1}{\partial z} = 0 \quad \text{on } \Delta^2, \quad \text{or} \quad \frac{\partial F_2}{\partial z} = 0 \quad \text{on } \Delta^2. \tag{7.8}$$

Since

$$\det \begin{pmatrix} \dfrac{\partial F_1}{\partial z} & \dfrac{\partial F_1}{\partial w} \\ \dfrac{\partial F_2}{\partial z} & \dfrac{\partial F_2}{\partial w} \end{pmatrix}$$

does not vanish at every point of Δ^2, from (7.7) and (7.8) we see that one of the following two relations holds:

(i) $\partial F_1/\partial z = \partial F_2/\partial w = 0$ on Δ^2,

(ii) $\partial F_1/\partial w = \partial F_2/\partial z = 0$ on Δ^2.

If relation (i) holds, then $F_1(z,w) = F_1(w)$, i.e., F_1 is independent on z. Then $F^{-1} \circ (1,g) \circ F$ is of form $(A_g, 1)$ and of type (I) for every $g \in G$. Thus setting $\mathscr{A}_G^* = \{A_g \mid (A_g, 1) = F^{-1} \circ (1,g) \circ F, g \in G\}$, we see that

$$S_{\rho(\tau)} \cong D_\tau/G_\tau \cong \Delta/\mathscr{A}_G^*$$

for any $\tau \in \Delta$, which concludes that all the fibers S_t are biholomorphically equivalent.

If relation (ii) holds, then $F_1(z,w) = F_1(z)$, and $F^{-1} \circ (1,g) \circ F$ is of form $(1, B_g)$ and of type (I) for every $g \in G$. Thus we have

$$S_{\rho(\tau)} \cong D_\tau/G_\tau \cong \Delta/\mathscr{B}_G^*$$

for any $\tau \in \Delta$, where $\mathscr{B}_G^* = \{B_g \mid (1, B_g) = F^{-1} \circ (1,g) \circ F, g \in G\}$. Hence all the fibers S_t are biholomorphically equivalent.

7.2. Let M be compact, and assume that there exists a biholomorphic map $F = (F_1, F_2) : \Delta^2 \to \tilde{M}$, and $\mathscr{G}^* = F^{-1}\mathscr{G}F$ has an element of type (II). Let $\mathscr{G}_0^*$ be the set all elements of type (I) in $\mathscr{G}^*$, which is a normal subgroup of $\mathscr{G}^*$ of index two. Using $\mathscr{G}_0^*$ in place of $\mathscr{G}^*$, the same way as in §7.1 we see that $F_1(z,w) = F_1(w)$ or $F_1(z,w) = F_1(z)$. If $F_1(z,w) = F_1(w)$, then $F^{-1} \circ (1,g) \circ F$ is of form $(A_g, 1)$ and of type (I) for every $g \in G$. Hence by the same reasoning as above we obtain

$$S_{\rho(\tau)} \cong D_\tau/G_\tau \cong \Delta/\mathscr{A}_G^*$$

for any $\tau \in \Delta$, which concludes that all the fibers S_t are biholomorphically equivalent.

Similarly if $F_1(z,w) = F_1(z)$, then we see that

$$S_{\rho(\tau)} \cong D_\tau/G_\tau \cong \Delta/\mathscr{B}_G^*.$$

Hence all the fibers S_t are biholomorphically equivalent.

This completes the proof of Theorem 2 in the case where M is compact.

7.3. In the case where the base surface R is not compact, a proof of Theorem 2 is given Imayoshi [9], pp. 587–596.

If all the fibers $S_t = \pi^{-1}(t)$ are biholomorphically equivalent, then the representation Ψ of (M, π, R) into $T(G)$ is constant, and so $\tilde{M} = \Delta \times D_0 \cong (\Delta)^2$. This completes the proof of Theorem 2.

Finally we note that a proof of Corollary 2 is given in Imayoshi [9], p. 587.

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