

ON CAUCHY-RIEMANN CIRCLE BUNDLES

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Abstract

Building on ideas of R. Mizner, [17]–[18], and C. Laurent-Thiébaud, [14], we study the CR geometry of real orientable hypersurfaces of a Sasakian manifold. These are shown to be CR manifolds of CR codimension two and to possess a canonical connection D (parallelizing the maximally complex distribution) similar to the Tanaka-Webster connection (cf. [21]) in pseudohermitian geometry. Examples arise as circle subbundles $S^1 \rightarrow N \xrightarrow{\pi} M$, of the Hopf fibration, over a real hypersurface M in the complex projective space. Exploiting the relationship between the second fundamental forms of the immersions $N \rightarrow S^{2n+1}$ and $M \rightarrow \mathbb{C}P^n$ and a horizontal lifting technique we prove a CR extension theorem for CR functions on N . Under suitable assumptions $[\text{Ric}_D(Z, \bar{Z}) + 2g(Z, (I - a)\bar{Z})] \geq 0$, $Z \in T_{1,0}(N)$, where a is the Weingarten operator of the immersion $N \rightarrow S^{2n+1}$ on the Ricci curvature Ric_D of D , we show that the first Kohn-Rossi cohomology group of M vanishes. We show that whenever $\text{Ric}_D(Z, \bar{W}) - 2g(Z, \bar{W}) = (\mu \circ \pi)g(Z, \bar{W})$ for some $\mu \in C^\infty(M)$, M is a pseudo-Einstein manifold.

1. Introduction and statement of main results

Let M be a differentiable manifold, of class C^∞ and real dimension $2n + k$. A subbundle $T_{1,0}(M)$, of the complexified tangent bundle $T(M) \otimes \mathbb{C}$, of complex rank n , is said to be a *CR structure* on M if

$$(1.1) \quad T_{1,0}(M)_x \cap T_{0,1}(M)_x = (0), \quad x \in M,$$

$$(1.2) \quad Z, W \in \Gamma^\infty(T_{1,0}(M)) \Rightarrow [Z, W] \in \Gamma^\infty(T_{1,0}(M)).$$

The pair (n, k) is the *type* of the CR structure. A pair $(M, T_{1,0}(M))$ is a *CR manifold* (of type (n, k)). We set $T_{0,1}(M) = \overline{T_{1,0}(M)}$, where an overbar denotes complex conjugation. If $E \rightarrow M$ is a vector bundle over M then E_x denotes the fibre over $x \in M$ and $\Gamma^\ell(E)$ denotes the space of sections of class C^ℓ , $\ell \in \mathbb{N} \cup \{\infty, \omega\}$ (eventually defined on an open subset $U \subseteq M$, to be understood from the context). The integer n (respectively k) is the *CR dimension* (respectively the *CR codimension*) of the CR manifold $(M, T_{1,0}(M))$. A CR manifold of type $(n, 1)$, i.e. of CR codimension $k = 1$, is said to be of *hypersurface type*. The terminology is motivated by the fact that every real hypersurface M in $\mathbb{C}P^{n+1}$ carries a naturally induced CR structure $T_{1,0}(M)$ (of type $(n, 1)$) given by

Received June 7, 2004; revised October 18, 2004.

$T_{1,0}(M)_x = [T_x(M) \otimes_{\mathbf{R}} \mathbf{C}] \cap T^{1,0}(\mathbf{C}^{n+1})_x$, $x \in M$, where $T^{1,0}(\mathbf{C}^{n+1})$ is the holomorphic tangent bundle, i.e. if z^1, \dots, z^{n+1} are complex coordinates on \mathbf{C}^{n+1} , then $T^{1,0}(\mathbf{C}^{n+1})$ is the span of $\{\partial/\partial z^j : 1 \leq j \leq n+1\}$. Such CR manifolds are referred to as *embedded*.

An important class of CR manifolds (of arbitrary type (n, k)) consists of the *proper CR submanifolds*, in the sense of A. Bejancu, [3]. Given a Hermitian manifold (M_0, J, g_0) , with the complex structure J and the Hermitian metric g_0 , a pair (M, \mathcal{D}) consisting of a submanifold M of M_0 and of a smooth distribution, $\mathcal{D} : x \in M \mapsto \mathcal{D}_x \subseteq T_x(M)$, is a *CR submanifold* if i) \mathcal{D} is *J-invariant*, i.e. $J_x(\mathcal{D}_x) = \mathcal{D}_x$, $x \in M$, and ii) the orthogonal complement \mathcal{D}^\perp (with respect to $g := j^*g_0$, $j : M \subset M_0$) of \mathcal{D} in $T(M)$ is *J-anti-invariant*, i.e. $J_x(\mathcal{D}_x^\perp) \subseteq T_x(M)^\perp$, $x \in M$. Here $\mathcal{V}(j) = T(M)^\perp$ is the normal bundle (of the given immersion $j : M \subset M_0$).

Let (M, \mathcal{D}) be a CR submanifold of the Hermitian manifold (M_0, J, g_0) . Set $n := \dim_{\mathbf{C}} \mathcal{D}_x$, $k := \dim_{\mathbf{R}} \mathcal{D}_x^\perp$, for any $x \in M$ (so that $\dim M = 2n + k$). A CR submanifold (M, \mathcal{D}) is *generic* if k and the codimension of M in M_0 coincide. A CR submanifold (M, \mathcal{D}) is said to be *proper* if both the holomorphic and totally real distributions are nonzero ($n \neq 0$ and $k \neq 0$).

By a result in [5] (referred to as the *Blair-Chen theorem*, in the sequel), every proper CR submanifold (M, \mathcal{D}) of a Hermitian manifold is a CR manifold. Indeed, let J_M be the restriction of J to \mathcal{D} . Moreover, let $J_M^{\mathbf{C}}$ be the extension (by complex linearity) of J_M to $\mathcal{D} \otimes \mathbf{C}$ and set $T_{1,0}(M) := \text{Eigen}(J_M^{\mathbf{C}}, i)$, i.e. $T_{1,0}(M)_x$ is the eigenspace of $(J_M^{\mathbf{C}})_x$ corresponding to the eigenvalue $i = \sqrt{-1}$, $x \in M$. Then, by the previously quoted result (cf. [5], or Theorem 2.1 in [29], p. 83), $(M, T_{1,0}(M))$ is a CR manifold of type (n, k) .

A CR structure on a manifold is a reformulation (in terms of fibre bundles) of the *tangential Cauchy-Riemann* equations induced (on a real submanifold) by the Cauchy-Riemann equations (on the ambient complex manifold). If $\Omega = \{\rho > 0\} \subset \mathbf{C}^{n+1}$ is a smooth domain, then $M := \partial\Omega$ inherits a CR structure $T^{1,0}(M)$ (as a real hypersurface in \mathbf{C}^{n+1}) locally spanned by $L_\alpha = \sum_{j=1}^{n+1} a_\alpha^j(z) \partial/\partial z^j$, $1 \leq \alpha \leq n$, where $\sum_{j=1}^{n+1} a_\alpha^j(z) \partial\rho/\partial z^j = 0$, $1 \leq \alpha \leq n$ (i.e. the L_α 's are purely tangential first order differential operators). The restriction u to M of a holomorphic function f on an open neighborhood of M satisfies the *tangential Cauchy-Riemann equations*

$$(1.3) \quad L_{\bar{\alpha}} u \equiv \sum_{j=1}^{n+1} \overline{a_{\bar{\alpha}}^j(z)} \frac{\partial u}{\partial \bar{z}^j} = 0, \quad 1 \leq \alpha \leq n.$$

If $(M, T_{1,0}(M))$ is an arbitrary CR manifold (of arbitrary, but fixed, type) let us consider the differential operator

$$\bar{\partial}_M : C^\infty(M, \mathbf{C}) \rightarrow \Gamma^\infty(T_{0,1}(M)^*)$$

given by $(\bar{\partial}_M u)\bar{Z} := \bar{Z}(u)$, for any $Z \in \Gamma^\infty(T_{1,0}(M))$, $u \in C^\infty(M, \mathbf{C})$. The equations

$$(1.4) \quad \bar{\partial}_M u = 0$$

are also referred to as the tangential Cauchy-Riemann equations (although not any longer confined to the CR codimension 1 case) and coincide locally with (1.3) when $M = \partial\Omega$.

When M is a real submanifold of a complex manifold V (endowed with the induced CR structure), a fundamental problem of complex analysis in several variables is whether, given a solution u to (1.4), there is a holomorphic function extending u (e.g. in a neighborhood of M). The solutions to the given problem (perhaps under additional assumptions on the geometry of M , e.g. imposed to its *Levi form*) occupy a large space in the specialized literature (cf. e.g. A. Boggess, [6], p. 189–260).

The holomorphic extension problem may be solved in the real analytic case by a result of G. Tomassini, [22]. A CR manifold is *real analytic* if M is a C^ω manifold and its CR structure is locally spanned by real analytic sections. Let $M \subset \mathbf{C}^n$ be a real analytic embedded CR manifold. Let $u : M \rightarrow \mathbf{C}$ be a real analytic *CR function*, i.e. a C^ω solution to (1.4). Then, by the *Tomassini theorem* (cf. op. cit.), there is a neighborhood U of M in \mathbf{C}^n and a unique holomorphic function $f : U \rightarrow \mathbf{C}$ whose trace on M is u .

Let N_0 be a manifold carrying an almost contact metric (a.ct.m.) structure (ϕ, ξ, η_0, g_0) . A pair (N, \mathcal{D}) consisting of a submanifold N of N_0 and a smooth distribution $\mathcal{D} : x \in N \mapsto \mathcal{D}_x \subseteq T_x(N)$ is said to be a *contact CR submanifold* of N_0 if i) \mathcal{D} is ϕ -invariant, i.e. $\phi_x(\mathcal{D}_x) \subseteq \mathcal{D}_x$, $x \in N$, and ii) the orthogonal complement \mathcal{D}^\perp of \mathcal{D} (with respect to $g := i^*g_0$) in $T(N)$ is ϕ -anti-invariant, i.e. $\phi_x(\mathcal{D}_x^\perp) \subseteq T_x(N)^\perp$, $x \in N$. Here $\mathcal{V}(i) = T(N)^\perp$ is the normal bundle of the immersion $i : N \subset N_0$.

Let (N, \mathcal{D}) be a contact CR submanifold of the a.ct.m. manifold N_0 . (N, \mathcal{D}) is *generic* if $\dim_{\mathbf{R}} \mathcal{D}_x^\perp = \text{codim}(N)$, $x \in N$. (N, \mathcal{D}) is *proper* if $\mathcal{D} \neq (0)$ and $\mathcal{D}^\perp \neq (0)$.

The concept of a contact CR submanifold (with the additional assumption that ξ , the contact field of the ambient manifold N_0 , is tangent to N) is due to K. Yano & M. Kon, [29], p. 48. Taking into consideration the Blair-Chen theorem, it is natural to ask whether a contact CR submanifold is a CR manifold (and of what type). Let (N, \mathcal{D}) be a contact CR submanifold of the Sasakian manifold N_0 , tangent to the contact field ξ of N_0 . A simple argument shows that either $\xi \in \mathcal{D}$ or $\xi \in \mathcal{D}^\perp$. However, to avoid the disjunction of cases, let $H(N)$ be the orthogonal complement of $\mathbf{R}\iota\xi$ in \mathcal{D} [where $\iota : T(N) \rightarrow \mathcal{D}$ is the canonical projection], i.e. $\mathcal{D} = H(N) \oplus \mathbf{R}\iota\xi$. Also, let $H(N)^\perp$ be the orthogonal complement of $H(N)$ in $T(N)$. $H(N)$ is ϕ -invariant, $H(N)^\perp$ is ϕ -anti-invariant, hence $(N, H(N))$ is a contact CR submanifold. Moreover $\xi \in H(N)^\perp$ and the restriction J of ϕ to $H(N)$ is a complex structure on $H(N)$. We establish

THEOREM 1. $T_{1,0}(N) := [T(N) \otimes \mathbf{C}] \cap T_{1,0}(N_0)$ is a CR structure on N , of type (n, k) , where $n := \dim_{\mathbf{C}} H(N)_x$ and $k := \dim_{\mathbf{R}} H(N)_x^\perp$, $x \in N$. Therefore $(N, T_{1,0}(N))$ is a CR manifold and $i : N \subset N_0$ is a CR immersion.

By a result in [13] (referred to as the *Ianus theorem*, in the sequel), every Sasakian manifold N_0 admits a natural CR structure $T_{1,0}(N_0) := \{X - i\phi X :$

$X \in \text{Ker}(\eta_0)\}$. A *CR immersion* is a C^∞ map $f : N \rightarrow N'$ of CR manifolds such that f is an immersion and a *CR map*, i.e. $(d_x f)T_{1,0}(N)_x \subseteq T_{1,0}(N')_{f(x)}$, $x \in N$. For simplicity Theorem 1 is stated for CR submanifolds of Sasakian manifolds (while the Ianuş theorem only requires that (ϕ, ζ, η_0) is normal).

An important class of contact CR submanifolds of an odd-dimensional sphere consists of the total spaces of the (circle) subbundles of the Hopf fibration, over a CR submanifold of the complex projective space. Precisely, let M be a CR submanifold in $\mathbb{C}P^n$ and let $N := \pi_0^{-1}(M)$, where $\pi_0 : S^{2n+1} \rightarrow \mathbb{C}P^n$ is the projection. Then N is a contact CR submanifold of S^{2n+1} (carrying the standard Sasakian structure) and $S^1 \rightarrow N \xrightarrow{\pi} M$ is a principal subbundle of the Hopf fibration $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$, where $\pi := \pi_0|_N$. $S^1 \rightarrow N \xrightarrow{\pi} M$ is said to be a *Cauchy-Riemann circle bundle* and is the central notion of this paper. Indeed, when the given contact CR submanifold N is (the total space of) a S^1 -bundle over a CR submanifold $M \subset \mathbb{C}P^n$, the pullback $\eta := i^*\eta_0$ of the contact form of S^{2n+1} is a connection form in $S^1 \rightarrow N \rightarrow M$, a fact which enables one to consider horizontal lifts of geometric objects on M (e.g. the first and the second fundamental forms of $j : M \subset \mathbb{C}P^n$, the induced and normal connections, the Weingarten operator, etc.). This leads to a precise description of the geometry of (the second fundamental form of) $j : M \subset \mathbb{C}P^n$ in terms of the geometry of (the second fundamental form of) $i : N \subset S^{2n+1}$ and successive applications. We have

THEOREM 2. *Let $n, k \in \mathbf{Z}$, $n, k > 0$, and let $S^1 \rightarrow N \rightarrow M$ be a Cauchy-Riemann circle bundle over a real analytic compact connected CR submanifold $M \subset \mathbb{C}P^{n+k-1}$, with $\dim N = 2n + k$ and $\dim_{\mathbf{R}} H(N)_x = 2n$, $x \in N$. If N is generic and the contact vector field of $S^{2(n+k)-1}$ lies in the invariant distribution of N , then there is a saturated subset $Q \subset S^{2(n+k)-1}$ such that $Q \supset N$, and there is a subbundle $S^1 \rightarrow N' \rightarrow M'$ of the Hopf fibration $S^1 \rightarrow S^{2(n+k)-1} \rightarrow \mathbb{C}P^{n+k-1}$, over a submanifold $M' \subset \mathbb{C}P^{n+k-1}$ of real dimension $2n + k$, such that $Q \supset N'$ and, for every basic CR function $u \in CR^\omega(N)$ there is an open and saturated set $U \subseteq S^{2(n+k)-1}$ such that $U \supset Q$ and there is a basic CR function $f \in CR^\omega(U)$ which extends u , i.e. $f|_N = u$.*

Theorem 2 is an immediate corollary of the Tomassini theorem and of a result by R. O. Wells (cf. [25]) according to which, given a compact and connected submanifold $M \subset V$ in a complex manifold V , if M is a generic CR manifold of type (n, k) , with $n > 0$, the holomorphic hull of M contains a manifold M' of higher dimension (i.e. $\dim M' = 2n + k + 1$). While Theorem 2 is a rather elementary consequence of the above mentioned results (of G. Tomassini and R. O. Wells, cf. op. cit.), it leads naturally to the question: which real submanifolds of a Sasakian manifold are generic? The following result establishes a necessary topological condition.

THEOREM 3. *Let (N, \mathcal{D}) be a contact CR submanifold of a Sasakian manifold N_0 . Let us assume that N is generic and that the tangential component of the*

contact vector ξ_0 of N_0 lies in the invariant distribution [i.e. $\xi := \tan(\xi_0) \in \mathcal{D}$]. Then the Euler-Poincaré characteristic of N vanishes [i.e. $\chi(N) = 0$].

Theorem 3 is a contact analogue of a result by R. O. Wells (cf. theorem 1 in [26], p. 124). If $M \subset \mathbf{CP}^n$ is a compact CR submanifold and $N := \pi_0^{-1}(M) \subset S^{2n+1}$, then it is immediate that $\chi(N) = 0$. Indeed, in this case N is tangent to the contact vector field ξ of S^{2n+1} (and $\xi(x) \neq 0$ for any $x \in N$).

Every nondegenerate CR manifold M , of hypersurface type, on which a contact form θ is fixed, admits a unique linear connection parallelizing the Levi distribution $\text{Ker}(\theta)$, its complex structure, the Webster metric g_θ , and whose torsion is pure. This is the *Tanaka-Webster connection* of (M, θ) (discovered independently by N. Tanaka, [21], and S. Webster, [24]). The discovery of the Tanaka-Webster connection revealed itself of a certain importance in understanding *pseudohermitian geometry*. For instance, the *Chern-Moser tensor* (which is well known to vanish if and only if M is locally CR equivalent to a sphere) may be expressed in terms of the curvature of the Tanaka-Webster connection. Similarly, the *Fefferman metric* (cf. C. Fefferman, [11]) may be expressed in terms of the connection forms and scalar curvature of the Tanaka-Webster connection (cf. J. M. Lee, [15]). These and other issues (for instance those tied to the geometry of Yang-Mills fields on CR manifolds, cf. H. Urakawa, [23], or to the geometry of CR immersions, cf. S. Dragomir, [8], etc.) lead to the fundamental problem of building an analogue of the Tanaka-Webster connection in higher CR codimension ($k \geq 2$). Indeed this is the case of real hypersurfaces $N \subset S^{2n+1}$ (any such N has CR codimension 2).

The first attempt to solve the problem belongs to R. Mizner (cf. [18]), who built a connection ∇ generalizing the Tanaka-Webster connection to the case of a CR manifold M of CR codimension ≥ 2 under the assumption that M is nondegenerate and the Levi distribution $H(M)$ admits a complement E which is trivial as a bundle over M . Since (as in the CR codimension one case) one of the requirements for building ∇ is that the Levi form be parallel, the nondegeneracy assumption is quite natural. As to the triviality of E , while it follows from the mere orientability of M in the CR codimension one case, it is not *a priori* clear what obstructions there are towards of $E \approx M \times \mathbf{R}^k$, when $k \geq 2$ (neither does R. Mizner investigate such obstructions).

In this paper we adopt an alternative to R. Mizner's ideas, that is we request that the connection parallelizes the induced metric (rather than the Levi form \mathcal{L}) and we replace, here as well as in the classical case, the symmetry assumption by a suitable "purity" axiom. Precisely, let N be a real orientable hypersurface in a Sasakian manifold $(N_0, (\phi, \xi, \eta_0, g_0))$ (for instance $N_0 = S^{2n+1}$), tangent to the contact vector field of N_0 . Let ν be a global unit normal vector field on N , and set $U := \phi\nu$. Then U is tangential (i.e. $U \in T(N)$) and orthogonal to ξ . Let $H(N)$ be the orthogonal complement of $\mathbf{R}U \oplus \mathbf{R}\xi$ in $T(N)$. We establish

THEOREM 4. *There is a unique linear connection D on N satisfying the following properties*

- 1) $H(N)$ is parallel with respect to D ,
- 2) $Dg = 0$, where $g = i^*g_0$, $i : N \subset N_0$,
- 3) $DJ = 0$, where $J = \phi|_{H(N)}$,
- 4) the torsion T_D of D is pure, i.e.

$$T_D(Z, W) = 0, \quad T_D(Z, \bar{W}) = -2i\mathcal{L}(Z, \bar{W}), \quad Z, W \in T_{1,0}(N),$$

$$\tau \circ J + J \circ \tau = 0, \quad \rho \circ J + J \circ \rho = 0,$$

where $\tau X := T_D(\xi, X)$, $\rho X := T_D(U, X)$, $X \in T(N)$, and

- 5) $D\xi = 0$, $DU = 0$.

In analogy with pseudohermitian geometry and the fundamental role played there by the Tanaka-Webster connection, we expect several applications (in the study of CR structures of real orientable hypersurfaces $N \subset S^{2n+1}$) of the canonical connection furnished by Theorem 4. For instance, when N fibres in circles over a real hypersurface in $\mathbb{C}P^n$, we establish the following ‘‘vanishing theorem’’

THEOREM 5. *Let $S^1 \rightarrow N \rightarrow M$ be a Cauchy-Riemann circle bundle over a compact orientable real hypersurface $M \subset \mathbb{C}P^n$. Assume that M satisfies the contact condition and*

$$(1.5) \quad \text{Ric}_D(Z, \bar{Z}) + 2g(Z, (1-a)\bar{Z}) \geq 0, \quad Z \in T_{1,0}(N),$$

where Ric_D is the Ricci tensor of the canonical connection D on N and a is the Weingarten operator corresponding to a choice of global unit normal ν on N . Then the first Kohn-Rossi cohomology group of M vanishes, i.e. $H^{0,1}(M; \bar{\partial}_M) = 0$.

Given a nondegenerate CR manifold M of CR dimension n , a contact form θ on M is *pseudo-Einstein* if the pseudohermitian Ricci tensor (of the Tanaka-Webster connection of (M, θ)) is proportional to the Levi form, i.e. $R_{\alpha\bar{\beta}} = \mu h_{\alpha\bar{\beta}}$ for some $\mu \in C^\infty(M)$ (a posteriori $\mu = (1/n)\rho$, where $\rho = h^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$ is the pseudohermitian scalar curvature of (M, θ)). Another application of Theorem 4 is

THEOREM 6. *Let $S^1 \rightarrow N \xrightarrow{\pi} M$ be a Cauchy-Riemann circle bundle over an orientable real hypersurface $M \subset \mathbb{C}P^n$ satisfying the contact condition. Then there is $\mu \in C^\infty(M)$ such that*

$$(1.6) \quad \text{Ric}_D(Z, \bar{W}) - 2g(Z, a\bar{W}) = (\mu \circ \pi)g(Z, \bar{W}), \quad Z, W \in T_{1,0}(N),$$

if and only if θ is pseudo-Einstein of pseudohermitian scalar curvature $\rho = (n-1)(\mu+2)$. Here $\theta(X) = -G(X, J_{\mathbb{C}P^n \nu_M})$, $X \in T(M)$. In particular, if (1.6) holds, then the CR structure of M has a vanishing first Chern class.

Acknowledgements. The author wishes to express his gratitude towards professor Sorin Dragomir for his constant guidance during the elaboration of this paper and to the Referee for pointing out certain corrections.

2. CR submanifolds

2.1. CR submanifolds of Hermitian manifolds

Let $j: M \subset M_0$ be a submanifold of a Hermitian manifold (M_0, J, g_0) and $g := j^*g_0$ the induced metric. Let $T(M)^\perp \rightarrow M$ be the normal bundle of the given immersion. We shall need the Gauss and Weingarten formulae

$$(2.1) \quad \nabla_X^0 Y = \nabla_X Y + h(X, Y)$$

$$(2.2) \quad \nabla_X^0 V = -a_V X + \nabla_X^\perp V$$

for any $X, Y \in \mathcal{X}(M)$ and $V \in \Gamma^\infty(T(M)^\perp)$. Here ∇^0 is the Levi-Civita connection of (M_0, g_0) , ∇ is the induced connection, h is the second fundamental form of the given immersion, a_V is the Weingarten operator (associated to the normal section V) and ∇^\perp is the normal connection. Let \tan_x and nor_x be the projections associated to the direct sum decomposition

$$T_x(M_0) = [(d_x j)T_x(M)] \oplus T_x(M)^\perp, \quad x \in M.$$

We set

$$PX = \tan(JX), \quad FX = \text{nor}(JX), \quad tV = \tan(JV), \quad fV = \text{nor}(JV).$$

Then

$$(2.3) \quad P^2 = -I - tF, \quad FP + fF = 0, \quad f^2 = -I - Ft, \quad Pt + tf = 0.$$

The identities (2.3) are commonly stated for submanifolds of Kählerian manifolds. However, an inspection of the proof (cf. e.g. [29], p. 77) shows that (2.3) hold when the ambient space is but Hermitian. We may state the following

THEOREM 7. *A submanifold M of a Hermitian manifold is a CR submanifold if and only if $FP = 0$. If this is the case then $\mathcal{D} := \text{Ker}(I + P^2)$ is J -invariant (while the orthogonal complement of \mathcal{D} in $T(M)$ is J -anti-invariant).*

Theorem 7 was first proved by K. Yano & M. Kon, (cf. Theor. 3.2 in [29], p. 87) for submanifolds of Kählerian manifolds, and the proof is a *verbatim* repetition of their arguments.

2.2. Contact CR submanifolds of a.ct.m. manifolds

Let N be a real $(2n + 1)$ -dimensional manifold. A synthetic object (ϕ, ξ, η) consisting of a tensor field ϕ of type $(1, 1)$, a tangent vector field ξ , and a 1-form η , is an almost contact structure on N if

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0.$$

An almost contact manifold is a (odd dimensional) manifold endowed with a fixed almost contact structure. Let $(N, (\phi, \xi, \eta))$ be an almost contact manifold. A Riemannian metric g is compatible with (ϕ, ξ, η) if $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$. Then $g(X, \xi) = \eta(X)$. An almost contact metric (a.ct.m.) structure

(ϕ, ξ, η, g) consists of an almost contact structure (ϕ, ξ, η) and a compatible metric g . Given an a.c.t.m. structure (ϕ, ξ, η, g) we set $\Omega(X, Y) := g(X, \phi Y)$. (ϕ, ξ, η, g) is a contact metric structure if $\Omega = d\eta$ (the *contact condition*). If this is the case $\eta \wedge \Omega^n$ is a volume form, hence η is a contact form on N . Each orientable real hypersurface N of a Hermitian manifold (M_0, J, g_0) carries a naturally induced a.c.t.m. structure. Indeed, let ν be a unit normal field on N . Then $\xi := J\nu$ is tangent to N . Moreover if $g := j^*g_0$, $\phi X := \tan(JX)$ and $\eta(X) := g(X, \xi)$, for $X \in T(N)$, then (ϕ, ξ, η, g) is an a.c.t.m. structure on N . An a.c.t.m. structure (ϕ, ξ, η) on N is normal if $N^{(1)} := [\phi, \phi] + 2(d\eta) \otimes \xi$ vanishes.

A Sasakian structure is a contact metric structure (ϕ, ξ, η, g) with $N^{(1)} = 0$. A Sasakian manifold is an odd dimensional manifold endowed with a fixed Sasakian structure. The underlying Riemannian metric is usually referred to as a Sasakian metric. Sasakian structures are characterized among a.c.t.m. structures by

$$(2.4) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

(cf. D. E. Blair, [4], p. 73). Here ∇ is the Levi-Civita connection of (N, g) . By the Ianuş theorem (cf. S. Ianuş, [13], or D. E. Blair, [4], p. 61) for each almost contact normal structure (ϕ, ξ, η) there is a naturally associated CR structure. Indeed $T_{1,0}(N) := \{X - i\phi X : X \in \text{Ker}(\eta)\}$ is an almost CR structure. Moreover, normality ($N^{(1)} = 0$) implies formal integrability.

Let (N, \mathcal{D}) be a contact CR submanifold of an a.c.t.m. manifold $(N_0, (\phi, \xi, \eta_0, g_0))$, tangent to the contact vector ($\xi \in T(N)$). Let ι and ι^\perp be the projections associated to the direct sum decomposition $T(N) = \mathcal{D} \oplus \mathcal{D}^\perp$. Next, for any $X \in T(N)$ and $V \in T(N)^\perp$ we set

$$PX = \tan(\phi X), \quad FX = \text{nor}(\phi X), \quad tV = \tan(\phi V), \quad fV = \text{nor}(\phi V).$$

Then

$$(2.5) \quad P^2 = -I - tF + \eta \otimes \xi, \quad FP + fF = 0, \quad f^2 = -I - Ft, \quad Pt + tf = 0.$$

If g is the induced metric ($g = i^*g_0$, $i : N \subset N_0$), then

$$(2.6) \quad g_0(FX, V) + g(X, tV) = 0.$$

We may state

THEOREM 8. *Let N be a submanifold of the a.c.t.m. manifold N_0 , tangent to the contact vector ($\xi \in T(N)$). Then N is a contact CR submanifold if and only if $FP = 0$. If this is the case $\mathcal{D}_1 := \text{Ker}(I + P^2 - \eta \otimes \xi)$ and $\mathcal{D}_2 := \text{Ker}(I + P^2)$ are ϕ -invariant and $\xi \in \mathcal{D}_1$, $\xi \in \mathcal{D}_2^\perp$.*

This was first proved by K. Yano & M. Kon (cf. [29]) though for a Sasakian ambient space only. The proof is *verbatim* repetition of the arguments in [29], p. 51.

Let us prove Theorem 1. To this end, let $H(N)$ be the orthogonal complement of $\mathbf{R}\iota\xi$ in \mathcal{D} , and $H(N)^\perp$ the orthogonal complement of $H(N)$ in $T(N)$.

Set $m := \dim_{\mathbf{R}} H(N)_x$ and $k := \dim_{\mathbf{R}} H(N)_x^\perp$, $x \in N$. Let $X \in H(N)$ and $Z \in H(N)^\perp$. As

$$(2.7) \quad H(N)^\perp = \mathcal{D}^\perp \oplus \mathbf{R}i\xi$$

it follows that $Z = W + \lambda i\xi$, for some $W \in \mathcal{D}^\perp$ and $\lambda \in C^\infty(N)$. Then (by $g(\mathcal{D}, \mathcal{D}^\perp) = 0$)

$$g(\phi X, Z) = \lambda g(\phi X, i\xi) = \lambda \eta(\phi X) = 0.$$

Here $\eta = i^*\eta_0$. Thus $\phi X \in H(N)$, that is $\phi H(N) \subseteq H(N)$. This proves that $H(N)$ is ϕ -invariant. To show that $H(N)^\perp$ is ϕ -anti-invariant, let $Z \in H(N)^\perp$ and $X \in T(N)$. Then, using the decomposition (2.7), $\phi W \in T(N)^\perp$, and

$$0 = \phi \xi = \phi i \xi + \phi i^\perp \xi \in \mathcal{D} \oplus T(N)^\perp \Rightarrow \phi i \xi = 0,$$

we have

$$g_0(\phi Z, X) = g_0(\phi(W + \lambda i\xi), X) = g_0(\phi W, X) = 0.$$

Therefore $\phi Z \in T(N)^\perp$, that is $\phi H(N)^\perp \subseteq T(N)^\perp$. Let $X \in H(N)$. Then

$$\phi^2 X = -X + \eta(X)\xi = -X$$

hence $\xi \in H(N)^\perp$. In particular, the real dimension of $H(N)_x$, $x \in N$, must be even, i.e. $m = 2n$. At this point we may check that $T_{1,0}(N) := [T(N) \otimes \mathbf{C}] \cap T_{1,0}(N_0)$ is a CR structure on N . Here $T_{1,0}(N_0) := \{A - i\phi A : A \in \text{Ker}(\eta_0)\}$ is the CR structure (of type $(l, 1)$, with $\dim N_0 = 2l + 1$) of N_0 . To this end, note that

$$(2.8) \quad T_{1,0}(N) = \text{Eigen}(J^{\mathbf{C}}, i).$$

Here $J^{\mathbf{C}}$ is the \mathbf{C} -linear extension of $J := \phi|_{H(N)}$ to $H(N) \otimes \mathbf{C}$. Indeed, let $X - iJX \in \text{Eigen}(J^{\mathbf{C}}, i)$, $X \in H(N)$. As $H(N) \subseteq \text{Ker}(\eta_0)$, it follows that $X - iJX \in T_{1,0}(N)$ and then

$$\text{Eigen}(J^{\mathbf{C}}, i) \subseteq [T(N) \otimes \mathbf{C}] \cap T_{1,0}(N_0).$$

To check the opposite inclusion, let $A - i\phi A \in T_{1,0}(N_0) \cap [T(N) \otimes \mathbf{C}]$. Then $A - i\phi A = X + iY$, for some $X, Y \in T(N)$. It follows that $A = X$ and $\phi A = -Y$, that is $A \in T(N) \cap \text{Ker}(\eta_0)$. Therefore

$$(2.9) \quad A \in (\mathbf{R}i\xi)^\perp = \text{Ker}(\eta) \subset T(N).$$

Here $(\mathbf{R}i\xi)^\perp$ is the orthogonal complement of $\mathbf{R}i\xi$ in $T(N)$. Let $E(N)$ be the orthogonal complement of $\mathbf{R}i\xi$ in $H(N)^\perp$, so that

$$H(N)^\perp = E(N) \oplus \mathbf{R}i\xi, \quad \mathcal{D}^\perp = E(N) \oplus \mathbf{R}i^\perp \xi.$$

Then

$$(2.10) \quad (\mathbf{R}i\xi)^\perp = H(N) \oplus E(N).$$

Now (2.9) and (2.10) yield $A = A_H + A_E$, for some $A_H \in H(N)$ and $A_E \in E(N)$. Therefore

$$-Y = \phi A = \phi A_H + \phi A_E$$

hence $\phi A_H = -Y$ and $\phi A_E = 0$ (because of $\phi A_E \in \phi \mathcal{D}^\perp \subseteq T(N)^\perp$). It follows that

$$0 = \phi^2 A_E = -A_E + \eta_0(A_E)\xi = -A_E.$$

Summing up, we have $A_E = 0$, that is $A = A_H \in H(N)$. Then $A - i\phi A \in \text{Eigen}(J^C, i)$ and (2.8) is proved. In particular (by (2.8)) $T_{1,0}(N)$ has constant rank n . Moreover, by the very definition of $T_{1,0}(N)$

$$T_{1,0}(N) \cap \overline{T_{1,0}(N)} \subseteq T_{1,0}(N_0) \cap T_{0,1}(N_0) = (0).$$

Both $T(N) \otimes \mathbf{C}$ and $T_{1,0}(N_0)$ are formally integrable and then so is $T_{1,0}(N)$. Q.e.d.

2.3. Real orientable hypersurfaces in a.ct.m. manifolds

As an application of Theorem 1 we look at real orientable hypersurfaces in an a.ct.m. manifold whose underlying almost contact structure is normal. We may state

THEOREM 9. *Let N_0 be a $(2n + 1)$ -dimensional a.ct.m. manifold, whith (ϕ, ξ, η_0) normal. Let N be a real orientable hypersurface in N_0 , tangent to the contact vector ($\xi \in T(N)$). Let v be a unit normal field on N . Then $U := \phi v$ is tangent to N . Let $H(N)$ be the orthogonal complement of $\mathbf{R}U \oplus \mathbf{R}\xi$ in $T(N)$. Then $(N, H(N))$ is a contact CR submanifold of N_0 . Consequently $T_{1,0}(N) := \{X - i\phi X : X \in H(N)\}$ is a CR structure of type $(n - 1, 2)$.*

Proof. Note that U and ξ are orthogonal. Next

$$\begin{aligned} g_0(U, v) &= g_0(\phi v, v) = g_0(\phi^2 v, \phi v) + \eta_0(\phi v)\eta_0(v) \\ &= g_0(-v + \eta_0(v)\xi, U) = -g_0(v, U), \end{aligned}$$

hence U is tangent to N . Let $X \in H(N)$. We have

$$g_0(\phi X, v) = -g_0(X, \phi v) = -g_0(X, U) = 0.$$

It follows that $\phi X \in T(N)$, i.e. $\phi H(N) \subseteq T(N)$. Moreover

$$\begin{aligned} g(\phi X, U) &= g_0(\phi X, \phi v) = g_0(X, v) - \eta_0(X)\eta_0(v) = 0, \\ g(\phi X, \xi) &= \eta(\phi X) = 0. \end{aligned}$$

Thus $\phi X \in H(N)$, that is $H(N)$ is ϕ -invariant. Now, let $Y \in H(N)^\perp = \mathbf{R}U \oplus \mathbf{R}\xi$. As $\phi\xi = 0$ and $\phi U = -v \in T(N)^\perp$ it follows that $\phi Y \in T(N)^\perp$, that is $H(N)^\perp$ is ϕ -anti-invariant. Q.e.d.

By Theorem 9, given a smooth domain $\Omega_0 \subset S^{2n+1}$, its boundary $N = \partial\Omega_0$ is a CR manifold of CR codimension 2, in contrast with domains in \mathbf{C}^n (where the CR codimension and the codimension of the boundary are both 1). A geometric study of CR manifolds of CR codimension 2 was begun by R. Mizner,

[17] (using Cartan connections); the case of real hypersurfaces in Sasakian manifolds is not considered. As to analysis on domains in a CR manifold, the subject starts with the *Hartogs-Bochner theorem* on a CR manifold, due to G. M. Henkin, [12]. He proves that, given a 1-concave generic CR submanifold $M \subset \mathbf{C}^n$, a bounded domain $\Omega \subset M$ and a smooth domain Ω_0 contained in Ω and such that $\Omega \setminus \Omega_0$ is connected, each CR function $f \in CR^\infty(\partial\Omega_0)$ extends to a function $F \in C^\infty(\overline{\Omega}_0) \cap CR^\infty(\Omega_0)$. The assumption that the diameter of Ω_0 is “sufficiently small” (adopted in [12]) was removed subsequently by C. Laurent-Thiébaud, [14]. The assumption that M is 1-concave concerns the Levi form and the geometry of the boundary $N = \partial\Omega_0$ is not studied. However, for a smooth domain $\Omega_0 \subset S^{2n+1}$, it is natural to expect that the geometry of the CR manifold $N = \partial\Omega_0$ is tied to the function theory on Ω_0 , and conversely. In the next section we look at the equations induced on N by the tangential Cauchy-Riemann equations of S^{2n+1} . Note that the *Hartogs-Bochner theorem* (cf. *op. cit.*) holds on $M = S^{2n+1} \subset \mathbf{C}^{n+1}$ (as the Levi form of M is positive definite). Finally, to underline the novelty of the situation, note that the pseudohermitian tools (e.g. the Tanaka-Webster connection, the Chern-Moser tensor, etc., cf. [21] and [24]) are not available in the CR codimension 2 case.

3. Traces of CR functions on contact CR submanifolds

3.1. Traces of CR functions

Let $(N_0, (\phi, \xi, \eta_0, g_0))$ be a $(2n+1)$ -dimensional a.ct.m. manifold with (ϕ, ξ, η_0) normal. Then $T_{1,0}(N_0) = \{X - i\phi X : X \in \text{Ker}(\eta_0)\}$ is a CR structure of type $(n, 1)$ on N_0 . For a CR manifold M , let $CR^k(M)$ be the space of all CR functions of class C^k , $k \in \mathbf{N} \cup \{\infty, \omega\}$, i.e. solutions f of class C^k to the tangential Cauchy-Riemann equations (1.4). Then

$$CR^k(N_0) = \{f \in C^k(N_0) : (X + i\phi X)f = 0, \forall X \in \text{Ker}(\eta_0)\}.$$

Let (N, \mathcal{D}) be a contact CR submanifold of N_0 . By the very definition $T_{1,0}(N) \subset T_{1,0}(N_0)$, hence the restriction $C^\infty(N_0) \rightarrow C^\infty(N)$ descends to a map $CR^\infty(N_0) \rightarrow CR^\infty(N)$. When N is a real hypersurface, for any $x \in N$ there is an open neighborhood $U \subseteq N_0$ and a real valued function $\rho \in C^\infty(U)$ such that $N \cap U = \{x \in U : \rho(x) = 0\}$. Let $\{T_\alpha\}$ be a (local) frame of $T_{1,0}(N_0)$ on U and set $V := \{x \in U : T_n(\rho)_x \neq 0\}$. Then the trace u on V of $f \in CR^\infty(N_0)$ is a solution to $T_j^-(u) - (T_j^-(\rho)/T_n^-(\rho))T_n^-(u) = 0$, $1 \leq j \leq n-1$. For instance, let $N_0 = \mathbf{H}_n = \mathbf{C}^n \times \mathbf{R}$ be the Heisenberg group with the CR structure spanned by $T_\alpha = \partial/\partial z^\alpha + i\bar{z}^\alpha(\partial/\partial t)$, $1 \leq \alpha \leq n$ (the Lewy operators). Consider the submanifold $N = \Sigma^{2n} := \{x \in \mathbf{H}_n : |x| = 1\}$, where $|x| := (|z|^4 + t^2)^{1/4}$ is the Heisenberg norm. \mathbf{H}_n is a Sasakian manifold in a natural way hence Theorem 8 applies. As

$$T_\alpha(\rho) = \frac{1}{2}|x|^{-3}\bar{z}^\alpha\bar{\phi}, \quad \phi(z, t) := |z|^2 - it \in CR^\infty(\mathbf{H}_n),$$

each $u \in CR^\infty(\Sigma^{2n})$ is a solution to $T_j^-(u) - (z^j/z^n)T_n^-(u) = 0$. Therefore, the tangential Cauchy-Riemann equations on $\Sigma^{2n} \cap \{z^n \neq 0\}$ read

$$\frac{\partial u}{\partial \bar{z}^j} - \frac{z^j}{z^n} \frac{\partial u}{\partial \bar{z}^n} = 0.$$

For a contact CR submanifold N of a Sasakian manifold N_0 , it is an open problem whether $CR^1(N_0) \rightarrow CR^1(N)$ is an epimorphism (the local version of the same problem is open, as well). A partial answer is provided by Theorem 2.

3.2. The Boothby-Wang fibration

Let N_0 be a Sasakian manifold. A local chart (U, φ) of N_0 is cubic (of side $2a$ and center $x_0 \in N_0$) if $\varphi(x_0) = 0$ and $\varphi(U) = \{(t^1, \dots, t^{2n+1}) : |t^j| < a, 1 \leq j \leq 2n+1\}$. Let $(U, \varphi = (x^1, \dots, x^{2n+1}))$ be a cubic chart of N_0 . Let $1 \leq p \leq 2n+1$ and $t = (t^{p+1}, \dots, t^{2n+1})$ such that $|t^{p+j}| < a, 1 \leq j \leq 2n-p+1$. Then $\Sigma_t := \{y \in U : x^{p+j}(y) = t^{p+j}, 1 \leq j \leq 2n-p+1\}$ is a p -dimensional slice of (U, φ) . The contact vector ξ is regular if N_0 admits a C^∞ atlas $\{(U, \varphi)\}$ such that the intersection of U with any maximal integral curve of ξ is a 1-dimensional slice of (U, φ) . By a result of R. Palais (cf. [20]) if ξ is regular the quotient space N_0/ξ admits a natural structure of a C^∞ manifold such that the projection $\pi_0 : N_0 \rightarrow M_0 := N_0/\xi$ is C^∞ . The contact vector of the standard sphere S^{2n+1} is regular. By a result of W. M. Boothby & H. C. Wang (cf. [7]), if N_0 is a compact Sasakian manifold with ξ regular, there is a free action of S^1 on N_0 making N_0 into a S^1 -principal bundle over $M_0 = N_0/\xi$, with projection $\pi_0 : N_0 \rightarrow M_0$. The contact vector ξ is tangent to the S^1 -orbits and η_0 is a connection 1-form. Moreover, the Sasakian structure of N_0 gives rise to a complex structure J and a Kählerian metric G_0 on M_0 given by

$$(JX)^\dagger = \phi X^\dagger, \quad G_0(X, Y)^v = g_0(X^\dagger, Y^\dagger), \quad X, Y \in T(M_0).$$

Here X^\dagger is the horizontal lift of X with respect to η_0 . Also, if $f \in C^\infty(M_0)$ we set $f^v := f \circ \pi_0$ (the vertical lift of f). Then $\pi_0 : (N_0, g_0) \rightarrow (M_0, G_0)$ is a Riemannian submersion and one may apply the results of B. O'Neill, [19]. In particular, if $h : T(N_0) \rightarrow H(N_0)$ is the projection associated to the direct sum decomposition $T(N_0) = H(N_0) \oplus \text{Ker}(d\pi_0)$ then $h\nabla_{X^\dagger}^0 Y^\dagger$ is π_0 -related to $\nabla_X^{M_0} Y$. Here ∇^0 and ∇^{M_0} are the Levi-Civita connections of (N_0, g_0) and (M_0, G_0) . Consequently (cf. K. Yano & M. Kon, [29], p. 100)

$$(3.1) \quad \nabla_{X^\dagger}^0 Y^\dagger = (\nabla_X^{M_0} Y)^\dagger - G_0(X, JY)^v \xi, \quad X, Y \in \mathcal{X}(M_0).$$

For the sake of completeness we give a short proof of (3.1). We have $\nabla_{X^\dagger}^0 Y^\dagger = A + \lambda \xi$, for some $A \in H(N_0)$ and $\lambda \in C^\infty(N_0)$. Then

$$(d\pi_0)A = (d\pi_0)\nabla_{X^\dagger}^0 Y^\dagger = \nabla_X^{M_0} Y = (d\pi_0)(\nabla_X^{M_0} Y)^\dagger$$

i.e. $A - (\nabla_X^{M_0} Y)^\dagger \in \text{Ker}(d\pi_0) \cap H(N_0) = (0)$. Recall that, on a Sasakian manifold $\nabla_{Z^\dagger}^0 \xi = -\phi Z$. On the other hand $g_0(Y^\dagger, \xi) = \eta_0(Y^\dagger) = 0$, hence

$$\begin{aligned} \lambda &= g_0(\nabla_{X^\dagger}^0 Y^\dagger, \xi) = X^\dagger(g_0(Y^\dagger, \xi)) - g_0(Y^\dagger, \nabla_{X^\dagger}^0 \xi) \\ &= g_0(Y^\dagger, \phi X^\dagger) = g_0(Y^\dagger, (JX)^\dagger) = G_0(Y, JX)^v = -G_0(X, JY)^v. \quad \text{Q.e.d.} \end{aligned}$$

3.3. Cauchy-Riemann fibrations

Let N_0 be a compact Sasakian manifold with ξ regular and $S^1 \rightarrow N_0 \rightarrow M_0 = N_0/\xi$ the corresponding Boothby-Wang fibration. Let $j : M \subset M_0$ be a submanifold and $N := \pi_0^{-1}(M)$, so that $S^1 \rightarrow N \xrightarrow{\pi} M$ is a principal subbundle of $S^1 \rightarrow N_0 \rightarrow M_0$ (with $\pi := \pi_0|_N$). We adopt the following notations: B (respectively h) and A_V (respectively a_W) are the second fundamental form of $j : M \subset M_0$ (respectively of $i : N \subset N_0$) and the Weingarten operator corresponding to the normal section $V \in \Gamma^\infty(\mathcal{V}(j))$, $\mathcal{V}(j) = T(M)^\perp$, (respectively to $W \in \Gamma^\infty(\mathcal{V}(i))$, $\mathcal{V}(i) = T(N)^\perp$). Moreover $g = i^*g_0$ and $G = j^*G_0$ are the induced metrics on N and M and ∇^\perp , $\nabla^{\perp, M}$ are the normal connections in $\mathcal{V}(i)$ and $\mathcal{V}(j)$. We also set $\eta = i^*\eta_0$ and $\mathcal{H} := \text{Ker}(\eta) = \text{Ker}(\eta_0) \cap T(N) = H(N_0) \cap T(N)$. \mathcal{H} is a connection-distribution in the principal bundle $S^1 \rightarrow N \rightarrow M$. Clearly ξ is tangent to N and spans $\text{Ker}(d\pi)$. Moreover

$$(3.2) \quad T(M)^\dagger = \mathcal{H}, \quad (T(M)^\perp)^\dagger = T(N)^\perp.$$

The Gauss formulae of $M \subset M_0$ and $N \subset N_0$

$$\nabla_X^{M_0} Y = \nabla_X^M Y + B(X, Y), \quad \nabla_{X^\dagger}^0 Y^\dagger = \nabla_{X^\dagger} Y^\dagger + h(X^\dagger, Y^\dagger),$$

together with (3.1), lead to

$$\begin{aligned} \nabla_{X^\dagger} Y^\dagger + h(X^\dagger, Y^\dagger) &= (\nabla_X^{M_0} Y)^\dagger - G_0(X, JY)^v \xi \\ &= (\nabla_X^M Y)^\dagger + B(X, Y)^\dagger - G(X, P_M Y)^v \xi \end{aligned}$$

where $P_M X := \tan(JX)$, $X \in T(M)$. We obtain

$$(3.3) \quad \nabla_{X^\dagger} Y^\dagger = (\nabla_X^M Y)^\dagger - G(X, P_M Y)^v \xi,$$

$$(3.4) \quad h(X^\dagger, Y^\dagger) = B(X, Y)^\dagger.$$

The Weingarten formulae of $M \subset M_0$ and $N \subset N_0$

$$\nabla_X^{M_0} V = -A_V X + \nabla_X^{\perp, M} V, \quad \nabla_{X^\dagger}^0 V^\dagger = -a_{V^\dagger} X^\dagger + \nabla_{X^\dagger}^\perp V^\dagger,$$

together with (3.1) lead to

$$\begin{aligned} -a_{V^\dagger} X^\dagger + \nabla_{X^\dagger}^\perp V^\dagger &= (\nabla_X^{M_0} V)^\dagger - G_0(X, JV)^v \xi \\ &= -(A_V X)^\dagger + (\nabla_X^{\perp, M} V)^\dagger - G(X, t_M V)^v \xi \end{aligned}$$

where $t_M V := \tan(JV)$, $V \in T(M)^\perp$. We obtain

$$(3.5) \quad a_{V^\dagger} X^\dagger = (A_V X)^\dagger + G(X, t_M V)^v \xi,$$

$$(3.6) \quad \nabla_{X^\dagger}^\perp V^\dagger = (\nabla_X^{\perp, M} V)^\dagger.$$

Set $F_M X := \text{nor}(JX)$ and $f_M V := \text{nor}(JV)$. Then

$$(3.7) \quad P X^\dagger = (P_M X)^\dagger, \quad F X^\dagger = (F_M X)^\dagger,$$

$$(3.8) \quad t V^\dagger = (t_M V)^\dagger, \quad f V^\dagger = (f_M V)^\dagger.$$

By a result of K. Yano & M. Kon (cf. Theor. 2.1 in [28], p. 375) N is a contact CR submanifold of N_0 if and only if M is a CR submanifold of M_0 . If this is the case, that is N is both a contact CR submanifold and the total space of a S^1 -principal bundle over a CR submanifold M , we call $S^1 \rightarrow N \rightarrow M$ a *Cauchy-Riemann (C-R) circle bundle*. Given a C-R circle bundle $S^1 \rightarrow N \rightarrow M$, the distribution $\mathcal{D} := \text{Ker}(\mathbf{I} + \mathbf{P}^2 - \eta \otimes \xi)$ is ϕ -invariant and \mathcal{D}^\perp (the orthogonal complement of \mathcal{D} in $T(N)$) is ϕ -anti-invariant. Also $H(M) := \text{Ker}(\mathbf{I} + \mathbf{P}_M^2)$ is J -invariant, while $H(M)^\perp$ (the orthogonal complement of $H(M)$ in $T(M)$) is J -anti-invariant.

Let us consider a foliated manifold (M, \mathcal{F}) , with the foliation \mathcal{F} . A subset $A \subseteq M$ is *saturated* if A is a union of leaves of \mathcal{F} , i.e. every leaf of \mathcal{F} intersecting A is contained in A . A smooth function $f : M \rightarrow \mathbf{C}$ is a *basic function* if $X(f) = 0$ for every $X \in T(\mathcal{F})$.

The total space N_0 of the Boothby-Wang fibration $S^1 \rightarrow N_0 \rightarrow M_0$ is a foliated manifold, in a natural way. Indeed, the vertical distribution $\text{Ker}(d\pi_0) = \mathbf{R}\xi$ is integrable and so, by the Frobenius theorem, there is a foliation \mathcal{V}_0 on N_0 tangent to $\text{Ker}(d\pi_0)$. The leaves of \mathcal{V}_0 are the maximal integral curves of the contact vector field ξ . If $S^1 \rightarrow N \rightarrow M$ is a C-R (circle) bundle, then N is a saturated subset of (N_0, \mathcal{V}_0) . Therefore \mathcal{V}_0 induces a foliation \mathcal{V} on N . We shall need the following lemma

LEMMA 1. *Let $S^1 \rightarrow N \rightarrow M$ be a C-R fibration. Consider the invariant distributions $\mathcal{D} = \text{Ker}(\mathbf{I} + \mathbf{P}^2 - \eta \otimes \xi)$ and $H(M) = \text{Ker}(\mathbf{I} + \mathbf{P}_M^2)$. Then (N, \mathcal{D}) is generic if and only if $(M, H(M))$ is generic.*

Proof. Note that $\xi \in \mathcal{D}$. Let $H(N)$ be the orthogonal complement of $\mathbf{R}\xi$ in \mathcal{D} , so that $\mathcal{D} = H(N) \oplus \mathbf{R}\xi$. Moreover $T(N) = \mathcal{D} \oplus \mathcal{D}^\perp$ and $T(N) = \mathcal{H} \oplus \mathbf{R}\xi$ imply

$$(3.9) \quad \mathcal{H} = H(N) \oplus \mathcal{D}^\perp.$$

Hence it is enough to prove that

$$(3.10) \quad H(M)^\dagger = H(N).$$

Indeed (3.10)–(3.9) yield $(H(M)^\perp)^\dagger = \mathcal{D}^\perp$ and lemma 1 is proved (as $(T(M)^\perp)^\dagger = T(N)^\perp$). Now we prove (3.10). Let $X \in H(M)$. Then

$$(\mathbf{I} + \mathbf{P}^2 - \eta \otimes \xi)X^\dagger = [(\mathbf{I} + \mathbf{P}_M^2)X]^\dagger - \eta(X^\dagger)\xi = 0$$

as $X \in \text{Ker}(\mathbf{I} + \mathbf{P}_M^2)$ and $X^\dagger \in H(M)^\dagger \subset T(M)^\dagger = \mathcal{H} = \text{Ker}(\eta)$. It follows that $X^\dagger \in \text{Ker}(\mathbf{I} + \mathbf{P}^2 - \eta \otimes \xi) = H(N)$ and so $H(M)^\dagger \subseteq H(N)$. The opposite inclusion is a bit more difficult. Let $\tilde{X} \in H(N) \subset \mathcal{H} = T(M)^\dagger$. Then $\tilde{X} = X^\dagger$ for some $X \in T(M)$ and

$$0 = (\mathbf{I} + \mathbf{P}^2 - \eta \otimes \xi)\tilde{X} = [(\mathbf{I} + \mathbf{P}_M^2)X]^\dagger$$

yields $X \in H(M)$, i.e. $\tilde{X} \in H(M)^\dagger$. Q.e.d.

Now we may prove Theorem 2. Let $S^1 \rightarrow N \rightarrow M$ be a C-R fibration over a compact, connected, real analytic CR submanifold. Let us assume that N is generic and proper. First, let us show that *if (N, \mathcal{D}) is a contact CR submanifold of an a.c.t.m. manifold $(N_0, (\phi, \xi, \eta_0, g_0))$ such that $\xi \in \mathcal{D}$, then $\mathcal{D} = \text{Ker}(\mathbf{I} + \mathbf{P}^2 - \eta \otimes \xi)$* . Indeed, let $X \in \mathcal{D}$ be orthogonal to ξ . Then

$$(\mathbf{I} + \mathbf{P}^2 - \eta \otimes \xi)X = (\mathbf{I} + \mathbf{P}^2)X = 0,$$

as \mathbf{P} is a complex structure in $H(N)$. Conversely, let us consider $X \in \text{Ker}(\mathbf{I} + \mathbf{P}^2 - \eta \otimes \xi)$, orthogonal to ξ . As $\mathbf{P}(T(N)) \subseteq \mathcal{D}$ and $\mathfrak{t}(T(N)^\perp) \subseteq \mathcal{D}^\perp$, one has $\iota = -\mathbf{P}^2 + \eta \otimes \xi$. Then

$$0 = (\mathbf{I} + \mathbf{P}^2 - \eta \otimes \xi)X = X - \iota X$$

hence $X \in \mathcal{D}$. Q.e.d.

Let (N, \mathcal{D}) be a contact CR submanifold of an almost contact manifold N_0 , tangent to ξ and generic. Then ξ is transverse to \mathcal{D}^\perp . Indeed, if there is $x_0 \in N$ such that $(\iota\xi)_{x_0} = 0$, then $H(N)_{x_0}^\perp = \mathcal{D}_{x_0}^\perp$ hence (by genericity) $k = \text{codim } N = \dim N_0 - (2n + k)$, a contradiction. It follows that, whenever N is generic, one has $(\iota\xi)_x \neq 0$ for any $x \in N$ and then the dimension of the ambient space must be $\dim N_0 = 2(n + k) - 1$ (as in the hypothesis of Theorem 2).

Lemma 1 implies that M is a generic proper CR submanifold of \mathbf{CP}^{n+k-1} . By a result of R. O. Wells, [26], there is a connected set $Q_M \subset \mathbf{CP}^{n+k-1}$ such that $Q_M \supset M$ and there is a submanifold $M' \subset \mathbf{CP}^{n+k-1}$ of dimension $2n + k$ such that $Q_M \supset M'$ and each holomorphic function on a neighborhood of M extends to a holomorphic function on a neighborhood of Q_M . Set $Q := \pi_0^{-1}(Q_M)$. Clearly Q is a saturated set (i.e. a union of leaves of the vertical foliation on $S^{2(n+k)-1}$) and $Q \supset N$, $Q \supset N'$, where $N' := \pi_0^{-1}(M')$. As N is locally diffeomorphic to $M \times S^1$ and M is real analytic, it follows that N is real analytic as well. Let $u \in CR^\omega(N)$ be a basic (i.e. constant along the leaves of \mathcal{V}) CR function. There is a C^ω function $\tilde{u} : M \rightarrow \mathbf{C}$ such that $\tilde{u} \circ \pi = u$, where $\pi := \pi_0|_N$. For any $v \in C^1(M)$ we have

$$(3.11) \quad \pi^* \bar{\partial}_M v = \bar{\partial}_N (v \circ \pi).$$

In particular $v \in CR^1(M)$ if and only if $v \circ \pi \in CR^1(N)$. Here $\bar{\partial}_M$ and $\bar{\partial}_N$ are, respectively, the tangential Cauchy-Riemann operators on M and N . We postpone the proof of (3.11). To end the proof of Theorem 2, note that (by (3.11)) $0 = \bar{\partial}_N u = \pi^* \bar{\partial}_M \tilde{u}$ hence $\tilde{u} \in CR^\omega(M)$. By the Tomassini theorem, there is an open neighborhood U_0 of M in \mathbf{CP}^{n+k-1} and there is a unique holomorphic function $f_0 \in \mathcal{O}(U_0)$ such that $f_0|_M = \tilde{u}$. By the aforementioned result of R. O. Wells (cf. [26]) there is a neighborhood \tilde{U} of Q_M in \mathbf{CP}^{n+k-1} and there is a holomorphic function $\tilde{f} \in \mathcal{O}(\tilde{U})$ which extends f_0 , i.e. $\tilde{f}|_{\tilde{U} \cap U_0} = f_0$. Set $U := \pi_0^{-1}(\tilde{U})$ and $f := \tilde{f} \circ \pi_0 \in C^\omega(U)$. Then U is open and saturated in $S^{2(n+k)-1}$ and $U \supset Q$. At this point, we need the identity

$$(3.12) \quad \pi_0^* \bar{\partial} v = \bar{\partial}_{S^{2(n+k)-1}} (v \circ \pi_0),$$

for any $v \in C^1(\mathbf{CP}^{n+k-1})$. Here $\bar{\partial}$ is the Cauchy-Riemann operator on \mathbf{CP}^{n+k-1} . By (3.12) one has $\bar{\partial}_{S^{2(n+k)-1}} f = \pi_0^* \bar{\partial} \tilde{f} = 0$, i.e. $f \in CR^\omega(U)$. Finally $f \circ i = \tilde{f} \circ \pi_0 \circ i = f \circ j \circ \pi = f_0 \circ j \circ \pi = \tilde{u} \circ \pi = u$, i.e. $f|_N = u$. It remains that we prove (3.11) and (3.12). If (M, \mathcal{D}) is a CR submanifold of a Hermitian manifold, the *tangential Cauchy-Riemann operator* is the differential operator $\bar{\partial}_M : C^\infty(M) \rightarrow \Omega^{0,1}(M)$ defined as follows. A complex valued 1-form $\omega \in \Omega^1(M) = \Gamma^\infty(T^*(M) \otimes \mathbf{C})$ is a $(0,1)$ -form if $T_{1,0}(M) \lrcorner \omega = 0$ and $\mathcal{D}^\perp \lrcorner \omega = 0$. Let $\bigwedge^{0,1}(M)$ be the vector bundle of all $(0,1)$ -forms on M and $\Omega^{0,1}(M) := \Gamma^\infty(\bigwedge^{0,1}(M))$. If $u : M \rightarrow \mathbf{C}$ is a C^1 function then $\bar{\partial}_M u$ is the unique $(0,1)$ -form on M coinciding with du on $T_{0,1}(M)$. Similar notions may be produced for the case of a contact CR submanifold (N, \mathcal{D}) of a Sasakian manifold. Of course, the basic difference (with respect to CR manifolds of hypersurface type) is that we deal with the higher CR codimension by using a fixed complement to the invariant distribution. As before, $\omega \in \Omega^1(N)$ is a $(0,1)$ -form if $T_{1,0}(N) \lrcorner \omega = 0$ and (with the notations of Theorem 1) $H(N)^\perp \lrcorner \omega = 0$. Then the tangential Cauchy-Riemann operator $\bar{\partial}_N : C^\infty(N) \rightarrow \Omega^{0,1}(N)$ is given by $(\bar{\partial}_N u) \bar{Z} = \bar{Z}(u)$, for every $Z \in T_{1,0}(N)$. Note that $\pi : N \rightarrow M$ is a CR map. Indeed, by (3.10)

$$T_{1,0}(M)^\dagger = \{X^\dagger - iJX^\dagger : X \in H(M)\} = T_{1,0}(N)$$

where $X^\dagger := (d_x \pi : \mathcal{H}_x \rightarrow T_{\pi(x)}(M))^{-1} X$. Then

$$\begin{aligned} \bar{\partial}_N(u \circ \pi)_x \bar{Z}_x &= \bar{Z}_x(u \circ \pi) = [(d_x \pi) \bar{Z}_x](u) \\ &= (\bar{\partial}_M u)_{\pi(x)}(d_x \pi) \bar{Z}_x = (\pi^* \bar{\partial}_M u)_x \bar{Z}_x, \end{aligned}$$

for any $u \in C^\infty(M)$, $x \in N$ and $Z \in \Gamma^\infty(T_{1,0}(N))$. Next

$$\begin{aligned} T(N) &= \mathcal{H} \oplus \mathbf{Ker}(d\pi) = T(M)^\dagger \oplus \mathbf{R}\xi \\ &= H(M)^\dagger \oplus [H(M)^\perp]^\dagger \oplus \mathbf{R}\xi = H(N) \oplus [H(M)^\perp]^\dagger \oplus \mathbf{R}\xi, \end{aligned}$$

yields

$$(3.13) \quad H(N)^\perp = [H(M)^\perp]^\dagger \oplus \mathbf{R}\xi$$

by (3.10) and the fact that the horizontal lift is a field of linear isometries (i.e. $G(X, Y)^v = g(X^\dagger, Y^\dagger)$). Then (by (3.13))

$$H(N)^\perp \lrcorner \pi^* \bar{\partial}_M u = (\bar{\partial}_M u)(d\pi)H(N)^\perp = (\bar{\partial}_M u)H(M)^\perp = 0$$

and (3.11) is proved. The proof of (3.12) is similar. If (z^1, \dots, z^{n+k-1}) are local complex coordinates in \mathbf{CP}^{n+k-1} then $\bar{\partial} v = Z_{\bar{j}}(v) d\bar{z}^j$, where $Z_{\bar{j}} = \partial/\partial \bar{z}^j$ and $v \in C^1(\mathbf{CP}^{n+k-1})$. The projection $\tilde{\pi}_0 : \mathbf{C}^{n+k} \setminus \{0\} \rightarrow \mathbf{CP}^{n+k-1}$ is holomorphic hence $\pi_0 : S^{2(n+k)-1} \rightarrow \mathbf{CP}^{n+k-1}$ is a CR map. Then

$$\bar{\partial}_{S^{2(n+k)-1}}(v \circ \pi_0) \bar{Z} = \bar{Z}(v \circ \pi_0) = [(d\pi_0) \bar{Z}](v) = \bar{Z}(\pi^{\bar{j}}) Z_{\bar{j}}(v)$$

where $\pi^{\bar{j}} := \bar{z}^j \circ \pi_0$. Finally $\bar{\partial}_{S^{2(n+k)-1}}(v \circ \pi_0) = Z_{\bar{j}}(v)(\pi_0^* d\bar{z}^j) = \pi_0^* \bar{\partial} v$ and Theorem 2 is completely proved.

R. O. Wells underlines (cf. [26]) the importance of the genericity property in the theory of the functions on a real submanifold of a complex manifold, and that there are some topological obstructions in order to have this property. Indeed, he proves (by essentially using the fact that the Euler-Poincarè characteristic $\chi(M)$ of a compact submanifold M is got by evaluating the Euler class of the normal bundle at the fundamental class of M) that a necessary condition in order to have genericity (for a compact orientable m -dimensional submanifold M , in a compact complex manifold V , homologous to zero in $H_m(V, \mathbf{Z})$) is that $\chi(M) = 0$. In the case of a contact CR submanifold N of a Sasakian manifold (N not necessarily tangent to the contact vector field) the genericity condition implies again $\chi(N) = 0$ (cf. Theorem 3). The proof does not require algebraic topology technics.

Let us prove Theorem 3. Let (N, \mathcal{D}) be a contact CR submanifold of a Sasakian manifold $(N_0, (\phi, \xi_0, \eta_0, g_0))$, under the assumptions of Theorem 3. As N is generic it follows that N_0 must be $(p + 2q)$ -dimensional, where $p = \dim_{\mathbf{R}} \mathcal{D}_x$ and $q = \dim_{\mathbf{R}} \mathcal{D}_x^\perp$, $x \in N$. Assume that $\xi_{x_0} = 0$, for some $x_0 \in N$. Then $\mathcal{D}_{x_0} \subseteq H(N_0)_{x_0} = \text{Ker}(\eta_0)_{x_0}$ (indeed, if $v \in \mathcal{D}_{x_0}$ then $g_{0,x_0}(v, \xi_{0,x_0}) = g_{x_0}(v, \xi_{x_0}) = 0$), hence ϕ_{x_0} descends to a complex structure on \mathcal{D}_{x_0} so that p must be even. Then N_0 is even dimensional, a contradiction. Therefore ξ is a smooth vector field on N , without zeros, so that $\chi(N) = 0$. Q.e.d.

4. Canonical connections on real orientable hypersurfaces of Sasakian manifolds

4.1. The Levi form

Let $(M, T_{1,0}(M))$ be a CR manifold of arbitrary type (n, k) . The *Levi form* of M is

$$L_x : T_{1,0}(M)_x \times T_{1,0}(M)_x \rightarrow [T_x(M) \otimes_{\mathbf{R}} \mathbf{C}] / [H(M)_x \otimes_{\mathbf{R}} \mathbf{C}],$$

$$L_x(v, w) := i\pi_x([V, \bar{W}]_x),$$

for $v, w \in T_{1,0}(M)_x$, $x \in M$, where $V, W \in \Gamma^\infty(T_{1,0}(M))$ are chosen such that $V_x = v$, $W_x = w$ and

$$\pi : T(M) \otimes \mathbf{C} \rightarrow [T(M) \otimes \mathbf{C}] / [H(M) \otimes \mathbf{C}]$$

is the projection. Then $(M, T_{1,0}(M))$ is *nondegenerate* if L is nondegenerate. Let us consider the conormal bundle

$$H(M)_x^\perp := \{\omega \in T_x^*(M) : \text{Ker}(\omega) \supseteq H(M)_x\}, \quad x \in M.$$

Assume M to be oriented. When $k = 1$, i.e. M is of hypersurface type, $H(M)^\perp$ is an oriented line bundle, hence trivial. Thus $H(M)^\perp$ admits global nowhere zero sections $\theta \in \Gamma^\infty(H(M)^\perp)$, each of which is referred to as a *pseudohermitian structure* on M . The Levi form may be recast as

$$L_\theta(V, \bar{W}) := -i(d\theta)(V, \bar{W}), \quad V, W \in T_{1,0}(M),$$

and it is easily seen that L and L_θ coincide up to a bundle isomorphism $[T(M) \otimes \mathbf{C}]/[H(M) \otimes \mathbf{C}] \approx H(M)^\perp$. If L_θ is nondegenerate then θ is a *contact form*, i.e. $\theta \wedge (d\theta)^n$ is a volume form on M .

If M is a CR manifold of type (n, k) and CR codimension $k \geq 2$, the conormal bundle $H(M)^\perp$ has rank k and, in general, the orientability assumption doesn't guarantee triviality (the topological obstructions towards $H(M)^\perp \approx M \times \mathbf{R}^k$ were not investigated). Given a real orientable hypersurface N in a Sasakian manifold N_0 (by our Theorem 9) N is a CR manifold. Let L be its Levi form. N has CR codimension $k = 2$, hence L may not be computed in terms of a pseudohermitian structure. However the conormal bundle $H(N)^\perp$ admits the global frame $\{\eta, u\}$, where $\eta = i^* \eta_0$ and $u(X) := g(X, U)$, $X \in T(N)$, hence $H(N)^\perp \approx N \times \mathbf{R}^2$. We shall need the bundle isomorphism

$$\Phi : [T(N) \otimes \mathbf{C}]/[H(N) \otimes \mathbf{C}] \rightarrow \mathbf{R}U \oplus \mathbf{R}\xi,$$

$$\pi_x(v) \mapsto \eta_x(v)\xi_x + u_x(v)U_x, \quad v \in T_x(N) \otimes_{\mathbf{R}} \mathbf{C}, \quad x \in N.$$

Then $H(N) \rfloor \eta = 0$ and $H(N) \rfloor u = 0$ yield

$$(4.1) \quad \Phi_x(L_x(v, w)) = -2i\{(d\eta)(V, \overline{W})_x \xi_x + (du)(V, \overline{W})_x U_x\}.$$

Next, as N_0 satisfies the contact condition

$$(4.2) \quad (d\eta)(X, Y) = g(X, \mathbf{P}Y),$$

for any $X, Y \in T(N)$. Also

$$\begin{aligned} 2(du)(X, Y) &= (\nabla_X u)Y - (\nabla_Y u)X, \\ (\nabla_X u)Y &= g_0(\nabla_X^0 U, Y). \end{aligned}$$

Using the well known characterization of Sasakian structures (cf. e.g. D. E. Blair, [4], p. 73)

$$(4.3) \quad (\nabla_X^0 \phi)Y = g_0(X, Y)\xi - \eta_0(Y)X, \quad X, Y \in T(N_0),$$

and the Weingarten formula, we have

$$\begin{aligned} \nabla_X^0 U &= \nabla_X^0 \phi v = \phi \nabla_X^0 v + g_0(X, v)\xi - \eta_0(v)X \\ &= \phi \nabla_X^0 v = \phi\{-aX + \nabla_X^\perp v\} = -\phi aX, \end{aligned}$$

(as $\nabla^\perp v = 0$). Summing up, on any real orientable hypersurface (of a Sasakian manifold) tangent to the contact vector field

$$(4.4) \quad \nabla_X^0 U = -\phi aX,$$

so that

$$(4.5) \quad (du)(X, Y) = \frac{1}{2}g(X, (a\mathbf{P} + \mathbf{P}a)Y),$$

for any $X, Y \in T(N)$. We may define

$$\mathcal{L}(X, Y) := g(X, Y)\xi + \frac{1}{2}\{g(aX, Y) + g(a\mathbf{P}X, \mathbf{P}Y)\}U$$

for $X, Y \in H(N)$. By the identities (4.1)–(4.2) and (4.5) it follows that

$$\Phi_x(L_x(v, w)) = -2\mathcal{L}(V, \overline{W})_x$$

(and \mathcal{L} is referred to as the Levi form of N , as well). Note that \mathcal{L} is nondegenerate as its component along ξ is precisely the first fundamental form (restricted to $H(N)$).

If M is a CR manifold of hypersurface type, then M is *strictly pseudoconvex* if L_θ is positive definite, for some pseudohermitian structure θ . On the other hand, for a real orientable hypersurface N in a Sasakian manifold, \mathcal{L} is a *vector valued* form, hence there is no obvious analog to strict pseudoconvexity.

4.2. Canonical connections

Let us prove Theorem 4. Given a linear connection D on N we say its torsion T_D is *pure* if

$$T_D(Z, W) = 0, \quad T_D(Z, \overline{W}) = -2i\mathcal{L}(Z, \overline{W}), \quad Z, \overline{W} \in T_{1,0}(N),$$

where \mathcal{L} is the Levi form of N , and

$$\tau \circ J + J \circ \tau = 0, \quad \rho \circ J + J \circ \rho = 0,$$

where τ, ρ are the vector valued 1-forms on N given by

$$\tau X := T_D(\xi, X), \quad \rho X := T_D(U, \xi), \quad X \in T(N).$$

We establish the following more general statement.

PROPOSITION 1. *For any $\lambda \in \Omega^1(N)$ there is a unique linear connection D on N satisfying the following axioms 1) $H(N)$ is parallel which respect to \mathcal{D} , 2) $Dg = 0$, where $g = i^*g_0$ is the first fundamental form of $i : N \subset N_0$, 3) $DJ = 0$, where $J = \phi|_{H(N)}$ is the complex structure of $H(N)$, 4) the torsion T_D of D is pure, and 5) $D\xi = \lambda \otimes U$ and $DU = -\lambda \otimes \xi$.*

Note first that $J = \phi$ along $H(N)$ and let us extend J to the whole of $T(N)$ by setting $JU := 0$ and $J\xi := 0$. Then $J = P$ on $T(N)$. Also, note that $T(N) = H(N) \oplus \mathbf{R}U \oplus \mathbf{R}\xi$ and $T_{1,0}(N) = \{X - i\phi X : X \in H(N)\}$. At this point, we may check the uniqueness of a linear connection D obeying to the axioms 1)–5) in Proposition 1. By axiom 2)

$$(4.6) \quad X(g(Y, Z)) = g(D_X Y, Z) + g(Y, D_X Z).$$

Let us set $Y = \xi$ and use $g(X, \xi) = \eta(X)$. Then

$$(4.7) \quad X(\eta(Z)) = g(D_X \xi, Z) + \eta(D_X Z).$$

Choose $Z \in H(N)$. Then (by $D_X Z \in H(N)$)

$$g(D_X \xi, Z) = 0,$$

i.e. the component of $D_X \xi$ along $H(N)$ vanishes. Now we set $Z = \xi$ in (4.7) so that (by $\eta(\xi) = 1$)

$$2\eta(D_X\xi) = 0,$$

i.e. the component of $D_X\xi$ along ξ vanishes. We may conclude that

$$(4.8) \quad D_X\xi \in \mathbf{R}U, \quad X \in T(N).$$

Set $Y = U$ in (4.6). Then

$$(4.9) \quad X(u(Z)) = g(D_XU, Z) + u(D_XZ).$$

Next, for $Z \in H(N)$ (due to $H(N) \lrcorner u = 0$)

$$g(D_XU, Z) = 0,$$

i.e. the component of D_XU along $H(N)$ vanishes. Set $Z = U$ in (4.9) so that (by $u(U) = 1$)

$$2u(D_XU) = 0,$$

i.e. the component of D_XU along U vanishes. Summing up

$$(4.10) \quad D_XU \in \mathbf{R}\xi, \quad X \in T(N).$$

Let $Z, W \in T_{1,0}(N)$. By axiom 3)

$$T_D(\bar{Z}, W) = 2i\{g(\bar{Z}, W)\xi + g(a\bar{Z}, W)U\},$$

hence

$$D_{\bar{Z}}W - D_W\bar{Z} - [\bar{Z}, W] = 2i\{g(\bar{Z}, W)\xi + g(a\bar{Z}, W)U\}.$$

It follows that

$$(4.11) \quad D_{\bar{Z}}W = [\bar{Z}, W]_{T_{1,0}(N)}, \quad D_W\bar{Z} = -[\bar{Z}, W]_{T_{0,1}(N)},$$

where $V_{T_{1,0}(N)}$ and $V_{T_{0,1}(N)}$ are the projections of a $V \in T(N) \otimes \mathbf{C}$ on $T_{1,0}(N)$ and $T_{0,1}(N)$, respectively. If $V \in T_{1,0}(N)$

$$g(D_ZW, \bar{V}) = Z(g(W, \bar{V})) - g(W, D_Z\bar{V}),$$

and by (4.11)

$$(4.12) \quad g(D_ZW, \bar{V}) = Z(g(W, \bar{V})) + g(W, [\bar{V}, Z]_{T_{0,1}(N)}).$$

Set $\lambda(X) := u(D_X\xi)$. As $g(\xi, U) = 0$ and $u(D_X\xi) = -\eta(D_XU)$ we get (by (4.8) and (4.10))

$$(4.13) \quad D_X\xi = \lambda(X)U, \quad D_XU = -\lambda(X)\xi,$$

for any $X \in T(N)$. Next, for $X \in T(N)$

$$(4.14) \quad \tau X = D_\xi X - \lambda(X)U - [\xi, X].$$

Applying J to (4.14) we have

$$J\tau X = JD_\xi X - J[\xi, X],$$

and (replacing X by JX)

$$\tau JX = D_\xi JX - \lambda(JX)U - [\xi, JX].$$

Adding up the last two identities and using $\tau \circ J + J \circ \tau = 0$ and $DJ = 0$

$$0 = 2JD_\xi X - J\mathcal{L}_\xi X - \mathcal{L}_\xi JX - \lambda(JX)U,$$

where \mathcal{L}_X denotes the Lie derivative in the direction X . If $X \in H(N)$

$$(4.15) \quad 2D_\xi X = -J^2\mathcal{L}_\xi X - J\mathcal{L}_\xi JX, \quad X \in H(N).$$

Indeed, for any $X \in T(N)$

$$X = X_H + \eta(X)\xi + u(X)U$$

for some $X_H \in H(N)$. We have $JX = JX_H = \phi X_H$ so that $J^2X = \phi^2 X_H = -X_H$. Therefore

$$J^2X = -X + \eta(X)\xi + u(X)U.$$

Let us now observe that (by (4.2))

$$\eta(\mathcal{L}_\xi X) = \eta([\xi, X]) = -2(d\eta)(\xi, X) = -2g(\xi, \mathbf{P}X) = 0,$$

i.e. $\mathcal{L}_\xi X \in H(N) \oplus \mathbf{R}U$. Moreover (by (4.5)) for $X \in H(N)$ and $Y = \xi$ one has $2(du)(X, \xi) = g(X, \mathbf{P}a\xi)$ so that

$$u([\xi, X]) = -2(du)(\xi, X) = g(X, \mathbf{P}a\xi)$$

i.e. $u(\mathcal{L}_\xi X) = g(\mathbf{P}a\xi, X)$. Then

$$J^2\mathcal{L}_\xi X = -\mathcal{L}_\xi X + g(\mathbf{P}a\xi, X)U$$

which together with (4.15) leads to

$$2D_\xi X = \mathcal{L}_\xi X - g(\mathbf{P}a\xi, X)U - J\mathcal{L}_\xi JX.$$

Note that for $X \in H(N)$ one has $(\mathcal{L}_\xi J)JX = -\mathcal{L}_\xi X - J\mathcal{L}_\xi JX$. Then our last identity may be also written

$$(4.16) \quad D_\xi X = \mathcal{L}_\xi X + \frac{1}{2}(\mathcal{L}_\xi J)JX - \frac{1}{2}g(\mathbf{P}a\xi, X)U,$$

for any $X \in H(N)$. On the other hand, by the Gauss equation of N in N_0 and using $\nabla_V^0 \xi = -\phi V$ (for any $V \in T(N_0)$) we obtain

$$\nabla_X \xi = -\mathbf{P}X, \quad g(aX, \xi) = -\mathbf{F}X.$$

Yet

$$(4.17) \quad \mathbf{F}X = -u(X)v,$$

so that

$$(4.18) \quad \nabla_X \xi = -\mathbf{P}X, \quad \eta(aX) = u(X).$$

Let us observe that

$$\begin{cases} aX \in H(N) \oplus \mathbf{R}U, & X \in H(N) \\ aU = \xi + Y, & Y \in H(N) \oplus \mathbf{R}U \\ a\xi \in H(N) \oplus \mathbf{R}U \end{cases}$$

Then $g(Pa\xi, X) = 0$. Consider

$$T_\xi : H(N) \rightarrow H(N), \quad T_\xi := \frac{1}{2}(\mathcal{L}_\xi J)JX, \quad X \in H(N).$$

The identity (4.16) becomes

$$(4.19) \quad D_\xi X = \mathcal{L}_\xi X + T_\xi X,$$

for any $X \in H(N)$. For $X \in T(N)$ (by (4.13))

$$(4.20) \quad \rho X = D_U X + \lambda(X)\xi - [U, X].$$

By applying J to (4.20) we obtain

$$J\rho X = JD_U X - J[U, X],$$

hence (replacing X by JX)

$$\rho JX = D_U JX + \lambda(JX)\xi - [U, JX].$$

Adding up the last two identities and using $\rho \circ J + J \circ \rho = 0$ and $DJ = 0$ we obtain

$$0 = 2JD_U X - J\mathcal{L}_U X - \mathcal{L}_U JX + \lambda(JX)\xi.$$

If $X \in H(N)$ (as $H(N)$ is D -parallel)

$$(4.21) \quad 2D_U X = -J^2 \mathcal{L}_U X - J\mathcal{L}_U JX, \quad X \in H(N).$$

Now (by (4.2))

$$\eta(\mathcal{L}_U X) = \eta([U, X]) = -2(d\eta)(U, X) = -2g(U, PX) = 0,$$

i.e. $\mathcal{L}_U X \in H(N) \oplus \mathbf{R}U$. Moreover (by (4.5))

$$u(\mathcal{L}_U X) = u([U, X]) = -2(du)(U, X) = -g(U, (aP + Pa)X),$$

i.e. $u(\mathcal{L}_U X) = -u(aPX)$ (as $PU = \tan\{\phi U\} = -\tan\{v\} = 0$). Then

$$J^2 \mathcal{L}_U X = -\mathcal{L}_U X - u(aPX)U,$$

which together with (4.21) leads to

$$2D_U X = \mathcal{L}_U X - J\mathcal{L}_U JX + u(aPX)U.$$

Note that for $X \in H(N)$ one has $(\mathcal{L}_U J)JX = -\mathcal{L}_U X - J\mathcal{L}_U JX$. Let us consider

$$T_U : H(N) \rightarrow H(N) \oplus \mathbf{R}U, \quad T_U X := \frac{1}{2}(\mathcal{L}_U J)JX, \quad X \in H(N).$$

Then

$$(4.22) \quad D_U X = \mathcal{L}_U X + T_U X + \frac{1}{2}u(aPX)U.$$

Uniqueness is proved. We proceed by proving the existence of a connection D obeying to 1)–5). To this end, we consider the differential operator

$$D_X : \Gamma^\infty(T(N)) \rightarrow \Gamma^\infty(T(N)), \quad X \in \mathcal{X}(N),$$

defined by (4.11), (4.12), (4.13), (4.19) and (4.22), for a fixed $\lambda \in \Omega^1(N)$. It may be easily checked that D is a linear connection on N . The only property requesting a bit of care is $D_X fY = X(f)Y + fD_X Y$. Let $f \in C^\infty(N)$. Then for any $Z, W \in T_{1,0}(N)$

$$D_{f\bar{Z}}W = [f\bar{Z}, W]_{T_{1,0}(N)} = (f[\bar{Z}, W] - W(f)\bar{Z})_{T_{1,0}(N)} = fD_{\bar{Z}}W,$$

(as $\bar{Z}_{T_{1,0}(N)} = 0$). Also

$$D_\xi(fX) = T_\xi(fX) + \mathcal{L}_\xi(fX) = fD_\xi X + \xi(f)X,$$

$$D_U(fX) = T_U(fX) + \mathcal{L}_U(fX) + \frac{1}{2}u(aP(fX)) = fD_U X + U(f)X,$$

etc.

4.3. The torsions τ and ρ

We establish a few useful properties of τ and ρ , when $\lambda \equiv 0$ (from now D is the connection furnished by Theorem 4). Let $X \in T(N)$ and $Z \in H(N)$. By (4.3)

$$(\nabla_X^0 \phi)Z = g(X, Z)\xi.$$

By the Gauss formula (of N in N_0)

$$\begin{aligned} g(X, Z)\xi &= (\nabla_X^0 \phi)Z = \nabla_X^0 PZ - \phi \nabla_X^0 Z \\ &= \nabla_X PZ + g(aX, PZ)v - \phi\{\nabla_X Z + g(aX, Z)v\} \\ &= \nabla_X PZ + g(aX, PZ)v - P\nabla_X Z - F\nabla_X Z - g(aX, Z)U, \end{aligned}$$

where from (by identifying the tangential components)

$$(4.23) \quad \nabla_X PZ = P\nabla_X Z + g(X, Z)\xi + g(aX, Z)U,$$

for any $X \in T(N)$, $Z \in H(N)$, and (by identifying the normal components)

$$g(aX, PZ)v - F\nabla_X Z = 0.$$

Next (by $FX = -u(X)v$, for any $X \in T(N)$) we obtain

$$(4.24) \quad g(aX, PZ) + u(\nabla_X Z) = 0,$$

for any $X \in T(N)$, $Z \in H(N)$.

Let $X \in H(N)$. By (4.19) and (4.22)

$$\tau X = T_\xi X, \quad \rho X = T_U X + \frac{1}{2}u(aPX)U.$$

On the other hand $T_D(\xi, U) = -[\xi, U]$ so that

$$\tau U = -\rho\xi = -[\xi, U].$$

By (4.5)

$$u([\xi, U]) = -2(du)(\xi, U) = -g(\xi, (aP + Pa)U) = 0,$$

while by (4.2)

$$\eta([\xi, U]) = -2(d\eta)(\xi, U) = -2g(\xi, PU) = 0.$$

Finally, for $Y \in H(N)$ (by using (4.4) and (4.18))

$$g([\xi, U], Y) = g(\nabla_\xi U, Y) - g(\nabla_U \xi, Y) = -g(\phi a \xi, Y) + g(PU, X) = 0,$$

i.e. the component of $[\xi, U]$ along $H(N)$ vanishes. Then $[\xi, U] = 0$ so that $\tau U = \rho \xi = 0$. Summing up, the vector valued 1-forms τ and ρ satisfy

$$(4.25) \quad \begin{cases} \tau X = T_\xi X, & X \in H(N), \\ \tau \xi = \tau U = 0 \end{cases}$$

$$(4.26) \quad \begin{cases} \rho X = T_U X + \frac{1}{2}u(aPX)U, & X \in H(N), \\ \rho \xi = \rho U = 0 \end{cases}.$$

We also establish

LEMMA 2. For every $X, Y \in H(N)$

$$(4.27) \quad g(T_\xi X, Y) = g(X, T_\xi Y), \quad g(T_U X, Y) = g(X, T_U Y).$$

The proof is a straightforward calculation based on (4.4), (4.18) and (4.23). Using $g(u(aPX)U, Y) = 0$, for any $X, Y \in H(N)$, and the previous lemma we obtain

COROLLARY 1. τ, ρ are self adjoint along the maximally complex distribution, i.e.

$$(4.28) \quad g(\tau X, Y) = g(X, \tau Y), \quad g(\rho X, Y) = g(X, \rho Y).$$

for any $X, Y \in H(N)$.

Note that $\eta(\mathcal{L}_\xi X) = 0$, $u(\mathcal{L}_\xi X) = 0$, for any $X \in H(N)$, i.e. $\mathcal{L}_\xi H(N) \subseteq H(N)$ so that $\tau H(N) \subseteq H(N)$. On the other hand (by $\eta(\mathcal{L}_U X) = 0$ and $u(\mathcal{L}_U X) = -u(aPX)$, for any $X \in H(N)$) one has $u(\rho X) = u(aPX)$ and then $u(T_U X) = \frac{1}{2}u(aPX)$, for $X \in H(N)$, i.e. $\rho H(N) \subseteq H(N) \oplus \mathbf{R}U$. Moreover, if $X, Y \in T(N)$, by the uniqueness of the direct sum decomposition $T(N) = H(N) \oplus \mathbf{R}U \oplus \mathbf{R}\xi$

$$\begin{aligned} g(\rho X, Y) &= g(\rho X_H, Y) \\ &= g(\rho X_H, Y_H) + \eta(Y)g(\rho X_H, \xi) + u(Y)g(\rho X_H, U) \\ &= g(X_H, \rho Y_H) + u(Y)u(aPX_H) \\ &= g(X, \rho Y_H) - \eta(X)g(\xi, \rho Y_H) - u(X)g(U, \rho Y_H) + u(Y)u(aPX_H) \end{aligned}$$

where X_H is the $H(N)$ -component of X . Summing up

COROLLARY 2. For any $X, Y \in T(N)$

$$g(\rho X, Y) = g(X, \rho Y) - u(X)u(aPY) + u(Y)u(aPX)$$

and

$$g(\tau X, Y) = g(X, \tau Y),$$

i.e. τ is self-adjoint [as an endomorphism of $(T(N), g)$].

We proceed by deriving an useful relation between the canonical connection D and the Levi-Civita connection ∇ of N . As both D and ∇ are metric connections

$$(4.29) \quad \begin{aligned} 2g(D_X Y, Z) &= 2g(\nabla_X Y, Z) + g(T_D(X, Y), Z) \\ &\quad + g(T_D(Z, X), Y) + g(T_D(Z, Y), X) \end{aligned}$$

holds for any $X, Y, Z \in T(N)$. By the purity axiom

$$(4.30) \quad T_D(X, Y) = 2g(X, PY)\xi - \{g(aPX, Y) - g(aX, PY)\}U,$$

for any $X, Y \in H(N)$. Also, for $Z \in H(N)$ and $X, Y \in T(N)$

$$T_D(Z, X) = T_D(Z, X_H) + \eta(X)T_D(Z, \xi) + u(X)T_D(Z, U),$$

where from

$$T_D(Z, X) = 2g(Z, PX_H)\xi - \{g(aPZ, X_H) - g(aZ, PX_H)\}U - \eta(X)\tau Z - u(X)\rho Z.$$

The torsion expression together with (4.29) leads to

$$(4.31) \quad \begin{aligned} 2g(D_X Y, Z) &= 2g(\nabla_X Y, Z) + g(T_D(X, Y), Z) \\ &\quad + 2g(Z, PX_H)\eta(Y) + 2g(Z, PY_H)\eta(X) \\ &\quad - \{g(aPZ, X_H) - g(aZ, PX_H)\}u(Y) \\ &\quad - \{g(aPZ, Y_H) - g(aZ, PY_H)\}u(X) \\ &\quad - \eta(X)g(\tau Z, Y) - u(X)g(\rho Z, Y) \\ &\quad - \eta(Y)g(\tau Z, X) - u(Y)g(\rho Z, X), \end{aligned}$$

for any $X, Y \in T(N)$, $Z \in H(N)$. If $Y \in H(N)$ then (by $D_X Y \in H(N)$)

$$(4.32) \quad \begin{aligned} 2D_X Y &= 2(\nabla_X Y)_H + (T_D(X, Y))_H + u(X)\{(aP + Pa)Y\}_H \\ &\quad + 2\eta(X)PY - \eta(X)\tau Y - u(X)(\rho Y)_H. \end{aligned}$$

Now, since $X = X_H + \eta(X)\xi + u(X)U \in T(N)$

$$(T_D(X, Y))_H = \{T_D(X_H, Y) + \eta(X)\tau Y + u(X)\rho Y\}_H$$

and (by $T_D(X_H, Y) \in \mathbf{R}U \oplus \mathbf{R}\xi$)

$$(T_D(X, Y))_H = \eta(X)\tau Y + u(X)(\rho Y)_H,$$

where from

$$(4.33) \quad 2D_X Y = 2(\nabla_X Y)_H + u(X)\{(aP + Pa)Y\}_H + 2\eta(X)PY.$$

On the other hand, due to $u((aP + Pa)Y) = u(aPY)$ and $\eta((aP + Pa)Y) = 0$

$$\{(aP + Pa)Y\}_H = (aP + Pa)Y - u(aPY)U$$

Moreover (by (4.23))

$$\eta(\nabla_X Y) = X(g(Y, \xi)) - g(Y, \nabla_X \xi) = -g(X, PY)$$

and (by (4.2))

$$u(\nabla_X Y) = X(g(Y, U)) - g(Y, \nabla_X U) = -g(aX, PY)$$

so that

$$(\nabla_X Y)_H = \nabla_X Y + g(X, PY)\xi + g(aX, PY)U.$$

Let us substitute in (4.33) so that to obtain the identity

$$(4.34) \quad D_X Y = \nabla_X Y + g(X, PY)\xi + \frac{1}{2}\{2g(aX, PY) - u(X)u(aPY)\}U \\ + \frac{1}{2}\{2\eta(X)PY + u(X)(aP + Pa)Y\},$$

for any $X \in T(N)$, $Y \in H(N)$. When $X \in H(N)$ (respectively $X = U$, or $X = \xi$) we obtain

$$(4.35) \quad D_X Y = \nabla_X Y + g(X, PY)\xi + g(aX, PY)U,$$

$$(4.36) \quad D_U Y = \nabla_U Y + \frac{1}{2}\{u(aPY)U + (aP + Pa)Y\},$$

$$(4.37) \quad D_\xi Y = \nabla_\xi Y + PY,$$

for any $X, Y \in H(N)$.

5. Applications

5.1. A curvature formula

Let us consider a C-R fibration over a real orientable hypersurface $M \subset \mathbf{CP}^n$, i.e.

$$\begin{array}{ccccc} S^1 & \longrightarrow & N^{2n} & \xrightarrow{i} & S^{2n+1} & \longleftarrow & S^1 \\ & & \downarrow \pi & & \downarrow \pi_0 & & \\ & & M^{2n-1} & \xrightarrow{j} & \mathbf{CP}^n & & \end{array}$$

under the further assumption that the natural almost contact structure of M (as real hypersurface of \mathbf{CP}^n) satisfies the *contact condition* $d\eta_M = \Omega_M$, where

$\eta_M(X) := G(X, J_{\mathbb{C}\mathbb{P}^n} \nu_M)$ is the contact form, ν_M is a (globally defined) unit normal field on M , and $J_{\mathbb{C}\mathbb{P}^n}$ is the complex structure of $\mathbb{C}\mathbb{P}^n$. Also, we set $\Omega_M(X, Y) = G(X, P_M Y)$, for $X, Y \in T(M)$. In general, if $(M, T_{1,0}(M))$ is a CR manifold of hypersurface type and if $\theta \in \Gamma^\infty(H(M)^\perp)$ is a pseudohermitian structure, then the *real Levi form* is

$$G_\theta(X, Y) := (d\theta)(X, JY), \quad X, Y \in H(M).$$

As $G_\theta(JX, JY) = G_\theta(X, Y)$ (as consequence of the formal integrability of $T_{1,0}(M)$) it follows that L_θ and the (\mathbb{C} -linear) extension of G_θ actually coincide (on $T_{1,0}(M) \otimes T_{0,1}(M)$). If M is nondegenerate there is (cf. e.g. S. Dragomir, [9]) a globally defined nowhere zero vector field $T \in \mathcal{X}(M)$, transverse to the Levi distribution $H(M)$, uniquely determined by $\theta(T) = 1$, $T \lrcorner d\theta = 0$. T is the *characteristic direction* of $d\theta$. The *Webster metric* g_θ is

$$g_\theta = G_\theta \quad \text{on } H(M) \otimes H(M),$$

$$g_\theta(X, T) = \theta(X), \quad X \in T(M).$$

g_θ is a semi-Riemannian metric on M (of signature $(2r+1, 2s)$, where (r, s) is the signature of the Levi form L_θ). If M is strictly pseudoconvex and L_θ is positive definite, then g_θ is a Riemannian metric on M and (J, T, θ, g_θ) is a contact metric structure (which is normal if and only if the Tanaka-Webster connection of (M, θ) has a vanishing pseudohermitian torsion, cf. [9]). For the given C-R fibration $S^1 \rightarrow N \rightarrow M$ the 1-form $\theta := -\eta_M$ is a pseudohermitian structure on M . Moreover, for $X, Y \in H(M)$

$$G_\theta(X, Y) = (d\theta)(X, J_M Y) = -(d\eta_M)(X, J_M Y) = -\Omega_M(X, J_M Y),$$

hence $G_\theta = G$ on $H(M) \otimes H(M)$. In particular M is strictly pseudoconvex. Let T the characteristic direction of $d\theta$. In general, given an a.ct.m. structure (ϕ, ξ, η, g) , the underlying contact field is not characteristic for $d\eta$. However, in the case at hand, if M satisfies the contact condition then

LEMMA 3.

$$T = -J_{\mathbb{C}\mathbb{P}^n} \nu_M.$$

Proof. Let $U_M := J_{\mathbb{C}\mathbb{P}^n} \nu_M \in \mathcal{X}(M)$. Then

$$\theta(-U_M) = \eta_M(U_M) = G(U_M, U_M) = 1,$$

$$-U_M \lrcorner d\theta = U_M \lrcorner d\eta_M = U_M \lrcorner \Omega_M = 0,$$

because $P_M U_M = 0$. Then $T = -U_M$. Q.e.d.

We also have $g_\theta(T, T) = 1$, $G(T, T) = G(U_M, U_M) = 1$ and $g_\theta(X, T) = 0 = G(X, T)$, for any $X \in H(M)$. Then, as $T = -U_M = J_{\mathbb{C}\mathbb{P}^n} \nu_M$ and $\theta = -\eta_M$, it follows that $g_\theta = G$, hence the Levi-Civita connection of the Webster metric and the induced connection actually coincide. For the remainder of this paper, we

adopt the following notations: D is the canonical connection given by Theorem 4, ∇ and ∇^M are respectively the Levi-Civita connection of N and M , and D^M is the Tanaka-Webster connection of M . Summing up, these connections are tied by the following identities

$$(5.1) \quad D_X Y = \nabla_X Y + g(X, \mathbf{P}Y)\xi + \frac{1}{2}\{2g(aX, \mathbf{P}Y) - u(X)u(a\mathbf{P}Y)\}U \\ + \frac{1}{2}\{2\eta(X)\mathbf{P}Y + u(X)(a\mathbf{P} + \mathbf{P}a)Y\},$$

$$(5.2) \quad D_X Y = \nabla_X Y + g(X, \mathbf{P}Y)\xi + g(aX, \mathbf{P}Y)U,$$

$$(5.3) \quad D_U Y = \nabla_U Y + \frac{1}{2}\{u(a\mathbf{P}Y)U + (a\mathbf{P} + \mathbf{P}a)Y\},$$

$$(5.4) \quad D_\xi Y = \nabla_\xi Y + \mathbf{P}Y,$$

for any $X, Y \in H(N)$, and

$$(5.5) \quad \nabla_{X^\dagger} Y^\dagger = (\nabla_X^M Y)^\dagger - G(X, \mathbf{P}_M Y)^v \xi,$$

for any $X, Y \in T(M)$. Moreover, we need (cf. e.g. [1])

$$(5.6) \quad \nabla_X^M Y = D_X^M Y + \{G(X, \mathbf{P}_M Y) - G(X, \tau_M Y)\}T \\ + \theta(Y)\tau_M X + \theta(X)\mathbf{P}_M Y + \theta(Y)\mathbf{P}_M X$$

(relating the Levi-Civita and Tanaka-Webster connections of M) where τ_M (the *pseudohermitian torsion*) is given by $\tau_M X = T_{D^M}(T, X)$. Also, we shall make use of

$$\tau_M \circ \mathbf{P}_M + \mathbf{P}_M \circ \tau_M = 0, \quad \tau_M H(M) \subseteq H(M), \\ G(\tau_M X, Y) = G(X, \tau_M Y), \quad \text{for any } X, Y \in T(M),$$

where T_{D^M} is the torsion of D^M . Let ν be a global unit normal field on N . Note that $(T(M)^\perp)^\dagger = T(N)^\perp$ yields $\nu_M^\dagger = \nu$, so that $T^\dagger = -U = -\phi\nu$. Moreover (by (3.5))

$$aX^\dagger = (AX)^\dagger - G(X, T)^v \xi$$

where $A = A_{\nu_M}$ is the Weingarten operator of M associated to the normal section ν_M . In particular

$$aX^\dagger = (AX)^\dagger, \quad X \in H(M), \\ aU = -aT^\dagger = -(AT)^\dagger + \xi.$$

Moreover $u(X^\dagger) = -(\theta(X))^v$, hence

$$\begin{cases} u(X^\dagger) = 0, & X \in H(M), \\ u(T^\dagger) = -1. \end{cases}$$

Let $X \in T(M)$, $Y \in H(M)$. Then $X^\dagger \in H(N) \oplus \mathbf{R}U$, $Y^\dagger \in H(N)$, and $\eta(X^\dagger) = 0$ so that (by (5.1) and (5.5)–(5.6))

$$(5.7) \quad \begin{aligned} D_{X^\dagger} Y^\dagger &= (D_X^M Y)^\dagger - \frac{1}{2}(\theta(X)\theta(AP_M Y))^v U \\ &\quad + \frac{1}{2}(\theta(X))^v \{2P_M Y - (AP_M + P_M A) Y\}^\dagger \\ &\quad - \{G(X, P_M Y) - G(X, \tau_M Y) - G(AX, P_M Y)\}^v U, \end{aligned}$$

for any $X \in T(M)$, $Y \in H(M)$. In particular

$$(5.8) \quad D_{X^\dagger} Y^\dagger = (D_X^M Y)^\dagger - \{G(X, P_M Y) - G(X, \tau_M Y) - G(AX, P_M Y)\}^v U,$$

and

$$(5.9) \quad D_U Y^\dagger = -(D_T^M Y)^\dagger + \frac{1}{2}(\theta(AP_M Y))^v U - \frac{1}{2}\{2P_M Y - (AP_M + P_M A) Y\}^\dagger$$

for any $X, Y \in H(M)$. Also (by (5.4) and (4.18))

$$D_\xi X^\dagger = \nabla_\xi X^\dagger + (P_M X)^\dagger = [\xi, X^\dagger] \in H(N).$$

As $\text{Ker}(d\pi) = \mathbf{R}\xi$ we have

$$d\pi[\xi, X^\dagger] = [d\pi(\xi), d\pi(X^\dagger)] = 0, \quad X \in T(M),$$

and then $[\xi, X^\dagger] \in (H(N) \oplus \mathbf{R}U) \cap \mathbf{R}\xi = (0)$, i.e. $[\xi, X^\dagger] = 0$ for any $X \in T(M)$. Thus $D_\xi X^\dagger = 0$. Note also that

$$\tau X^\dagger = D_\xi X^\dagger - [\xi, X^\dagger] = T_\xi X^\dagger$$

and then (by $H(M)^\dagger = H(N)$) it follows that $T_\xi = 0$, i.e. $\tau = 0$. If $X, Y \in T(N)$ (by (5.5))

$$[X^\dagger, Y^\dagger] = [X, Y]^\dagger - 2G(X, P_M Y)^v \xi.$$

In particular $[U, X^\dagger] = -[T, X]^\dagger$. Let $X, Y, Z \in H(M)$. A straightforward calculation (based on (5.8)) leads to

$$(5.10) \quad \begin{aligned} D_{X^\dagger} D_{Y^\dagger} Z^\dagger &= (D_X^M D_Y^M Z)^\dagger - \{G(X, P_M D_Y^M Z) - G(AX, P_M D_Y^M Z) \\ &\quad - G(X, \tau_M D_Y^M Z) + X(G(Y, P_M Z)) - X(G(AY, P_M Z)) \\ &\quad - X(G(Y, \tau_M Z))\}^v U. \end{aligned}$$

Using (5.7), $D_\xi Z^\dagger = 0$, and $[X, Y]^\dagger \in H(N) \oplus \mathbf{R}U$

$$(5.11) \quad \begin{aligned} D_{[X^\dagger, Y^\dagger]} Z^\dagger &= (D_{[X, Y]}^M Z)^\dagger - \frac{1}{2}(\theta([X, Y])\theta(AP_M Z))^v U \\ &\quad + \frac{1}{2}(\theta([X, Y]))^v \{2P_M Z - (AP_M + P_M A) Z\}^\dagger \\ &\quad - \{G([X, Y], P_M Z) - G([X, Y], \tau_M Z) - G(A[X, Y], P_M Z)\}^v U. \end{aligned}$$

At this point, a rather lengthy (yet once again straightforward) calculation (based on (5.10)–(5.11) and $D^M G = 0$) relates the curvature tensor fields of D and D^M

$$\begin{aligned}
(5.12) \quad R^D(X^\uparrow, Y^\uparrow)Z^\uparrow &= (R^{D^M}(X, Y)Z)^\uparrow \\
&- \{G(D_X^M Y, P_M Z) - G(D_Y^M X, P_M Z) - G([X, Y], P_M Z) \\
&- G(D_X^M A Y, P_M Z) + G(D_Y^M A X, P_M Z) + G(A[X, Y], P_M Z) \\
&- G(D_X^M \tau_M Y, Z) + G(D_Y^M \tau_M X, Z) + G(\tau_M[X, Y], Z)\}^v U \\
&+ \frac{1}{2}(\theta([X, Y])\theta(AP_M Z))^v U \\
&- \frac{1}{2}(\theta([X, Y]))^v \{2P_M Z - (AP_M + P_M A)Z\}^\uparrow.
\end{aligned}$$

Note that (by $T_{D^M}(X, Y) \in \mathbf{RT}$)

$$G(D_X^M Y, P_M Z) - G(D_Y^M X, P_M Z) - G([X, Y], P_M Z) = G(T_{D^M}(X, Y), P_M Z) = 0.$$

Also

$$\theta(AX) = G(T, AX) = -G(T, \nabla_X^{\text{CP}^n} v_M) = G(T, \nabla_X^{\text{CP}^n} J_{\text{CP}^n} T) = 0,$$

for any $X \in H(M)$. Then (by (5.6))

$$G(D_X^M A Y, P_M Z) = G(\nabla_X^M A Y, P_M Z),$$

so that

$$\begin{aligned}
&G(D_X^M A Y - D_Y^M A X - A[X, Y], P_M Z) \\
&= G(\nabla_X^M A Y - \nabla_Y^M A X - A\nabla_X^M Y + A\nabla_Y^M X, P_M Z) \\
&= G((\nabla_X^M A)Y - (\nabla_Y^M A)X, P_M Z).
\end{aligned}$$

On the other hand, we need to recall (cf. K. Yano & M. Kon, [29], p. 153) the *Codazzi equation* of M (as a real hypersurface of a complex space form of holomorphic sectional curvature 4)

$$(\nabla_X^M A)Y - (\nabla_Y^M A)X = \theta(X)P_M Y - \theta(Y)P_M X + 2G(X, P_M Y)T = 2G(X, P_M Y)T,$$

for any $X, Y \in H(M)$. Therefore

$$G(D_X^M A Y - D_Y^M A X - A[X, Y], P_M Z) = 0.$$

Next, the identities

$$\begin{aligned}
D_X^M \tau_M Y &= D_X^M D_T^M Y - D_X^M [T, Y], \\
D_Y^M \tau_M X &= D_Y^M D_T^M X - D_Y^M [T, X], \\
\tau_M [X, Y] &= D_T^M [X, Y] - [T, [X, Y]],
\end{aligned}$$

lead to

$$\begin{aligned}
& G(D_X^M \tau_M Y - D_Y^M \tau_M X - \tau_M[X, Y], Z) \\
&= G(R^{D^M}(X, T)Y - R^{D^M}(Y, T)X + D_{[X, T]}^M Y - D_{[Y, T]}^M X, Z) \\
&\quad + G(D_X^M[Y, T] - D_Y^M[X, T] + [T, [X, Y]], Z) \\
&= G(R^{D^M}(X, T)Y - R^{D^M}(Y, T)X, Z).
\end{aligned}$$

Finally (by $\theta([X, Y]) = -2(d\theta)(X, Y) = 2G(X, \mathbf{P}_M Y)$) (5.12) becomes

$$\begin{aligned}
(5.13) \quad R^D(X^\uparrow, Y^\uparrow)Z^\uparrow &= (R^{D^M}(X, Y)Z)^\uparrow \\
&\quad + G(R^{D^M}(X, T)Y - R^{D^M}(Y, T)X, Z)^v U \\
&\quad - G(X, \mathbf{P}_M Y)^v \{2\mathbf{P}_M Z - (A\mathbf{P}_M + \mathbf{P}_M A)Z\}^\uparrow,
\end{aligned}$$

for any $X, Y, Z \in H(M)$. One should note (by $\theta(AZ) = 0$, for any $Z \in H(M)$) that $u(aZ^\uparrow) = 0$, i.e. $u \circ a|_{H(N)} = 0$. In particular $u(aPX) = 0$ for any $X \in H(N)$, where from

$$\begin{cases} \rho X = T_U X, & X \in H(N), \\ T_U H(N) \subseteq H(N). \end{cases}$$

5.2. A vanishing theorem

Let us prove Theorem 5. To this end, we consider a local orthonormal frame $\{X_a\} = \{X_j, J_M X_j\}$ of $H(M)$ (here $G(X_a, X_b) = \delta_{ab}$), so that $\{T, X_a\}$ is a (local) orthonormal frame of $T(M)$. The *Ricci curvature* of D^M is

$$\begin{aligned}
\text{Ric}_{D^M}(Y, Z) &:= \text{trace}\{V \mapsto R^{D^M}(V, Y)Z\} \\
&= \sum_{k=1}^n g(R^{D^M}(X_k, Y)Z, X_k).
\end{aligned}$$

On the other hand, $\{X_a^\uparrow\} = \{X_j^\uparrow, J_M X_j^\uparrow\}$ is an orthonormal frame of $H(N)$, hence $\{\xi, U, X_a^\uparrow\}$ is an orthonormal frame in $T(N)$. Then (by (5.13))

$$\begin{aligned}
\text{Ric}_D(Y^\uparrow, Z^\uparrow) &= \sum_{a=1}^{2n-2} g(R^D(X_a^\uparrow, Y^\uparrow)Z^\uparrow, X_a^\uparrow) \\
&= \sum_{a=1}^{2n-2} \{g((R^{D^M}(X_a, Y)Z)^\uparrow, X_a^\uparrow)\} \\
&\quad - \sum_{a=1}^{2n-2} \{G(X_a, \mathbf{P}_M Y)^v G([2\mathbf{P}_M Z - (A\mathbf{P}_M + \mathbf{P}_M A)Z], X_a)^v\} \\
&= \sum_{a=1}^{2n-2} G(R^{D^M}(X_a, Y)Z, X_a)^v - G([2\mathbf{P}_M - (A\mathbf{P}_M + \mathbf{P}_M A)]Z, \mathbf{P}_M Y)^v,
\end{aligned}$$

where from

$$(5.14) \quad \text{Ric}_D(Y^\uparrow, Z^\uparrow) = \text{Ric}_{D^M}(Y, Z)^v - G([2P_M - (AP_M + P_MA)]Z, P_M Y)^v.$$

Then, for $V = Y - iP_M Y$, $W = Z - iP_M Z \in T_{1,0}(M)$

$$\text{Ric}_D(V^\uparrow, \bar{W}^\uparrow) = \text{Ric}_{D^M}(V, \bar{W})^v - G([2P_M - (AP_M + P_MA)]\bar{W}, P_M V)^v.$$

On the other hand

$$\begin{aligned} & G([2P_M - (AP_M + P_MA)]\bar{W}, P_M V) \\ &= G(2P_M \bar{W}, P_M V) - G(AP_M \bar{W}, P_M V) - G(P_MA \bar{W}, P_M V) \\ &= G(2\bar{W}, V) - 2G(A\bar{W}, V), \end{aligned}$$

so that

$$G([2P_M - (AP_M + P_MA)]\bar{W}, P_M V) = 2G((I - A)\bar{W}, V),$$

leading to

$$(5.15) \quad \text{Ric}_D(V^\uparrow, \bar{W}^\uparrow) = \text{Ric}_{D^M}(V, \bar{W})^v - 2G((I - A)\bar{W}, V).$$

The identity (5.15) is the key ingredient in the proof of Theorems 5 and 6. Let us recall that a complex valued q -form $\omega \in \Omega^q(M)$ is a $(0, q)$ -form if

$$T \lrcorner \omega = 0, \quad T_{1,0}(M) \lrcorner \omega = 0.$$

The *tangential Cauchy-Riemann operator* (on $(0, q)$ -forms, $q \geq 1$) is the differential operator

$$\bar{\partial}_M : \Omega^{0,q}(M) \rightarrow \Omega^{0,q+1}(M)$$

defined as follows. Let $\Lambda^{0,q}(M) \rightarrow M$ be the fibre bundle of the $(0, q)$ -forms and set $\Omega^{0,q}(M) := \Gamma^\infty(\Lambda^{0,q}(M))$. If $\omega \in \Omega^{0,q}(M)$ then $\bar{\partial}_M \omega$ is the unique $(0, q+1)$ -form coinciding with $d\omega$ on $T_{0,1}(M) \otimes \cdots \otimes T_{0,1}(M)$ ($q+1$ -terms). As well known

$$(5.16) \quad C^\infty(M, \mathbb{C}) \xrightarrow{\bar{\partial}_M} \Omega^{0,1}(M) \xrightarrow{\bar{\partial}_M} \cdots \xrightarrow{\bar{\partial}_M} \Omega^{0,n-1}(M) \xrightarrow{\bar{\partial}_M} 0$$

is a cochain complex (the *tangential Cauchy-Riemann complex*) and its cohomology

$$H^{0,q}(M, \bar{\partial}_M) := H^{0,q}(\Omega^{0,\cdot}(M), \bar{\partial}_M) = \frac{\text{Ker}\{\bar{\partial}_M : \Omega^{0,q}(M) \rightarrow \cdot\}}{\bar{\partial}_M \Omega^{0,q-1}(M)}$$

is the so called *Kohn-Rossi cohomology* of $(M, T_{1,0}(M))$. By a result of J. M. Lee (cf. [16]), if M is compact and strictly pseudoconvex, and the (pseudo-hermitian) Ricci tensor of the Tanaka-Webster connection satisfies $\text{Ric}_{D^M}(W, \bar{W}) \geq 0$, for any $W \in T_{1,0}(M)$, then $H^{0,1}(M, \bar{\partial}_M) = 0$. This is clearly an analog of the well known Bochner theorem that the first Betti number of a compact Riemannian manifold of positive semi-definite Ricci curvature vanishes (yet

$H^{0,1}(M, \bar{\partial}_M)$ is not a topological invariant). In the same spirit, by our assumption (1.5) and by (5.15) we may conclude as in Theorem 5. Q.e.d.

5.3. C-R fibrations over a pseudo-Einstein manifold

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold, of hypersurface type, of CR dimension n . Let θ be a contact form over M and D^M the Tanaka-Webster connection of (M, θ) . If $\{T_{\alpha\bar{\beta}}\}$ is a local frame of $T_{1,0}(M)$ we set $R_{\alpha\bar{\beta}} = \text{Ric}_{D^M}(T_{\alpha}, T_{\bar{\beta}})$ (the *pseudohermitian Ricci tensor* of (M, θ)). Then (M, θ) is a *pseudo-Einstein manifold* if the pseudohermitian Ricci tensor is proportional to the Levi form (a CR analog to the Einstein condition in Riemannian geometry), i.e. $R_{\alpha\bar{\beta}} = \lambda g_{\alpha\bar{\beta}}$, for some $\lambda \in C^\infty(M)$. Here $g_{\alpha\bar{\beta}} = L_\theta(T_{\alpha}, T_{\bar{\beta}})$. *A posteriori* $\lambda = \rho/n$, where $\rho = g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}$ is the *pseudohermitian scalar curvature* of (M, θ) . Examples are abundant. The sphere S^{2n+1} endowed with the standard contact structure (as hypersurface in \mathbb{C}^{n+1}) is a pseudo-Einstein manifold. The total space $U(M)$ (carrying the contact structure induced by the almost hermitian structure of $T(M)$) of the tangent sphere bundle over a real space-form M of (constant) sectional curvature 1 is a pseudo-Einstein manifold (cf. E. Barletta & S. Dragomir, [1]). If $f: M \rightarrow S^{2n+1}$ is a pseudohermitian immersion with a flat normal Tanaka-Webster connection, of a compact strictly pseudoconvex CR manifold M into a sphere, then M admits a global pseudo-Einstein structure (cf. E. Barletta & S. Dragomir, [2]).

As well known (cf. J. M. Lee, [16]) there are obstructions towards the existence of a globally defined pseudo-Einstein structure on a given nondegenerate CR manifold (for instance, if $(M, T_{1,0}(M))$ admits a contact pseudo-Einstein structure then the first Chern class of the CR structure must vanish ($c_1(T_{1,0}(M)) = 0$). The *Lee conjecture* states that every compact strictly pseudoconvex CR manifold satisfying $c_1(T_{1,0}(M)) = 0$ should actually possess a globally defined pseudo-Einstein structure (a CR analog of the Calabi problem). Positive answers to the Lee conjecture are known under additional assumptions (e.g. when M admits a 1-parameter group of CR automorphisms transverse to $H(M)$, cf. J. M. Lee, [16], or when the contact vector field T is regular in the sense of Palais, cf. S. Dragomir, [10]). Theorem 6 furnishes new examples of pseudo-Einstein manifolds, as base spaces of C-R fibrations whose total space satisfies the assumption (1.6). Let us prove Theorem 6. By (5.15) for $V = T_\alpha$ and $W = T_{\bar{\beta}}$

$$(5.17) \quad \text{Ric}_D(T_\alpha^\dagger, T_{\bar{\beta}}^\dagger) - 2G(T_\alpha, AT_{\bar{\beta}}) = R_{\alpha\bar{\beta}} - 2g_{\alpha\bar{\beta}}$$

where $g_{\alpha\bar{\beta}} := G(T_\alpha, T_{\bar{\beta}})$. Assume that (1.6) holds. Then (by (3.5))

$$(5.18) \quad R_{\alpha\bar{\beta}} = (\mu + 2)g_{\alpha\bar{\beta}}$$

so that (M, θ) is a pseudo-Einstein manifold of pseudohermitian scalar curvature $\rho = g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}} = (n-1)(\mu+2)$. Conversely, if (5.18) holds, then (by $T_{1,0}(N) = T_{1,0}(M)^\dagger$ and (5.17)) we obtain (1.6). Finally, if this is the case, by a result by J. M. Lee (cf. [16]) $c_1(T_{1,0}(M)) = 0$. Q.e.d.

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