U. DURSUN
KODAI MATH. J.
28 (2005), 191–198

NULL 2-TYPE SUBMANIFOLDS OF THE EUCLIDEAN SPACE E^5 WITH PARALLEL NORMALIZED MEAN CURVATURE VECTOR

Uğur Dursun

Abstract

We classify 3-dimensional null 2-type submanifolds of the Euclidean space E^5 with parallel normalized mean curvature vector under certain hypothesis.

1. Introduction

The theory of finite type was introduced by B. Y. Chen in 1983 ([2]) and since then it has become a useful tool in the study of submanifolds. That concept is the natural extension of minimal submanifolds, to which many mathematicians have devoted themselves in the last decades.

The problem of the classification of null 2-type hypersurfaces or, in general, submanifolds are quite interesting in the theory of finite type. In [3], B. Y. Chen has given a classification of null 2-type surfaces in the Euclidean space E^3 and proved that they are circular cylinders. Later, in [4], he proved that a surface M in the Euclidean space E^4 is of null 2-type with parallel normalized mean curvature vector if and only if M is an open portion of a circular cylinder in a hyperplane of E^4, and that the helical cylinders are the only surfaces of null 2-type and constant mean curvature of the Euclidean space E^4.

Also in [5], S. J. LI showed that a surface M in E^m with parallel normalized mean curvature vector is of null 2-type if and only if M is an open portion of a circular cylinder.

In [1], A. Ferrandez and P. Lucas have shown that Euclidean hypersurfaces of null 2-type and having at most two distinct principal curvatures are locally isometric to a generalized cylinder.

In this paper we investigate the classification of 3-dimensional null 2-type submanifolds of the Euclidean space E^5 with parallel normalized mean curvature vector. We prove that a 3-dimensional submanifold M of the Euclidean space E^5 having two distinct principal curvatures in the parallel mean curvature direction and having a second fundamental form of a constant square length is of null

MSC: 53C40.
Key words: Finite type submanifold; Null 2-type submanifold; Mean curvature vector.
Received August 5, 2004; revised October 22, 2004.

2-type if and only if M is locally isometric to one of $E \times S^2 \subset E^4 \subset E^5$, $E^2 \times S^1 \subset E^4 \subset E^5$ or $E \times S^1(a) \times S^1(a)$.

2. Preliminaries

Let M be an n-dimensional submanifold in an $(n+2)$-dimensional Euclidean space E^{n+2}. We denote by h, A, H, ∇ and $\nabla^{\perp}$, the second fundamental form, the Weingarten map, the mean curvature vector, the Riemannian connection and the normal connection of the submanifold M in E^{n+2}, respectively. We choose an orthonormal local frame $\{e_1, \dots, e_{n+2}\}$ on M such that $e_1, \dots, e_n$ are tangent to M and e_{n+1} is the direction of H, i.e., the normalized mean curvature vector. Denote by $\{\omega^1, \dots, \omega^{n+2}\}$ the dual frame and $\{\omega_B^A\}$, $A, B = 1, \dots, n+2$, the connection forms associated to $\{e_1, \dots, e_{n+2}\}$. We use the following convention on the range of indices: $1 \leq A, B, C, \dots \leq n+2$, $1 \leq i, j, k, \dots \leq n$, $n+1 \leq \beta, \nu, \gamma, \dots \leq n+2$. Denoting by D the Riemannian connection of E^{n+2}, we put $D_{e_k} e_i = \sum \omega_i^j(e_k) e_j + \sum h^{\beta}(e_i, e_k) e_{\beta}$ and $D_{e_k} e_{\nu} = \sum \omega_{\nu}^j(e_k) e_j + \sum \omega_{\nu}^{\beta}(e_k) e_{\beta}$. By Cartan's Lemma, we have

$$\omega_i^{\beta} = \sum_{j=1}^{n} h_{ij}^{\beta} \omega^j, \quad h_{ij}^{\beta} = h_{ji}^{\beta},$$

where h_{ij}^{β} are the coefficients of the second fundamental form in the direction e_{β}. The mean curvature vector H is given by

$$H = \frac{1}{n} \sum_{\beta=n+1}^{n+2} \operatorname{tr}(h^{\beta}) e_{\beta} \tag{1}$$

and the square length of the second fundamental form is defined by

$$\sigma = \sum_{\beta} \operatorname{tr}(h^{\beta})^2 = \sum_{i,j,\beta} (h_{ij}^{\beta})^2. \tag{2}$$

Using the connection equations:

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_j^k(e_i) e_k$$

we can obtain the Gauss and Codazzi equations for $n = 3$, respectively, as

$$\begin{aligned} & e_{\ell}(\omega_i^j(e_k)) - e_k(\omega_i^j(e_{\ell})) \\ & \quad = \sum_{r=1}^{3} \{\omega_i^r(e_{\ell}) \omega_r^j(e_k) - \omega_i^r(e_k) \omega_r^j(e_{\ell}) + \omega_i^j(e_r)[\omega_k^r(e_{\ell}) - \omega_{\ell}^r(e_k)]\} \\ & \qquad + \sum_{\nu=4}^{5} (h_{ik}^{\nu} h_{j\ell}^{\nu} - h_{jk}^{\nu} h_{i\ell}^{\nu}), \quad 1 \leq i < j \leq 3, \ 1 \leq \ell < k \leq 3, \end{aligned} \tag{3}$$

and

$$
\begin{aligned}
& e_j(h^{\nu}_{ik}) - e_k(h^{\nu}_{ij}) \\
& \qquad = \sum_{r=1}^{3} \{h^{\nu}_{ir}[\omega^r_k(e_j) - \omega^r_j(e_k)] + h^{\nu}_{rk}\omega^r_i(e_j) - h^{\nu}_{rj}\omega^r_i(e_k)\} \\
& \qquad\quad + \sum_{\beta=4}^{5} (h^{\beta}_{ij}\omega^{\nu}_{\beta}(e_k) - h^{\beta}_{ik}\omega^{\nu}_{\beta}(e_j)), \quad \nu = 4,5;\ i = 1,2,3,\ 1 \le j < k \le 3.
\end{aligned} \tag{4}
$$

If M is a null 2-type submanifold of E^{n+2}, then we have the following decomposition of the position vector x of M in E^{n+2}:

$$x = x_0 + x_1, \quad \Delta x_0 = 0, \quad \Delta x_1 = c x_1, \tag{5}$$

for some non-constant vectors x_0 and x_1 on M, where c is a non-zero constant. Since we have $\Delta x = -nH$, then (5) implies

$$\Delta H = cH. \tag{6}$$

3. Null 2-type submanifolds

To achieve our goal we need the following lemmas.

Lemma 3.1 [4]. *Let M be a n-dimensional submanifold of a Euclidean space E^m. If there is a constant $c \neq 0$ such that $\Delta H = cH$, then M is either of 1-type or of null 2-type.*

Let $U = \{u \in M \mid H \neq 0 \text{ at } u\}$. We choose an orthonormal local frame field e_{n+1}, e_{n+2} normal to $U \subset M$ in E^5 so that e_{n+1} is parallel to H. Then the allied mean curvature vector $\mathscr{A}(H)$ is defined by

$$\mathscr{A}(H) = \operatorname{tr}(A_H A_{n+2}) e_{n+2} = \|H\| \operatorname{tr}(A_{n+1}A_{n+2}) e_{n+2}, \tag{7}$$

where $A_\nu = A_{e_\nu}$. If $H = 0$ at the point u in M, then $\mathscr{A}(H)$ is defined to be zero. In [2, p. 271] we have the formula

$$\Delta H = \Delta^{\nabla^\perp} H + \|A_{n+1}\|^2 H + \mathscr{A}(H) + \operatorname{tr}(\overline{\nabla} A_H), \tag{8}$$

where $\Delta^{\nabla^\perp} H$ is the Laplacian of H with respect to the normal connection $\nabla^\perp$, $\|A_{n+1}\|^2 = \operatorname{tr}(A_{n+1}A_{n+1})$ and

$$\overline{\nabla} A_H = \nabla A_H + A_{\nabla^\perp H}. \tag{9}$$

Lemma 3.2 [2, 3]. *Let M be a n-dimensional submanifold of a Euclidean space E^m. Then we have*

$$\operatorname{tr}(\overline{\nabla} A_H) = \frac{n}{2} \nabla \alpha^2 + 2 \operatorname{tr}(A_{\nabla^\perp H}), \tag{10}$$

where $\alpha^2 = \langle H, H \rangle$ and $\nabla \alpha^2$ is the gradient of α^2.

Lemma 3.3 [4]. *Let M be a n-dimensional submanifold of a Euclidean space E^m such that M is not of* 1*-type. Then M is of null* 2*-type if and only if we have*

$$\operatorname{tr}(\overline{\nabla} A_H) = 0 \tag{11}$$

and

$$\Delta^{\nabla^\perp} H + \|A_{n+1}\|^2 H + \mathscr{A}(H) = cH \tag{12}$$

for some constant c.

Lemma 3.4 [4]. *Let M be a n-dimensional submanifold of a Euclidean space E^m. Then we have*

$$\begin{aligned} \Delta^{\nabla^\perp} H &= \{\Delta\alpha + \alpha\langle\nabla^\perp e_{n+1}, \nabla^\perp e_{n+1}\rangle\} e_{n+1} \\ &\quad + \sum_{r=n+2}^{m} \{\alpha\langle\nabla^\perp e_{n+1}, \nabla^\perp e_r\rangle - 2\omega_{n+1}^r(\nabla\alpha) - \alpha \operatorname{tr}(\nabla\omega_{n+1}^r)\} e_r \end{aligned} \tag{13}$$

Let M be a 3-dimensional submanifold of a Euclidean space E^5, $(n = 3, m = 5)$, then we have

Lemma 3.5. *Let M be a* 3*-dimensional submanifold of the Euclidean space E^5 such that M is not of* 1*-type. Then M is of null* 2*-type if and only if we have*

$$2A_4(\nabla\alpha) = -3\alpha\nabla\alpha - 2\alpha\sum_{i=1}^{3}\omega_4^5(e_i)A_5(e_i), \tag{14}$$

$$\Delta\alpha + \alpha\|\omega_4^5\|^2 + \alpha\|A_4\|^2 = c\alpha, \tag{15}$$

$$\alpha \operatorname{tr}(A_4A_5) = 2\omega_4^5(\nabla\alpha) + \alpha \operatorname{tr}(\nabla\omega_4^5). \tag{16}$$

Considering (7) the proof of the lemma follows from Lemma 3.2, Lemma 3.3, and Lemma 3.4.

Proposition 3.1. *Let M be a* 3*-dimensional submanifold of the Euclidean space E^5 with parallel normalized mean curvature vector such that M is not of* 1*-type. If M is of null* 2*-type with the Weingarten map in the direction of the mean curvature vector H has two distinct eigenvalues, then the mean curvature α is constant on M.*

Proof. As the codimension is 2 and the normalized mean curvature vector, $e_4 = H/\alpha$, is parallel, then the other unit normal vector e_5 in the basis is also parallel. Therefore the normal space is flat. Hence we can have the diagonalized Weingarten maps in the direction e_4 and e_5, and $\omega_4^5 \equiv 0$ on M. Since A_4 has two distinct eigenvalues, say, $\lambda = h_{11}^4 \neq h_{22}^4 = h_{33}^4 = \mu$ and $h_{11}^5 = \nu$, $h_{22}^5 = \rho$, $h_{33}^5 = \tau$, we can write

$$A_4 = \operatorname{diag}(\lambda, \mu, \mu) \quad \text{and} \quad A_5 = \operatorname{diag}(\nu, \rho, \tau) \quad \text{with} \ \nu + \rho + \tau = 0.$$

However, from (16) we get

$$\operatorname{tr}(A_4A_5) = \lambda\nu + \mu(\tau+\rho) = (\mu-\lambda)(\tau+\rho) = 0. \tag{17}$$

As $\mu - \lambda \neq 0$ we have $\tau + \rho = 0$, that is, $\nu = 0$ and $\tau = -\rho$.

Assume that α is not constant. Let $V = \{p \in M : \nabla\alpha \neq 0 \text{ at } p\}$ which is open in M. From (14) it is seen that the vector $\nabla\alpha$ is an eigenvector of A_4 corresponding to the eigenvalue $-\frac{3}{2}\alpha$. Then we may say that $\nabla\alpha$ is parallel to e_1 or e_3 (the same as e_2). For the last case it could also be proved that the mean curvature α is constant by using the same way as in the first case. So $\lambda = -\frac{3}{2}\alpha$ and $\mu = \frac{9}{4}\alpha$ because of $3\alpha = \lambda + 2\mu$. Then we have

$$\omega_1^4 = -\frac{3}{2}\alpha\omega^1, \quad \omega_2^4 = \frac{9}{4}\alpha\omega^2, \quad \omega_3^4 = \frac{9}{4}\alpha\omega^3. \tag{18}$$

Also, by (15), we obtain

$$\Delta\alpha + \frac{99}{8}\alpha^3 = c\alpha. \tag{19}$$

Since $\nabla\alpha$ is parallel to e_1 we can have $e_2(\alpha) = e_3(\alpha) = 0$, that is, $e_2(\lambda) = e_3(\lambda) = e_2(\mu) = e_3(\mu) = 0$ and

$$d\alpha = e_1(\alpha)\omega^1. \tag{20}$$

By using the Codazzi equations for the normal direction e_4 we have the followings: $\omega_2^1(e_1) = 0$ for $i = 1,\ j = 1,\ k = 2$; $\omega_3^1(e_1) = 0$ for $i = 1,\ j = 1,\ k = 3$; $\omega_2^1(e_3) = 0$ for $i = 2,\ j = 1,\ k = 3$; $\omega_3^1(e_2) = 0$ for $i = 1,\ j = 2,\ k = 3$;

$$e_1(\alpha) = \frac{5}{3}\alpha\omega_2^1(e_2), \quad \text{for } i = 2,\ j = 1,\ k = 2; \tag{21}$$

and

$$e_1(\alpha) = \frac{5}{3}\alpha\omega_3^1(e_3), \quad \text{for } i = 3,\ j = 1,\ k = 3. \tag{22}$$

Applying the structure equations, it can be shown that $d\omega^1 = 0$. Hence we have locally

$$\omega^1 = du, \tag{23}$$

where u is a local coordinate on U.

From (20) and (23) we have $d\alpha \wedge du = 0$. This shows that α is a function of u, i.e., $\alpha = \alpha(u)$ and $d\alpha = \alpha'(u)\,du$. Thus, by (21) and (22) we have

$$\omega_2^1(e_2) = \omega_3^1(e_3) = \frac{3\alpha'}{5\alpha}. \tag{24}$$

Considering $\omega_2^1(e_1) = \omega_3^1(e_1) = \omega_2^1(e_3) = \omega_3^1(e_2) = 0$, from the Gauss equations for $i = 1,\ j = 2,\ \ell = 1,\ k = 2$ we get

$$e_1(\omega_2^1(e_2)) = (\omega_2^1(e_2))^2 + \lambda\mu. \tag{25}$$

Using (24) and the last statement we obtain

$$120\alpha\alpha'' - 192(\alpha')^2 + 675\alpha^4 = 0. \tag{26}$$

Let $y = (\alpha')^2$. Then it is easy to see that the equation (26) can be reduced to the following first order differential equation:

$$60\alpha y' - 192y + 675\alpha^4 = 0, \tag{27}$$

where y' denotes the first derivative of y with respect to α. For this equation we obtain the solution

$$(\alpha')^2 = \tilde{C}\alpha^{16/5} - \frac{225}{16}\alpha^4, \tag{28}$$

where $\tilde{C}$ is a constant.

On the other hand, by a straight forward calculation we obtain

$$\Delta\alpha = \frac{6(\alpha')^2}{5\alpha} - \alpha''. \tag{29}$$

The equations (19) and (29) show that

$$40\alpha\alpha'' - 48(\alpha')^2 - 495\alpha^4 + 40c\alpha^2 = 0. \tag{30}$$

By (26) and (30) we obtain

$$(\alpha')^2 = 45\alpha^4 - \frac{5}{2}c\alpha^2. \tag{31}$$

As a result, comparing (28) and (31) we deduce that α is locally constant on V, and (26) implies $\alpha = 0$ which contradicts to our assumption that M is not of 1-type. Therefore, V is empty and M has constant mean curvature α. □

For later use we need the connection forms ω_A^B of $E \times S^1(a) \times S^1(a) \subset E^5$. By a suitable choice of the Euclidean coordinates, its equation takes the following form

$$x(u_1, u_2, u_3) = (u_1, a\cos u_2, a\sin u_2, a\cos u_3, a\sin u_3),$$

where a is a nonzero constant. If we put

$$e_1 = \frac{\partial}{\partial u_1} = (1, 0, 0, 0, 0), \quad e_2 = \frac{1}{a}\frac{\partial}{\partial u_2} = (0, -\sin u_2, \cos u_2, 0, 0),$$

$$e_3 = \frac{1}{a}\frac{\partial}{\partial u_3} = (0, 0, 0, -\sin u_3, \cos u_3),$$

$$e_4 = \frac{1}{\sqrt{2}}(0, \cos u_2, \sin u_2, \cos u_3, \sin u_3),$$

$$e_5 = \frac{1}{\sqrt{2}}(0, \cos u_2, \sin u_2, -\cos u_3, -\sin u_3),$$

then, by a straight forward calculation we obtain

$$\omega^1 = du_1, \quad \omega^2 = a\, du_2, \quad \omega^3 = a\, du_3, \quad \omega_2^1 = \omega_3^1 = \omega_3^2 = \omega_1^4 = \omega_1^5 = \omega_5^4 = 0,$$

$$(32) \quad \omega_2^4 = -\frac{1}{a\sqrt{2}}\omega^2, \quad \omega_3^4 = -\frac{1}{a\sqrt{2}}\omega^3, \quad \omega_2^5 = \frac{1}{a\sqrt{2}}\omega^2, \quad \omega_3^5 = -\frac{1}{a\sqrt{2}}\omega^3.$$

THEOREM 3.1. *Let M be a 3-dimensional submanifold of the Euclidean space E^5 with parallel normalized mean curvature vector such that M is not of 1-type. Then M is of null 2-type having two distinct principal curvatures in the mean curvature direction and having a second fundamental form σ of a constant square length if and only if M is locally isometric to one of $E \times S^2 \subset E^4 \subset E^5$, $E^2 \times S^1 \subset E^4 \subset E^5$ or $E \times S^1(a) \times S^1(a)$.*

Proof. Let M be of null 2-type and let the Weingarten map in the direction H has two distinct principal curvatures. Then the mean curvature α on M is constant by Proposition 3.1. However, as in the proof of Proposition 3.1 we can have

$$A_4 = \operatorname{diag}(\lambda, \mu, \mu) \quad \text{and} \quad A_5 = \operatorname{diag}(0, \rho, -\rho).$$

By using (15) we have $\|A_4\|^2 = \lambda^2 + 2\mu^2 = c$ which is constant. Hence, remembering that α is constant, it is easily seen that the eigenvalues λ and μ of A_4 are constant. Since the square length of the second fundamental form is constant, then, by using (2) we obtain $\rho = \text{const.}$

As $h_{11}^4 = \lambda$, $h_{22}^4 = h_{33}^4 = \mu$ and $h_{11}^5 = 0$, $h_{22}^5 = -h_{33}^5 = \rho$, from the Codazzi equations (4), for $v = 4$ we obtain $\omega_j^1(e_i)(\lambda - \mu) = 0$, $i = 1, 2, 3$, $j = 2, 3$ which imply that

$$(33) \qquad \omega_2^1(e_1) = \omega_3^1(e_1) = \omega_2^1(e_2) = \omega_2^1(e_3) = \omega_3^1(e_2) = \omega_3^1(e_3) = 0.$$

Therefore, from the Codazzi equations (4), for $v = 5$ we get

$$(34) \qquad \rho\omega_3^2(e_1) = \rho\omega_3^2(e_2) = \rho\omega_3^2(e_3) = 0,$$

for $i = 2$, $j = 1$, $k = 3$; $i = 2$, $j = 2$, $k = 3$ and $i = 3$, $j = 2$, $k = 3$, respectively. However, by using the Gauss equations (3), for $i = 1$, $j = 2$, $l = 1$, $k = 2$ and for $i = 2$, $j = 3$, $l = 2$, $k = 3$, we obtain, respectively,

$$(35) \qquad \lambda\mu = 0,$$

$$(36) \qquad e_2(\omega_3^2(e_3)) - e_3(\omega_3^2(e_2)) = (\omega_3^2(e_2))^2 + (\omega_3^2(e_3))^2 + \mu^2 - \rho^2.$$

Since A_4 has two distinct eigenvalues, one of λ and μ is different from zero. Therefore we have the followings:

CASE 1. $\lambda \neq 0$ and $\mu = 0$. Then, by (34) we get $\rho = 0$ or $\omega_3^2(e_2) = \omega_3^2(e_1) = \omega_3^2(e_3) = 0$. Using the second part, (36) implies that $\rho = 0$. Therefore A_5 vanishes. Since the normal space is flat and $A_5 \equiv 0$, then M is contained in a hyperplane of E^5.

A classical result of B. Segre [6] states that the isoparametric hypersurfaces in E^{n+1} are E^n, S^{n+1} and $S^p \times E^{n-p}$, where S^p is the p-sphere of radius r in the Euclidean space E^{p+1} perpendicular to E^{n-p}. From this results, as A_4 has constant eigenvalues M is locally isometric to $S^1 \times E^2 \subset E^4 \subset E^5$.

CASE 2. $\mu \neq 0$, $\lambda = 0$ and $\rho = const. \neq 0$. From (34) we have

$$\omega_3^2(e_2) = \omega_3^2(e_1) = \omega_3^2(e_3) = 0. \tag{37}$$

Considering (33) and (37) it is seen that M is flat. However, from (36) we get $\rho = \mp\mu$. Also, we can write

$$\omega_1^4 = 0, \quad \omega_2^4 = \mu\omega^2, \quad \omega_3^4 = \mu\omega^3, \quad \omega_1^5 = 0, \quad \omega_2^5 = \pm\mu\omega^2, \quad \omega_3^5 = \mp\mu\omega^3$$

Considering that M has a flat normal connection it is seen that the connection forms ω_B^A coincide with the connection forms of $E \times S^1(a) \times S^1(a)$ given in (32). Therefore, as a result of the fundamental theorem of submanifolds, M is in fact isometric to $E \times S^1(a) \times S^1(a) \subset E^5$.

CASE 3. $\mu \neq 0$, $\lambda = 0$ and $\rho = 0$. Then $A_5 = 0$. That is, M is contained in a hyperplane of E^4. M is therefore isometric to $E \times S^2(a) \subset E^4$ because of [6].

The converses of all these cases are trivial. □

I would like to express my hearty thanks to Prof. Dr. Abdülkadir Özdeğer for his valuable suggestions about this work.

REFERENCES

[1] A. FERRANDEZ AND P. LUCAS, Null Finite Type Hypersurfaces in Space Forms, Kodai Math. J., **14** (1991), 406–419.

[2] B.-Y. CHEN, Total Mean Curvature and Submanifolds of Finite Type, World Scientific, Singapor-New Jersey-London, 1984.

[3] B.-Y. CHEN, Null 2-type Surfaces E^3 are circular cylinder, Kodai Math J., **11** (1988), 295–299.

[4] B.-Y. CHEN, Null 2-type surfaces in Euclean spaces, in Algebra, Analysis and Geometry, (Taipei, 1988) 1–18, World Sci. Publishing, Teaneck, NJ, 1989.

[5] S. J. LI, Null 2-type surfaces in E^m with parallel normalized mean curvature vector, Math. J. Toyama Univ., **17** (1994), 23–30.

[6] B. SEGRE, Famiglie di ipersuperficie isoparametrische negli spazi euclidei ad un qualunque numero di dimensioni, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **27** (1938), 203–207.

ISTANBUL TECHNICAL UNIVERSITY
FACULTY OF SCIENCE AND LETTERS
DEPARTMENT MATHEMATICS
34469 MASLAK, ISTANBUL, TURKEY
E-mail: udursun@itu.edu.tr