J. O. BUTTON
KODAI MATH. J.
28 (2005), 181–190

INVARIANT TRACE FIELDS OF ONCE-PUNCTURED TORUS BUNDLES

J. O. Button

Abstract

We show that there exist hyperbolic once-punctured torus bundles with 2-generator fundamental groups which have invariant trace fields of arbitrarily high degree. We also answer a question of Bowditch on stabilisers of Markoff triples.

1. Introduction

Given a Kleinian group Γ, that is a discrete subgroup of $PSL(2, \mathbf{C})$, we have the trace field $\mathbf{Q}(\operatorname{tr}\Gamma) \subseteq \mathbf{C}$ which is the field generated by the traces of elements in Γ (or rather by their representatives in $SL(2, \mathbf{C})$). Throughout this paper we are only interested in the case when Γ is finitely generated. Then the trace field is also finitely generated over $\mathbf{Q}$ by [6]; however in general $\mathbf{Q}(\operatorname{tr}\Gamma)$ will contain transcendental elements. We also have the invariant trace field $\mathbf{Q}(\operatorname{tr}\Gamma^{(2)})$ of Γ, where $\Gamma^{(2)} = \langle \gamma^2 \mid \gamma \in \Gamma \rangle$ and has finite index in Γ so that $\mathbf{Q}(\operatorname{tr}\Gamma^{(2)})$ is a subfield of $\mathbf{Q}(\operatorname{tr}\Gamma)$ that is also finitely generated. The invariant trace field has the advantage that it is an invariant of the commensurability class of Γ if Γ is non-elementary (see [9] Theorem 3.3.4).

Let M be a complete hyperbolic 3-manifold so that we can regard it as $\mathbf{H}^3/\Gamma$ where Γ is a torsion free Kleinian group. Assuming that Γ is non-elementary, we know that M has finite volume if and only if it is closed or is homeomorphic to the interior of a compact 3-manifold with all boundary components being tori (we call these the cusps). In this case $\mathbf{Q}(\operatorname{tr}\Gamma)$, and hence the invariant trace field $\mathbf{Q}(\operatorname{tr}\Gamma^{(2)})$, is a number field (see [9] Theorem 3.1.2), namely a finite extension over $\mathbf{Q}$ which has the form $\mathbf{Q}(\theta)$ for θ an algebraic integer which satisfies a unique irreducible monic polynomial in $\mathbf{Z}[t]$ called the minimal polynomial. Thus a question that immediately arises here is what number fields can arise as the trace field or invariant trace field of a finite volume hyperbolic 3-manifold. This is Problem 3.61 in [8] and also it is asked in [9] Section 5.6 where it is pointed out that this is a wide open question. As any number field $\mathbf{Q}(\theta)$ that occurs comes equipped with a pair of complex conjugate embeddings into $\mathbf{C}$ by considering Γ (or $\Gamma^{(2)}$) as a subgroup of $PSL(2, \mathbf{C})$ whose traces will

AMS Classification. 57N10, 11R04.

Received July 8, 2004; revised September 30, 2004.

not all be real, it must have at least one complex place. If $\mathbf{Q}(\theta)$ has exactly one complex place then there exist closed hyperbolic 3-manifolds M with $\pi_1 M$ being arithmetic and having invariant trace field $\mathbf{Q}(\theta)$; conversely the invariant trace fields of arithmetic Kleinian groups do have exactly one complex place. There are also examples of closed hyperbolic 3-manifolds whose invariant trace field is a quadratic extension of $\mathbf{Q}\left(\cos \frac{2\pi}{n}\right)$ for n a sufficiently large integer, for instance the polyhedral groups obtained from triangular prisms and the Fibonacci manifolds in [9] Section 4.7.3 and 4.8.2 respectively.

However when we consider cusped hyperbolic 3-manifolds, those which are arithmetic will all have an imaginary quadratic number field for their invariant trace field, thus we will not find number fields with degree greater than 2 by this means. There is a construction which produces invariant trace fields of cusped finite volume hyperbolic 3-manifolds that are of arbitrarily high degree in [9] Theorem 5.6.4; here it is shown that for any positive square free integers $d_1, \ldots, d_r$ the field $\mathbf{Q}(\sqrt{-d_1}, \ldots, \sqrt{-d_r})$ occurs as an invariant trace field, and this is achieved by building up 3-manifolds via cutting and pasting along incompressible thrice punctured spheres. This method will produce 3-manifolds that may well become more and more complicated topologically as the degree of the invariant trace field increases. Another approach might be to take an infinite family of well understood cusped 3-manifolds and show that the degrees of the trace fields in this family are unbounded. This is done in [7] for the family of hyperbolic twist knots and it is proved that the degree of the trace field is in fact the number of twists.

Another family of cusped hyperbolic 3-manifolds which appears in many guises throughout the literature is the family of hyperbolic once-punctured torus bundles. These can be regarded as amongst the most well understood of finite volume hyperbolic 3-manifolds. In this note we prove that there exist hyperbolic once-punctured torus bundles with invariant trace field (hence also their trace field) having arbitrarily high degree. The particular once-punctured torus bundles that we examine all have 2-generator fundamental groups, thus making them particularly basic examples of fibred hyperbolic 3-manifolds with (invariant) trace fields of arbitrarily high degree. In addition we show that their invariant trace fields have no real places. Another possible point of interest is provided by Thurston's famous conjecture that every finite volume hyperbolic 3-manifold has a finite cover which fibres over the circle: if this turns out to be true then the fields that can occur as invariant trace fields of finite volume hyperbolic 3-manifolds are precisely the ones that occur for hyperbolic fibre bundles. Also our construction also allows us to answer a question of Bowditch on stabilisers of Markoff triples.

2. Once-punctured torus bundles

We know by the work of Thurston that if we have an orientation preserving homeomorphism of a compact orientable surface S then the interior of the mapping torus M has a hyperbolic structure if and only if the homeomorphism is pseudo-Anosov. We can view this group-theoretically: we have

$$\pi_1 M = \langle t, x_1, \dots, x_n : R, t x_i t^{-1} = \phi_*(x_i) \rangle$$

where $\pi_1 S = \langle x_1, \dots, x_n \rangle$ is the fibre subgroup, which will be normal in $\pi_1 M$. Here R is empty if S has boundary and consists of the one standard relation of $\pi_1 S$ if S is closed, and ϕ_* is the automorphism of $\pi_1 S$ induced by the glueing homeomorphism ϕ of S. Each element of $\pi_1 M$ has a unique expression of the form kt^m where $k \in \pi_1 S$, and we multiply by the rule $k_1 t^m k_2 t^n = k_1 \phi_*^m(k_2) t^{m+n}$. We are considering the situation where the fibre is a once-punctured torus, so that $\pi_1 S = F_2$ is the free group on two elements, say x and y, with the commutator $z = xyx^{-1}y^{-1}$ a peripheral element representing the boundary curve. The mapping class group (group of orientation preserving self-homeomorphisms of S up to isotopy) is very well understood in this case. We can think of an element of the mapping class group as an outer automorphism of F_2; as all automorphisms are induced by homeomorphisms of S we have that the mapping class group can be identified with the orientation preserving outer automorphisms (those that send the commutator z to a conjugate of itself rather than its inverse). This group is well known to be $SL(2, \mathbf{Z})$, generated by the elements L, R and ε where

$$L(x, y) = (xy, y) \quad \text{corresponds to} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$R(x, y) = (x, yx) \quad \text{corresponds to} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\text{and} \quad \varepsilon(x, y) = (x^{-1}, y^{-1}) \quad \text{corresponds to} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The pseudo-Anosov elements are precisely those representing hyperbolic elements of $SL(2, \mathbf{Z})$ (those whose trace has modulus greater than 2). Every hyperbolic element of $SL(2, \mathbf{Z})$ has a "left-right" decomposition in that it is conjugate to an element of the form

$$\varepsilon^j R^{m_1} L^{n_1} \cdots R^{m_k} L^{n_k}$$

for $m_i, n_i > 0$ and $j = 0$ or 1. Also this is unique up to a cyclic permutation.

We need to show how, given a hyperbolic once-punctured torus bundle, it is straightforward to find its invariant trace field. We use [9] Corollary 4.3.2 which tells us that the invariant trace field of a hyperbolic surface bundle is the same as that of the fibre subgroup. But our fibre subgroup $\pi_1 S$ is a two generator group so we can use the following: Given $(A, B) \in SL(2, \mathbf{C})^2$, let

$$(a, b, c) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) \in \mathbf{C}^3$$

be the trace triple of this ordered pair. We can also define this to be the trace triple of an ordered pair of elements of $PSL(2, \mathbf{C})$, up to the change of sign of any two elements of the triple. If the quantity $a^2 + b^2 + c^2 - abc$ (which is

$\mathrm{tr}(ABA^{-1}B^{-1})+2$ and well defined in the case of $PSL(2,\mathbf{C})$) is not equal to 4 then the trace triple is a parametrisation of simultaneous conjugacy: that is, given $(A',B')\in SL(2,\mathbf{C})^2$ then there exists $K\in SL(2,\mathbf{C})$ with $KAK^{-1}=A'$ and $KBK^{-1}=B'$ if and only if the respective trace triples are equal (and this works for $PSL(2,\mathbf{C})$ too, up to the action of this Klein 4-group of sign changes). All this is well known, as is the fact that for any word $w(x,y)\in F_2$ there is a trace polynomial $p_w\in\mathbf{Z}[t_1,t_2,t_3]$ so that for $(A,B)\in SL(2,\mathbf{C})^2$ we have $\mathrm{tr}\,w(A,B)=p_w$ evaluated at its trace triple. In particular we see that the trace field of a two generator group $\Gamma=\langle A,B\rangle$ is $\mathbf{Q}(a,b,c)$ and, using [9] Lemma 3.5.7, the invariant trace field $k\Gamma=\mathbf{Q}(a^2,b^2,abc)$ provided that Γ is non-elementary and $a,b\neq 0$.

In order to deal with hyperbolic once-punctured torus bundles, first note that if we choose any generating pair (A,B) for the fibre subgroup $\pi_1 S\leq PSL(2,\mathbf{C})$ then the peripheral curve of S must be represented by a parabolic element, so that the trace of the commutator $ABA^{-1}B^{-1}$ must be -2 which implies that the trace triple of (A,B) satisfies $a^2+b^2+c^2=abc$. Moreover if ϕ_* is the automorphism of $\pi_1 S$ induced by the homeomorphism ϕ, we must have an element $T\in PSL(2,\mathbf{C})$ with $(TAT^{-1},TBT^{-1})=(\phi_*(A),\phi_*(B))$ so that its trace triple is the same as that of (A,B). But thinking of ϕ_* as an outer automorphism of the free group F_2, we can express its action on any trace triple by noting the results of the generators ε, L, R on trace triples, which are:

$$L(a,b,c)=(c,b,bc-a)$$
$$R(a,b,c)=(a,c,ac-b)$$

with ε leaving the trace triple unchanged (although it will change the element T). Therefore the trace triple corresponding to the chosen generating pair of $\pi_1 S$ is a fixed point under ϕ_*, at least up to the group of sign changes, and if we were to choose a different generating pair, say $(\alpha A,\alpha B)$ for an automorphism α of F_2, then this would be fixed by $\alpha\phi_*\alpha^{-1}$.

Much more powerfully, if we know that ϕ is pseudo-Anosov and we attempt to solve for trace triples fixed by ϕ_* then Thurston's work ensures that we will be able to find such a point (and its complex conjugate) which also satisfies the parabolic relation and such that the pair (A,B) obtained from the trace triple generates a discrete group that is free of rank 2 and is the fibre subgroup of a hyperbolic once-punctured torus bundle. We can show that these two trace triples will be the only fixed points of ϕ_* that represent generating pairs (A,B) giving rise to a discrete free group.

PROPOSITION 2.1. *If M is a compact orientable 3-manifold fibred over the circle by the surface S using the homeomorphism ϕ and $\theta:\pi_1 M\to G$ is a homomorphism into any group G which is injective on restriction to the fibre subgroup $\pi_1 S$ then either θ is itself injective or ϕ_* is a periodic outer automorphism.*

Proof. Setting $K=\mathrm{Ker}\,\theta$, we have

$$K = \frac{K}{\pi_1 S \cap K} \cong \frac{K\pi_1 S}{\pi_1 S}$$

which is a subgroup of $\mathbf{Z} = \pi_1 M/\pi_1 S$. So if K is non-trivial, it is a cyclic normal subgroup $\langle k \rangle$ of $\pi_1 M$. Thus for all $s \in \pi_1 S$ we have $sks^{-1} = k^{\pm 1}$, but writing $k = s't^m$ for $s' \in \pi_1 S$ and $m \neq 0$ (as $k \notin \pi_1 S$) we see by abelianising that k commutes with all of $\pi_1 S$. Thus the automorphism ϕ_*^m, which is conjugation by t^m, is seen to be also conjugation by $(s')^{-1}$ and so is inner. □

Corollary 2.2. *If we have a trace triple (a, b, c) fixed by ϕ_*, where ϕ is a pseudo-Anosov homeomorphism of the once-punctured torus, such that the generating pair (A, B) obtained from (a, b, c) generates a free discrete group then $\langle A, B \rangle$ is the fibre subgroup in $PSL(2, \mathbf{C})$ of the hyperbolic once-punctured torus bundle with glueing homeomorphism ϕ.*

Proof. We take the element $T \in PSL(2, \mathbf{C})$ where conjugation of (A, B) by T induces the automorphism ϕ_* of F_2. But $\langle A, B \rangle$ is free and discrete, and is a normal subgroup of $\Gamma = \langle T, A, B \rangle$. This implies that Γ must itself be discrete (for instance see [10] Sublemma 6.3.4); moreover Γ is also a homomorphic image of the fundamental group of the hyperbolic once punctured torus bundle obtained from ϕ_*. Thus by Proposition 2.1, as ϕ is not periodic we have that our homomorphism is an isomorphism of this fundamental group onto a discrete subgroup of $PSL(2, \mathbf{C})$. Thus by Mostow-Prasad-Marden rigidity, this isomorphism is just conjugation or anti-conjugation. □

Moreover we do not have to worry about ϕ_* fixing our trace triple only up to sign changes; if $\Gamma = \langle T, A, B \rangle$ is the fundamental group of the finite volume hyperbolic 3-manifold that we are trying to find then it is discrete and without 2-torsion, so that a result of Culler [4] states that we can find a lift $\bar{\Gamma} = \langle \bar{T}, \bar{A}, \bar{B} \rangle$ of Γ in $SL(2, \mathbf{C})$, namely a subgroup of $SL(2, \mathbf{C})$ isomorphic to Γ and projecting down onto it. Thus the action of $\bar{T}$ by conjugation on $\bar{A}$ and on $\bar{B}$ results in the same words in $\bar{A}$ and $\bar{B}$ as that of T on A and on B. Hence $(\bar{A}, \bar{B})$ and $(\phi_*(\bar{A}), \phi_*(\bar{B}))$ are conjugate in $SL(2, \mathbf{C})$ so their trace triples are the same, including signs.

The automorphisms we consider are $\alpha_k = L^{-1}R^{-k}$, with the associated homeomorphisms being pseudo-Anosov if $k \neq 0$. We have $\alpha_k(x, y) = (x^{k+1}y^{-1}, yx^{-k})$ so that y can be eliminated from the first relation, making the fundamental group of the fibre bundle 2-generator. On conjugation by $RL^{-1}R$ it is readily seen that the "left-right" form of α_k is RL^k, but we stick with this definition for ease of calculation.

We can show immediately by induction:

Lemma 2.3.

$$R^{-k}(a, b, c) = (a, p_k(a)b - p_{k-1}(a)c, p_{k-1}(a)b - p_{k-2}(a)c)$$

where the polynomial $p_k \in \mathbf{Z}[t]$ *satisfies the second order recurrence relation* $p_{k+1}(t) = tp_k(t) - p_{k-1}(t)$ *and* $p_0(t) = 1,\ p_1(t) = t$.

In fact it is easily checked that $p_k(2\cos\theta) = \sin(k+1)\theta/\sin\theta$ so they are merely the Chebyshev polynomials of the second kind (normalised to make them monic). We now look for the fixed points of $L^{-1}R^{-k}$.

Lemma 2.4. *If* $L^{-1}R^{-k}(a,b,c) = (a,b,c) \neq (0,0,0)$ *and* $a^2+b^2+c^2 = abc$ *then* $(a,b,c) = (a, 3-p_k(a), a)$ *and* a *is a root of the degree* $k+2$ *polynomial* $z_k(t)$, *where*

$$z_k(t) = (t^2-3)p_k(t) - p_{k-2}(t) + 6 - t^2.$$

Proof. We have $R^{-k}(a,b,c) = L(a,b,c)$, giving $a = c$ and hence

$$b(p_k(a)-1) = ap_{k-1}(a), \quad 2a^2+b^2 = a^2b.$$

Eliminating b from these two equations, cancelling a (as $a=0$ only gives rise to the trivial solution) and using the identity $p_{k-1}(a)p_{k+1}(a) = p_k^2(a) - 1$, we obtain

$$(p_k(a)-1)(p_k(a)-3) + ap_{k-1}(a) = 0.$$

Thus $(p_k(a)-1)(p_k(a)-3+b) = 0$ but if $p_k(a) = 1$ we have $p_{k-1}(a) = 0$, $p_{k-2}(a) = -1$. Then on returning to the original equations we would get $a = ba - a$ but $b = 2$ implies $4 = 0$. Hence we must have $b = 3 - p_k(a)$ which gives us a triple of $(a, 3-p_k(a), a)$. Then using the equation displayed above (and the recurrence relation) as well as the commutator condition provides us with the following simultaneous equations in a:

$$p_k^2 - 3p_k + p_{k-2} + 3 = 0$$

$$p_k^2 + (a^2-6)p_k + 9 - a^2 = 0$$

from which we obtain $z_k(a)$. □

Note: we can also look for fixed points of R^{-k} and certainly if we have $p_k(a) = 1$ and $p_{k-2}(a) = -1$ then we can take any $b, c \in \mathbf{C}$ subject to $a^2+b^2+c^2 = abc$. To see that this is easy to arrange, note that if $\theta = 2\pi n/k$ for $n = 1, \ldots, (k-1)/2$ (k odd) or $n = 1, \ldots, k/2-1$ (k even) then, setting $a = 2\cos\theta$, we have $p_k(a) = \sin(k+1)\theta/\sin\theta$ which gives us a fixed point.

But we can find fixed points of L^{-j} in exactly the same way: we just need $p_j(b) = 1$ and $p_{j-2}(b) = -1$, thus $b = 2\cos\phi$ for $\phi = 2\pi m/j$. The reason why we raise this is because it answers a question of Bowditch. We have the action of L and R on $\Phi = \{(a,b,c) \in \mathbf{C} : a^2+b^2+c^2 = abc\}$ via trace triples, and the group generated by L and R (or more correctly their images when quotiented out by the kernel of the action, namely $\langle\varepsilon\rangle$) is $PSL(2,\mathbf{Z})$. In [2] p. 724 it says: "Another question of interest seems to be which subgroups of $PSL(2,\mathbf{Z})$ can stabilise an element of $\Phi\backslash\{(0,0,0)\}$. The only examples I know are either finite or virtually cyclic." Of course only $(0,0,0)$ has stabiliser $PSL(2,\mathbf{Z})$ and in fact it is straightforward to show that this is the only point in Φ with a finite orbit. However we obtain:

PROPOSITION 2.5. *There exist points* $(a,b,c) \in \Phi \backslash \{(0,0,0)\}$ *whose stabilisers contain non-abelian free groups.*

Proof. As described above, given any $j,k \geq 3$ we can find $a \neq 0$ such that (a,b,c) is fixed by R^k for any b and c with $(a,b,c) \in \Phi$, and $b \neq 0$ such that (a,b,c) is fixed by L^j for any a and c with $(a,b,c) \in \Phi$. Taking that a and that b, we solve the quadratic equation for c to obtain a trace triple with stabiliser containing $\langle L^j, R^k \rangle \leq PSL(2,\mathbf{Z})$ which for $j,k \geq 3$ is a rank 2 free subgroup of $PSL(2,\mathbf{Z})$ of infinite index. □

Moving back to the hyperbolic once-punctured torus bundle M_k with monodromy $L^{-1}R^{-k}$ and fibre S_k, we can now find its invariant trace field.

LEMMA 2.6. *The invariant trace field* $k(\pi_1 M_k)$ *equals* $\mathbf{Q}(a)$ *or* $\mathbf{Q}(a^2)$, *where* a *is a root of the polynomial* z_k.

Proof. We know from before that $k(\pi_1 M_k) = k(\pi_1 S_k) = \mathbf{Q}(a^2, b^2, abc)$ as $a, b \neq 0$ because $\pi_1 S_k$ is torsion free, so our field is $\mathbf{Q}(a^2, 3 - p_k(a))$. This is $\mathbf{Q}(a)$ unless both p_k and the minimum polynomial of a contain only terms of even degree. □

Of course we only know that the minimum polynomial of a is a factor of z_k, not z_k itself. A quick computer check up to $k = 100$ suggests that z_k is irreducible over $\mathbf{Z}$ for even k, and for odd k it is irreducible except for the linear factor $a+2$. However we merely need to show that these minimum polynomials have arbitrarily high degree and we can do this by looking for irreducible factors over $\mathbf{Z}_2$.

THEOREM 2.7. *If* $k = 2^l - 2$ *where* $l-1$ *is prime then any factor of the polynomial* z_k *has degree equivalent to* 0, 1, 2, 3 *or* 4 *modulo* $l-1$.

Proof. We examine our polynomials $p_k(t)$ over $\mathbf{Z}_2$ and spot a possible pattern: for any k of the form $2^n - 1$, we find $p_k(t) \equiv t^k$ which will be shown in Lemma 2.10 below. Now over $\mathbf{Z}_2$ we have

$$z_k(t) \equiv (t^2+1)p_k(t) + p_{k-2}(t) + t^2$$

which, replacing tp_k with $p_{k+1} + p_{k-1}$ and $p_k + p_{k-2}$ with tp_{k-1}, becomes $z_k(t) \equiv tp_{k+1}(t) + t^2$. Hence putting $k = 2^l - 2$ and using our assumption we obtain

$$z_{2^l-2}(t) \equiv t^{2^l} + t^2.$$

However it is well known that over the field $\mathbf{Z}_p$ the polynomial $t^{p^j} - t$ is the product of all the irreducible polynomials whose degree divides j. Hence

$$z_{2^l-2}(t) \equiv (t^{2^{l-1}} + t)^2$$

is the product of the squares of all the irreducible polynomials over $\mathbf{Z}_2$ with degree dividing $l-1$, so for $l-1$ prime the irreducible factors will all be of degree $l-1$ except for t twice and $t+1$ twice. Then the irreducible factors of z_{2^l-2} over $\mathbf{Z}$ will each factor over $\mathbf{Z}_2$ into a product of these terms. □

COROLLARY 2.8. *If M_k is the hyperbolic once-punctured torus bundle with monodromy $L^{-1}R^{-k}$ then there exist invariant trace fields $k(\pi_1 M_k)$ with arbitrarily high degree.*

Proof. We choose $k = 2^l - 2$ for $l-1$ prime as above and we know that the invariant trace field $k(\pi_1 M_k)$ is $\mathbf{Q}(a)$ or $\mathbf{Q}(a^2)$ so its degree is the same as the minimum polynomial of a (or half that in the latter case if the minimum polynomial has only even coefficients). The minimum polynomial m_k of a divides z_k from Lemma 2.6 and from Theorem 2.7 we have that any factor of z_k has degree $(l-1)n + r$ for $n = 0, 1, 2, \ldots$ and $r = 0, 1, 2, 3$ or 4. We just need to eliminate m_k having degree 1, 2, 3 or 4. If we have degree 1 then the invariant trace field is real which cannot occur here. If it is degree 2 then a is a quadratic integer (as z_k is monic) so the fact that the trace triple consists of algebraic integers means that all traces in the fibre subgroup are also algebraic integers, because the trace polynomials p_w have integer coefficients. This implies that every trace in $\pi_1 M_k$ is also an algebraic integer, using Proposition 2.8 of [1] which states that if a subgroup of $SL(2, \mathbf{C})$ has integral traces then so does its normaliser. Thus we have a finite volume 3-manifold whose fundamental group has an imaginary quadratic invariant trace field with all traces algebraic integers, which means that $\pi_1 M_k$ is arithmetic. But the paper [3] classifies arithmetic hyperbolic once-punctured torus bundles: they are all a cyclic cover or the sister of a cyclic cover (meaning ε times a cyclic cover) of the bundles with monodromy RL, R^2L and R^2L^2 (note that the authors have swapped the definitions of L and R in [3] and in [9], and here we are using the notation of [9]). But our monodromy is (the inverse of) R^kL so is not one of these for $k > 2$.

As for m_k having degree 3 or 4, note that $z_k(t)$ contains only even powers of t for even k because $p_k(t)$ does. Thus $m_k(-t)$ also divides $z_k(t)$, giving rise to a factor of z_k of degree 6 in the first case which from Theorem 2.7 we know cannot happen for $l \geq 8$. Similarly for $l \geq 12$ we cannot have a factor of z_k of degree 8, so we must have $m_k(t) = t^4 + a_2 t^2 + a_0$. But p_k also consists only of even terms, thus making the invariant trace field $\mathbf{Q}(a^2)$ which is again an imaginary quadratic number field so that we are back in the arithmetic case. □

We can also show that in the above cases the invariant trace field has no real places.

PROPOSITION 2.9. *For positive $k \equiv 2 \mod 4$ the invariant trace field $k(\pi_1 M_k)$ has no real places.*

Proof. We need to show that the minimal polynomial m_k has no real or imaginary roots (imaginary in case the trace field is $\mathbf{Q}(a^2)$), so we show this is true of z_k. We can write z_k in some useful alternative forms: setting f_k to be the Chebyshev polynomials of the first kind that are normalised to make them monic, we have $f_k(t) = 2\cos k\theta$ for $t = 2\cos\theta$ with f_k satisfying the same recurrence relation as that for p_k. Induction and trigonometric identities then tell us that

$$\begin{aligned} z_k(t) &= f_{k+2}(t) + 6 - t^2 \\ &= 2\cos(k+2)\theta - 2\cos 2\theta + 4 \\ &= 4 - 4\sin\left(\frac{k}{2}+2\right)\theta \sin\frac{k\theta}{2}. \end{aligned}$$

Thus, setting $\theta \in \mathbf{R}$ so that $t \in [-2, 2]$, the last identity tells us that if z_k has any roots in this interval then the difference in the two sine arguments must be a multiple of 2π, giving only the solution $t = -2$ for k odd. Now put $\theta = i\phi$ where $\phi \in \mathbf{R}$ in the second equation, so $t = 2\cosh\phi \in [2, \infty)$ and $\cosh(k+2)\phi \geq \cosh 2\phi$ as k is positive, giving no solutions. Then for k even there are no solutions in $(-\infty, -2]$ either because z_k is an even function.

Finally we need to consider imaginary t, so we put $\theta = i\phi - \pi/2$ for $\phi \in \mathbf{R}$ to obtain $t = 2i\sinh\phi$ and we use the first equation. This tells us that

$$z_k(t) = \pm 2\cosh(k+2)\phi + 4\sinh^2\phi + 6$$

where the plus sign is taken when $k/2$ is odd, which is the case we want. □

Note that in [5] it is also shown that z_k has no real roots; the paper finds the correct complex place that gives rise to the discrete faithful representation of $\pi_1 M_k$ in $PSL(2, \mathbf{C})$.

We finish by confirming the lemma we needed earlier in Theorem 2.7.

LEMMA 2.10. *Over $\mathbf{Z}_2$ we have $p_k(t) \equiv t^k$ for $k = 2^n - 1$.*

Proof. Writing $[k, r] \in \mathbf{Z}_2$ for the coefficient of t^r in p_k where $k \in \mathbf{N}$, we have

$$[k, r] \equiv [k-1, r-1] + [k-2, r]$$

from the recurrence relation. This gives us by induction on k that $[k, r] \equiv 0$ for $k - r$ odd. We claim that for $k - r$ even we have

$$[k, r] \equiv \begin{pmatrix} \frac{k+r}{2} \\ r \end{pmatrix};$$

again this follows from induction on k and the standard relation of binomial coefficients. Now we need to establish that $[2^n - 1, r] \equiv 0$ for $0 \leq r \leq 2^n - 2$. This is fine for r even, so for $r = 2s + 1$ and $m = n - 1$ we show that

$$\begin{pmatrix} 2^m + s \\ 2s + 1 \end{pmatrix}$$

is even for $0 \leq s \leq 2^m - 2$. The determination of the exact power to which a prime p divides the binomial coefficient $\binom{a+b}{a}$ is derived on p. 31 of [11] and dates back to Kummer in 1852. One calculates the sum of a and b in base p and then it is the number of times we "carry over". In our case we are adding $a = 2s + 1$ and $b = 2^m - s - 1$ in binary; the latter number's representation is obtained by taking the binary representation of s (consisting of m entries) and swapping the ones and the zeroes. So taking the lowest power of 2 where a zero appears in the binary expansion of s, a one will appear in this place for $2s + 1$ and a one will appear as well for $2^m - s - 1$, so that in calculating $a + b$ we are guaranteed a carry over in this place, hence our binomial coefficient is divisible by 2 at least once. □

References

[1] H. Bass, Groups of integral representation type, Pacific J. Math. **86** (1980), 15–51.

[2] B. H. Bowditch, Markoff triples and quasi-Fuchsian groups, Proc. London Math. Soc. **77** (1998), 697–736.

[3] B. H. Bowditch, C. Machlachlan and A. W. Reid, Arithmetic hyperbolic surface bundles, Math. Ann. **302** (1995), 31–60.

[4] M. Culler, Lifting representations to covering groups, Adv. in Math. **59** (1986), 64–70.

[5] H. Helling, The trace field of a series of hyperbolic manifolds, SFB 343 Preprint 99-072, Bielefeld.

[6] R. D. Horowitz, Characters of free groups represented in the two-dimensional special linear group, Comm. Pure Appl. Math. **25** (1972), 635–649.

[7] J. Hoste and P. D. Shanahan, Trace fields of twist knots, J. Knot Theory Ramifications **10** (2001), 625–639.

[8] R. Kirby, Problems in low-dimensional topology, AMS/IP Stud. Adv. Math. 2.2, Geometric Topology, Athens, GA, 1993, 35–473.

[9] C. Maclachlan and A. W. Reid, The arithmetic of hyperbolic 3-manifolds, Graduate Texts in Mathematics **219**, Springer-Verlag, New York, 2003.

[10] J. P. Otal, The hyperbolization theorem for fibered 3-manifolds, SMF/AMS Texts and Monographs **7**, American Math. Soc., Soc. Math. de France, 2001.

[11] P. Ribenboim, The new book of prime number records, Springer-Verlag, New York, 1996.

Selwyn College
Cambridge
CB3 9DQ
England, U.K.
E-mail: jb128@dpmms.cam.ac.uk