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KODAI MATH. J.
28 (2005), 111–130

EXTREMAL DISKS AND EXTREMAL SURFACES OF GENUS THREE

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Abstract

A compact Riemann surface of genus $g \geq 2$ is said to be extremal if it admits an extremal disk, a disk of the maximal radius determined by g. If $g = 2$ or $g \geq 4$, it is known that how many extremal disks an extremal surface of genus g can admit. In the present paper we deal with the case of $g = 3$. Considering the side-pairing patterns of the fundamental polygons, we show that extremal surfaces of genus 3 admit at most two extremal disks and that 16 surfaces admit exactly two. Also we describe the group of automorphisms and hyperelliptic surfaces.

1. Introduction

Let S be a compact Riemann surface of genus $g \geq 2$ equipped with the metric induced by the hyperbolic metric of the unit disk $\Delta = \{z \in \mathbf{C}; |z| < 1\}$. The hyperbolic metric is derived from $ds = 2|dz|/(1 - |z|^2)$. Then S is said to be extremal if a disk of radius R_g is isometrically embedded in S, where R_g is the maximal radius determined by g as follows ([2]):

$$\cosh R_g = \frac{1}{2 \sin \beta_g},$$

where $\beta_g = \pi/(12g - 6)$. The embedded disk in S of radius R_g is called an extremal disk.

We know several results on extremal surfaces ([2]): an extremal surface of genus g has a regular $(12g - 6)$-gon as a fundamental region; there are finitely many extremal disks for each extremal surface; an extremal disk is the projection of the disk inscribed in the $(12g - 6)$-gon.

Our concern is the following problem.

2000 *Mathematics Subject Classification*: Primary 30F10; Secondary 30F35, 20H10.

Key words and phrases: extremal disk, extremal surface, the group of automorphisms, hyperelliptic surface, Weierstrass point.

This work was supported by Grant-in-Aid for Scientific Research (No. 15740095), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Received May 10, 2004; revised September 24, 2004.

PROBLEM. How many extremal disks does an extremal surface of genus g admit? Where are extremal disks embedded provided that an extremal surface admits more than one extremal disk?

In the case of $g \geq 4$, each extremal surface has a unique extremal disk ([4]). In the case of $g = 2$, there are 9 extremal surfaces up to conformal equivalence. One has 4 extremal disks, one has a unique extremal disk, and the others have two extremal disks. For each surface, the positions of embedded extremal disks are obtained ([6], [8]). In the case of $g = 3$, there are examples of an extremal surface which admits a unique extremal disk or two extremal disks ([4], [5]).

In the present paper we shall consider this problem for every extremal surface of genus 3.

As a fundamental polygon for an extremal surface of genus g we have a regular $(12g - 6)$-gon. When we treat an extremal surface, the side-pairing pattern of the regular polygon plays an important role. If $g = 3$, then the number of side-pairing patterns of the regular 30-gon which make a compact surface of genus 3 is 1726 ([1], [7]). In particular, there exist 927 side-pairing patterns up to mirror images. The 927 side-pairing patterns are explicitly given in [9]. Let P_j $(j = 1, 2, \dots, 927)$ be a regular 30-gon in Δ centered at the origin endowed with the j-th side-pairing pattern in [9]. If the mirror image of P_j is a different side-pairing pattern from the original one, we denote it by P_j'. Then the set $\mathscr{P}$ consisting of all P_j and P_j' (if it exists) has 1726 elements. Let S_j and S_j' be the surfaces derived from P_j and P_j', respectively. We denote by $\mathscr{S}$ the set of all S_j and S_j' (if it exists). Later, we shall show that the surfaces in $\mathscr{S}$ are distinct, so that $\#\mathscr{S} = 1726$.

2. Finding extremal disks for $g = 3$

In the following of this paper, we shall deal with the case of $g = 3$ and abbreviate the radius $R = R_3$ and the angle $\beta = \beta_3$. Here $R \approx 2.247$.

Let P be a regular 30-gon in Δ centered at the origin endowed with a side-pairing pattern and S the surface derived from P. We denote by $C_1, \dots, C_{30}$ the sides of P (Figure 1), by v_n the vertex between C_{n-1} and C_n, and by w_n the middle point of the side C_n, where subscripts are regarded as modulo 30. The hyperbolic length of a side of P, denoted by s, is

$$s = 2 \sinh^{-1}\left(\frac{2}{\sqrt{3}} \sin \beta \sinh R\right) \approx 1.076,$$

and the hyperbolic length between a vertex and the center of P, denoted by l, is

$$l = \sinh^{-1}\left(\frac{2}{\sqrt{3}} \sinh R\right) \approx 2.388.$$

The polygon P determines side-pairing transformations $A_1, \dots, A_{30}$, where A_n maps C_n onto some C_m. Since we take $v_1 = \tanh(l/2)e^{\beta i}$, $A_n = A_{n,m}$ is explicitly of the form

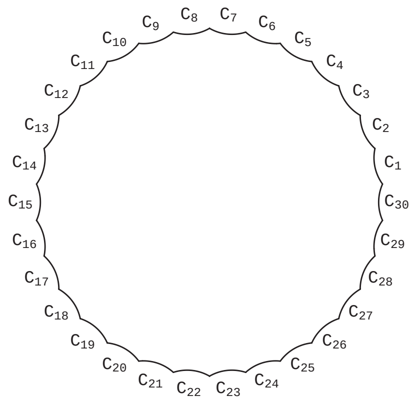

FIGURE 1. A Regular 30-gon

$$A_{n,m}(z) = \frac{i \cosh R e^{i(m-n)\beta} z - i \sinh R e^{i(m+n)\beta}}{i \sinh R e^{-i(m+n)\beta} z - i \cosh R e^{-i(m-n)\beta}}.$$

In particular, $A_{15,m} : C_{15} \to C_m$ is of the form

$$A_{15,m}(z) = \frac{\cosh R e^{im\beta} z + \sinh R e^{im\beta}}{\sinh R e^{-im\beta} z + \cosh R e^{-im\beta}}.$$

Let $\pi : \Delta \to S$ denote the projection. Let $p \in S$ be the center of an extremal disk. We shall consider the set $\{\rho(z,w) \mid \pi^{-1}(p) \ni z, w \ (z \neq w)\}$, where $\rho(z,w)$ is the hyperbolic distance between z and w. Note that this set is independent of the choice of S and of p. We denote the elements of the set by $\rho_1, \rho_2, \ldots (\rho_1 < \rho_2 < \cdots)$.

LEMMA 1. *For $z \in P$, if $\pi(z)$ is the center of an extremal disk of S, then $\rho(z, A_k(z)) \in \{\rho_j\}_{j=1}^{\infty}$ for every $k = 1, \ldots, 30$. Precisely, it follows that $\rho(z, A_k(z)) \in \{\rho_1, \rho_2, \ldots, \rho_{20}\}$, where $\rho_1 \approx 4.494$, $\rho_2 \approx 5.852$, $\rho_3 \approx 6.642$, $\rho_4 \approx 7.190$, $\rho_5 \approx 7.603$, $\rho_6 \approx 7.645$, $\rho_7 \approx 7.926$, $\rho_8 \approx 8.185$, $\rho_9 \approx 8.295$, $\rho_{10} \approx 8.395$, $\rho_{11} \approx 8.565$, $\rho_{12} \approx 8.701$, $\rho_{13} \approx 8.768$, $\rho_{14} \approx 8.807$, $\rho_{15} \approx 8.888$, $\rho_{16} \approx 8.944$, $\rho_{17} \approx 8.977$, $\rho_{18} \approx 8.988$, $\rho_{19} \approx 9.132$, $\rho_{20} \approx 9.176$.*

Remark 2. Let K be the Fuchsian group generated by the side-pairing transformations $A_1, \ldots, A_{30}$ of P. Consider the tessellation $\{A(P) \mid A \in K\}$ of Δ. Then ρ_1 is the hyperbolic distance between the centers of P and $A_k(P)$. ρ_j $(j = 2, 3, 4, 5, 7, 8, 10, 11, 12, 14, 15, 16, 17, 18)$ is given by a distance between the centers of P and $A_l A_k(P)$. ρ_j $(j = 6, 9, 11, 13, 18, 19, 20)$ is given by a distance between the centers of P and $A_m A_l A_k(P)$.

Proof of Lemma 1. The first statement is clear because $\pi(z)$ and $\pi(A_k(z))$ are the same center of an extremal disk. For any $z \in P$ and any side-pairing A_k, it follows that $\rho(z, A_k(z)) \le \rho(z,0) + \rho(0, A_k(0)) + \rho(A_k(0), A_k(z)) = 2\rho(z,0) + \rho(0, A_k(0)) \le 2l + 2R \approx 9.270$. Since $\rho_{21} \approx 9.357$, the second statement is proved. □

LEMMA 3. *Let K_n $(n = 1, \ldots, 30)$ be a pentagon with vertices w_{n-1}, v_n, v_{n+1}, w_{n+1} and the origin (Figure 2). For a fixed n, if $\pi(z)$ $(z \in K_n)$ is the center of an extremal disk of S, then $\rho(z, A_n(z)) = \rho_1 = 2R$.*

Proof. By Lemma 4 in [6] we have

$$\rho(z, A_n(z)) \le \max\{\rho(0, A_n(0)), \rho(w_{n-1}, A_n(w_{n-1})), \rho(v_n, A_n(v_n)), \rho(v_{n+1}, A_n(v_{n+1})), \rho(w_{n+1}, A_n(w_{n+1}))\}.$$

Here $\rho(0, A_n(0)) = 2R \approx 4.494$ and $\rho(w_{n-1}, A_n(w_{n-1})) \le \rho(w_{n-1}, 0) + \rho(0, A_n(v_n)) + \rho(A_n(v_n), A_n(w_{n-1})) = R + l + s/2 \approx 5.173$. Similarly, $\rho(w_{n+1}, A_n(w_{n+1})) \le R + l + s/2$. Since $A_n(v_n)$ and $A_n(v_{n+1})$ are vertices of P, $\rho(v_n, A_n(v_n)) \le \rho(v_n, v_{n+15}) = 2l \approx 4.776$ and also $\rho(v_{n+1}, A_n(v_{n+1})) \le 2l$. Therefore $\rho(z, A_n(z)) < \rho_2 \approx 5.852$. Since $\pi(z)$ is the center of an extremal disk, it follows that $\rho(z, A_n(z)) = \rho_1$. □

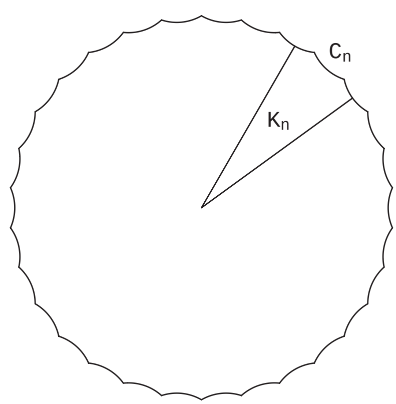

FIGURE 2. K_n

Now we shall find a point of P such that its projection is the center of an extremal disk. First, we shall find a point z of K_n such that $\rho(z, A_n(z)) = \rho_1$, where $A_n = A_{n,m} : C_n \to C_m$ (cf. Lemma 3). Using a formula

$$\sinh \frac{1}{2}\rho(z, A_n(z)) = \cosh \rho(z, \mathrm{ax}(A_n)) \sinh \frac{1}{2} T_{A_n},$$

where $\mathrm{ax}(A_n)$ and T_{A_n} denote the axis and the translation length of A_n, respectively, we can show that z with $\rho(z, A_n(z)) = \rho_1$ is on the following curves (cf. Theorem 3.4 in [8]):

$$L_n = L_{n,m} : \left| z - \frac{\tanh R}{2\cos(n-m)\beta} e^{i(n+m)\beta} \right| = \frac{\tanh R}{2|\cos(n-m)\beta|}$$

$$(m \not\equiv n + 15 \pmod{30}) \quad \text{or}$$

$$M_n = M_{n,m} : z = \frac{e^{2in\beta}}{\tanh R} - te^{i(n+m+15)\beta} \quad (t \in \mathbf{R}),$$

Two examples of $L_{n,m}$ and $M_{n,m}$ when $(n,m) = (15,1)$ or $(15,6)$ are given in Figure 3. Here, $M_{15,6}$, the dotted line, intersects with K_{15} only at the boundary of the polygon.

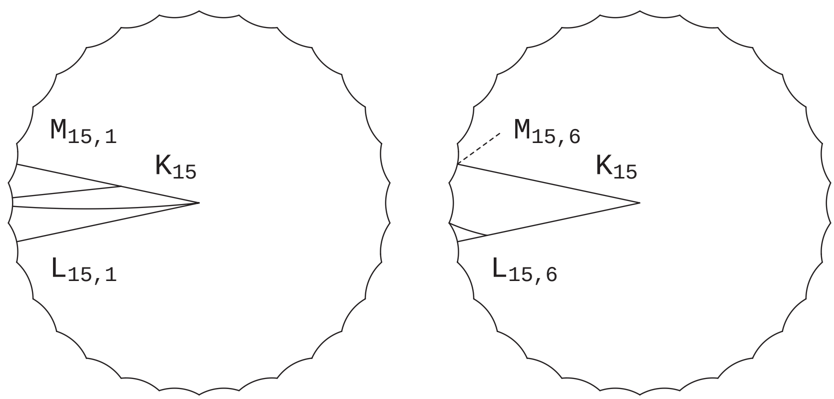

FIGURE 3. $L_{15,1}$, $M_{15,1}$, $L_{15,6}$, and $M_{15,6}$

For every n we draw L_n and M_n in K_n according to the side-pairings of P and find the intersection in $K_n \cap K_{n+1}$ of curves $L_n \cup M_n$ and $L_{n+1} \cup M_{n+1}$. Next, we select ζ from the intersection satisfying $\rho(\zeta, A_k(\zeta)) \in \{\rho_1, \rho_2, \ldots, \rho_{20}\}$ for every k (cf. Lemma 1). Then $\zeta \in P$ is a candidate for the point whose projection is the center of an extremal disk.

Through this process for every polygon P_j $(j = 1, \ldots, 927)$ by computer, it turns out that there exist 12 polygons admitting more than one ζ (16 polygons in $\mathscr{P}$). They are P_{75}, P_{184}, P_{202}, P_{437}, P_{481}, P_{483}, P_{489}, P_{594}, P_{614}, P_{617}, P_{785}, P_{879} ($P'_{184}, P'_{202}, P'_{437}, P'_{489}$) and each of them has exactly two points (the one is the origin) (Figures 6, 7, 8, 9). For example, P_{75} has the intersection consisting of 6 points, but only two points 0 and $(2\sin 3\beta)/\tanh R$ on the real axis are the candidates.

Put $p = \pi(\zeta)$ $(\zeta \neq 0)$ and $o = \pi(0)$. In order to prove that p is the center of an extremal disk, it is sufficient to show that there exists an automorphism T of S such that $T(p) = o$ because o is the center of an extremal disk. Put $\gamma(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then γ induces an automorphism of S if and only if $\gamma A_k \gamma^{-1} \in K$ for every k, where $K = \langle A_1, A_2, \ldots, A_{30} \rangle$, the Fuchsian group generated by the side-pairing transformations of P.

For each of the 12 polygons P_j, we shall show that $\gamma A_k \gamma^{-1}$ is an element of K.

P_{75}: $\zeta = \dfrac{2 \sin 3\beta}{\tanh R}$,

$\gamma A_{1,19} = A_{19,1}\gamma$, $\gamma A_{2,22} = A_{22,2}\gamma$, $\gamma A_{3,24} = A_{24,3}\gamma$,
$\gamma A_{4,17} = A_{19,1}A_{25,21}\gamma$, $\gamma A_{5,9} = A_{8,28}A_{27,6}\gamma$, $\gamma A_{6,27} = A_{27,6}\gamma$,
$\gamma A_{7,12} = A_{11,29}A_{28,8}\gamma$, $\gamma A_{8,28} = A_{28,8}\gamma$, $\gamma A_{10,16} = A_{15,30}A_{29,11}\gamma$,
$\gamma A_{11,29} = A_{29,11}\gamma$, $\gamma A_{13,26} = A_{9,5}A_{29,11}\gamma$, $\gamma A_{14,20} = A_{19,1}A_{30,15}\gamma$,
$\gamma A_{15,30} = A_{30,15}\gamma$, $\gamma A_{18,23} = A_{22,2}A_{1,19}\gamma$, $\gamma A_{21,25} = A_{24,3}A_{2,22}\gamma$.

P_{184}: $\zeta = \dfrac{2 \sin 4\beta}{\tanh R} i$,

$\gamma A_{1,5} = A_{4,10}A_{9,2}\gamma$, $\gamma A_{2,9} = A_{9,2}\gamma$,
$\gamma A_{4,10} = A_{10,4}\gamma$, $\gamma A_{6,23} = A_{17,8}A_{12,19}\gamma$,
$\gamma A_{8,17} = A_{28,7}\gamma$, $\gamma A_{11,15} = A_{28,7}A_{5,1}\gamma$,
$\gamma A_{13,26} = A_{17,8}A_{6,23}\gamma$, $\gamma A_{14,20} = A_{28,7}A_{30,24}A_{7,28}\gamma$,
$\gamma A_{21,25} = A_{17,8}A_{25,21}A_{7,28}\gamma$, $\gamma A_{22,29} = A_{17,8}A_{26,13}A_{7,28}\gamma$,
$\gamma A_{3,16} = A_{28,7}A_{9,2}\gamma$,
$\gamma A_{7,28} = A_{17,8}\gamma$,
$\gamma A_{12,19} = A_{28,7}A_{6,23}\gamma$,
$\gamma A_{18,27} = A_{17,8}A_{7,28}\gamma$,
$\gamma A_{24,30} = A_{17,8}A_{20,14}A_{8,17}\gamma$.

P_{202}: $\zeta = \dfrac{2 \sin 4\beta}{\tanh R} i$,

$\gamma A_{1,15} = A_{28,7}A_{15,1}A_{8,17}\gamma$, $\gamma A_{2,9} = A_{9,2}\gamma$, $\gamma A_{3,16} = A_{28,7}A_{9,2}\gamma$,
$\gamma A_{4,30} = A_{17,8}A_{10,14}\gamma$, $\gamma A_{5,21} = A_{28,7}A_{11,25}\gamma$, $\gamma A_{6,23} = A_{28,7}A_{13,26}\gamma$,
$\gamma A_{7,28} = A_{17,8}\gamma$, $\gamma A_{8,17} = A_{28,7}\gamma$, $\gamma A_{10,14} = A_{28,7}A_{4,30}\gamma$,
$\gamma A_{11,25} = A_{17,8}A_{5,21}\gamma$, $\gamma A_{12,19} = A_{28,7}A_{6,23}\gamma$, $\gamma A_{13,26} = A_{17,8}A_{6,23}\gamma$,
$\gamma A_{18,27} = A_{17,8}A_{7,28}\gamma$, $\gamma A_{20,24} = A_{17,8}A_{24,20}A_{7,28}\gamma$, $\gamma A_{22,29} = A_{17,8}A_{26,13}A_{7,28}\gamma$.

P_{437}: $\zeta = \dfrac{2 \sin 4\beta}{\tanh R} i$,

$\gamma A_{1,15} = A_{13,6}A_{25,11}A_{9,22}\gamma$, $\gamma A_{2,19} = A_{28,7}A_{9,22}\gamma$, $\gamma A_{3,26} = A_{17,8}A_{9,22}\gamma$,
$\gamma A_{4,10} = A_{10,4}\gamma$, $\gamma A_{5,21} = A_{28,7}A_{11,25}\gamma$, $\gamma A_{6,13} = A_{13,6}\gamma$,
$\gamma A_{7,28} = A_{17,8}\gamma$, $\gamma A_{8,17} = A_{28,7}\gamma$, $\gamma A_{9,22} = A_{28,7}A_{3,26}\gamma$,
$\gamma A_{11,25} = A_{17,8}A_{5,21}\gamma$, $\gamma A_{12,29} = A_{17,8}A_{6,13}\gamma$, $\gamma A_{14,20} = A_{28,7}A_{11,25}A_{6,13}\gamma$,
$\gamma A_{16,23} = A_{28,7}A_{3,26}A_{7,28}\gamma$, $\gamma A_{18,27} = A_{17,8}A_{7,28}\gamma$, $\gamma A_{24,30} = A_{17,8}A_{20,14}A_{8,17}\gamma$.

P_{481}: $\zeta = \dfrac{2 \sin 4\beta}{\tanh R} i$,

$\gamma A_{1,5} = A_{15,11}A_{8,17}\gamma$, $\gamma A_{2,29} = A_{17,8}A_{9,12}\gamma$,
$\gamma A_{4,30} = A_{17,8}A_{10,14}\gamma$, $\gamma A_{7,28} = A_{17,8}\gamma$,

$$\gamma A_{9,12} = A_{3,6}\gamma, \qquad \gamma A_{10,14} = A_{28,7}A_{4,30}\gamma,$$
$$\gamma A_{13,16} = A_{28,7}A_{6,3}\gamma, \qquad \gamma A_{18,27} = A_{17,8}A_{7,28}\gamma,$$
$$\gamma A_{20,24} = A_{17,8}A_{24,20}A_{7,28}\gamma, \qquad \gamma A_{21,25} = A_{17,8}A_{25,21}A_{7,28}\gamma,$$
$$\gamma A_{3,6} = A_{9,12}\gamma,$$
$$\gamma A_{8,17} = A_{28,7}\gamma,$$
$$\gamma A_{11,15} = A_{28,7}A_{5,1}\gamma,$$
$$\gamma A_{19,22} = A_{28,7}A_{23,26}A_{7,28}\gamma,$$
$$\gamma A_{23,26} = A_{17,8}A_{19,22}A_{8,17}\gamma.$$

P_{483}: $\zeta = \dfrac{2\sin 4\beta}{\tanh R} i,$

$$\gamma A_{1,15} = A_{28,7}A_{15,1}A_{8,17}\gamma, \qquad \gamma A_{2,29} = A_{17,8}A_{9,12}\gamma,$$
$$\gamma A_{4,10} = A_{10,4}\gamma, \qquad \gamma A_{5,11} = A_{11,5}\gamma,$$
$$\gamma A_{8,17} = A_{28,7}\gamma, \qquad \gamma A_{9,12} = A_{3,6}\gamma,$$
$$\gamma A_{14,30} = A_{17,8}A_{30,14}A_{7,28}\gamma, \qquad \gamma A_{18,27} = A_{17,8}A_{7,28}\gamma,$$
$$\gamma A_{20,24} = A_{17,8}A_{24,20}A_{7,28}\gamma, \qquad \gamma A_{21,25} = A_{17,8}A_{25,21}A_{7,28}\gamma,$$
$$\gamma A_{3,6} = A_{9,12}\gamma,$$
$$\gamma A_{7,28} = A_{17,8}\gamma,$$
$$\gamma A_{13,16} = A_{28,7}A_{6,3}\gamma,$$
$$\gamma A_{19,22} = A_{28,7}A_{23,26}A_{7,28}\gamma,$$
$$\gamma A_{23,26} = A_{17,8}A_{19,22}A_{8,17}\gamma.$$

P_{489}: $\zeta = \dfrac{2\sin 4\beta}{\tanh R} i,$

$$\gamma A_{1,15} = A_{28,7}A_{15,1}A_{8,17}\gamma, \qquad \gamma A_{2,9} = A_{9,2}\gamma,$$
$$\gamma A_{4,30} = A_{17,8}A_{10,14}\gamma, \qquad \gamma A_{5,11} = A_{11,5}\gamma,$$
$$\gamma A_{7,28} = A_{17,8}\gamma, \qquad \gamma A_{8,17} = A_{28,7}\gamma,$$
$$\gamma A_{12,29} = A_{17,8}A_{6,13}\gamma, \qquad \gamma A_{18,27} = A_{17,8}A_{7,28}\gamma,$$
$$\gamma A_{20,24} = A_{17,8}A_{24,20}A_{7,28}\gamma, \qquad \gamma A_{21,25} = A_{17,8}A_{25,21}A_{7,28}\gamma,$$
$$\gamma A_{3,16} = A_{28,7}A_{9,2}\gamma,$$
$$\gamma A_{6,13} = A_{13,6}\gamma,$$
$$\gamma A_{10,14} = A_{28,7}A_{4,30}\gamma,$$
$$\gamma A_{19,22} = A_{28,7}A_{23,26}A_{7,28}\gamma,$$
$$\gamma A_{23,26} = A_{17,8}A_{19,22}A_{8,17}\gamma.$$

P_{594}: $\zeta = \dfrac{\sin\beta}{\tanh R \sin 2\beta} i,$

$$\gamma A_{1,4} = A_{11,14}\gamma, \qquad \gamma A_{2,12} = A_{12,2}\gamma, \qquad \gamma A_{3,13} = A_{13,3}\gamma,$$
$$\gamma A_{5,16} = A_{16,5}\gamma, \qquad \gamma A_{6,27} = A_{18,9}\gamma, \qquad \gamma A_{7,21} = A_{21,7}\gamma,$$
$$\gamma A_{8,24} = A_{24,8}\gamma, \qquad \gamma A_{9,18} = A_{27,6}\gamma, \qquad \gamma A_{10,29} = A_{29,10}\gamma,$$
$$\gamma A_{11,14} = A_{1,4}\gamma, \qquad \gamma A_{15,28} = A_{29,10}A_{4,1}\gamma, \qquad \gamma A_{17,30} = A_{14,11}A_{5,16}\gamma,$$
$$\gamma A_{19,23} = A_{24,8}A_{6,27}\gamma, \qquad \gamma A_{20,25} = A_{24,8}A_{7,21}\gamma, \qquad \gamma A_{22,26} = A_{18,9}A_{7,21}\gamma.$$

P_{614}: $\zeta = \dfrac{2\sin 4\beta}{\tanh R} i,$

$$
\begin{array}{ll}
\gamma A_{1,25} = A_{17,8}A_{15,21}A_{8,17}\gamma, & \gamma A_{2,29} = A_{17,8}A_{9,12}\gamma, \\
\gamma A_{4,10} = A_{10,4}\gamma, & \gamma A_{5,11} = A_{11,5}\gamma, \\
\gamma A_{8,17} = A_{28,7}\gamma, & \gamma A_{9,12} = A_{3,6}\gamma, \\
\gamma A_{14,20} = A_{28,7}A_{30,24}A_{7,28}\gamma, & \gamma A_{15,21} = A_{28,7}A_{1,25}A_{7,28}\gamma, \\
\gamma A_{19,22} = A_{28,7}A_{23,26}A_{7,28}\gamma, & \gamma A_{23,26} = A_{17,8}A_{19,22}A_{8,17}\gamma,
\end{array}
$$

$$
\begin{array}{l}
\gamma A_{3,6} = A_{9,12}\gamma, \\
\gamma A_{7,28} = A_{17,8}\gamma, \\
\gamma A_{13,16} = A_{28,7}A_{6,3}\gamma, \\
\gamma A_{18,27} = A_{17,8}A_{7,28}\gamma, \\
\gamma A_{24,30} = A_{17,8}A_{20,14}A_{8,17}\gamma.
\end{array}
$$

P_{617}: $\zeta = \dfrac{2\sin 4\beta}{\tanh R} i,$

$$
\begin{array}{lll}
\gamma A_{1,15} = A_{28,7}A_{15,1}A_{8,17}\gamma, & \gamma A_{2,29} = A_{17,8}A_{9,12}\gamma, & \gamma A_{3,6} = A_{9,12}\gamma, \\
\gamma A_{4,20} = A_{28,7}A_{10,24}\gamma, & \gamma A_{5,21} = A_{28,7}A_{11,25}\gamma, & \gamma A_{7,28} = A_{17,8}\gamma, \\
\gamma A_{8,17} = A_{28,7}\gamma, & \gamma A_{9,12} = A_{3,6}\gamma, & \gamma A_{10,24} = A_{17,8}A_{4,20}\gamma, \\
\gamma A_{11,25} = A_{17,8}A_{5,21}\gamma, & \gamma A_{13,16} = A_{28,7}A_{6,3}\gamma, & \gamma A_{14,30} = A_{17,8}A_{30,14}A_{7,28}\gamma, \\
\gamma A_{18,27} = A_{17,8}A_{7,28}\gamma, & \gamma A_{19,22} = A_{28,7}A_{23,26}A_{7,28}\gamma, & \gamma A_{23,26} = A_{17,8}A_{19,22}A_{8,17}\gamma.
\end{array}
$$

S_{785}: $\zeta = \dfrac{1}{2\tanh R \sin 7\beta},$

$$
\begin{array}{ll}
\gamma A_{1,21} = A_{21,1}\gamma, & \gamma A_{2,5} = A_{25,28}\gamma, \\
\gamma A_{4,27} = A_{27,4}\gamma, & \gamma A_{6,20} = A_{21,1}A_{28,25}\gamma, \\
\gamma A_{8,16} = A_{15,30}A_{29,9}\gamma, & \gamma A_{9,29} = A_{29,9}\gamma, \\
\gamma A_{11,17} = A_{15,30}A_{7,12}A_{29,9}\gamma, & \gamma A_{13,19} = A_{21,1}A_{18,23}A_{30,15}\gamma, \\
\gamma A_{15,30} = A_{30,15}\gamma, & \gamma A_{18,23} = A_{21,1}A_{23,18}A_{1,21}\gamma,
\end{array}
$$

$$
\begin{array}{l}
\gamma A_{3,26} = A_{26,3}\gamma, \\
\gamma A_{7,12} = A_{9,29}A_{12,7}A_{29,9}\gamma, \\
\gamma A_{10,24} = A_{5,2}A_{29,9}\gamma, \\
\gamma A_{14,22} = A_{21,1}A_{30,15}\gamma, \\
\gamma A_{25,28} = A_{2,5}\gamma.
\end{array}
$$

P_{879}: $\zeta = \dfrac{\sin 2\beta}{\tanh R \sin 3\beta} i,$

$$
\begin{array}{lll}
\gamma A_{1,17} = A_{16,6}A_{10,13}\gamma, & \gamma A_{2,5} = A_{10,13}\gamma, & \gamma A_{3,11} = A_{11,3}\gamma, \\
\gamma A_{4,12} = A_{12,4}\gamma, & \gamma A_{6,16} = A_{16,6}\gamma, & \gamma A_{7,20} = A_{20,7}\gamma, \\
\gamma A_{8,25} = A_{25,8}\gamma, & \gamma A_{9,29} = A_{29,9}\gamma, & \gamma A_{10,13} = A_{2,5}\gamma, \\
\gamma A_{14,28} = A_{29,9}A_{5,2}\gamma, & \gamma A_{15,21} = A_{20,7}A_{6,16}\gamma, & \gamma A_{18,23} = A_{25,8}A_{14,28}A_{6,16}\gamma, \\
\gamma A_{19,26} = A_{25,8}A_{7,20}\gamma, & \gamma A_{22,27} = A_{29,9}A_{17,1}A_{7,20}\gamma, & \gamma A_{24,30} = A_{29,9}A_{8,25}\gamma.
\end{array}
$$

Hence we arrive at the following theorem:

Theorem 4. *The surfaces in $\mathscr{S}$ admitting more than one extremal disks are* S_{75}, S_{184}, S'_{184}, S_{202}, S'_{202}, S_{437}, S'_{437}, S_{481}, S_{483}, S_{489}, S'_{489}, S_{594}, S_{614}, S_{617}, S_{785}, *and* S_{879}. *Moreover, each of them has exactly two extremal disks.*

We see that the 16 *surfaces in Theorem* 4 *are respectively different by virtue of the following theorem.*

Theorem 5. *All surfaces in $\mathscr{S}$ differ from each other. Hence there are* 1726 *extremal surfaces of genus* 3.

Proof. The proof of this theorem is similar to that of Theorem 1 in [6]. (When a surface admits two extremal disks, we have an automorphism T which interchanges the centers of extremal disks. Hence we can use T in place of the hyperelliptic involution J.) □

3. The group of automorphisms and hyperelliptic surfaces

We shall determine the group of automorphisms and the hyperellipticity for each surface of $\mathscr{S}$. The next lemma is used for finding the fixed points of T.

Lemma 6. *Let $P \in \mathscr{P}$ be a polygon with a point ζ $(\neq 0)$ whose projection is the center of an extremal disk. Let γ be $\gamma(z) = (\zeta - z)/(1 - \bar{\zeta}z)$ and B an element of $\langle A_1, \dots, A_{30}\rangle$, the Fuchsian group generated by the side-pairings $A_1, \dots, A_{30}$ of P. If $B\gamma$ has a fixed point located in P, then B can be written as a product of at most two side-pairings of P.*

Proof. Let z be a fixed point of $B\gamma$ in P. Since $\gamma(z) = B^{-1}(z)$, we have $\rho(0, B(0)) = \rho(0, B^{-1}(0)) \leq \rho(0, B^{-1}(z)) + \rho(B^{-1}(z), B^{-1}(0)) = \rho(0, \gamma(z)) + \rho(z, 0) = \rho(\zeta, z) + \rho(z, 0) \leq 2l + l \approx 7.164 < \rho_4$. Since $B(0)$ is a pre-image of the center o of an extremal disk, $\rho(0, B(0)) = 0, \rho_1, \rho_2$, or ρ_3. From Remark 2 it follows that $B(0)$ is the center of a disk in the region $(\bigcup_{m,l} A_m A_l(P)) \cup (\bigcup_k A_k(P)) \cup P$. Therefore $B(0) = A_m A_l(0)$, $A_k(0)$, or 0 for some m, l or k, hence $B = A_m A_l$, A_k, or the identity. □

We shall consider S_{75} and S_{184} as examples of a surface with two extremal disks.

S_{75}: First, we shall show that the group of automorphisms $\mathrm{Aut}(S_{75})$ is isomorphic to the cyclic group $\mathbf{Z}_2$ of order 2. Let o and p be the centers of the extremal disks, where o is the projection of the origin and p is the projection of $\zeta = (2 \sin 3\beta)/\tanh R$. We have already shown that S_{75} has an involution T which maps p to o. Suppose that S_{75} has another automorphism T'. Then T' either fixes o or interchanges o with p. If $T'(o) = p$, TT' fixes o. Then we can take a lift of TT' such that it fixes the origin. From the side-pairings of P_{75} it follows that P_{75} does not admit any non-trivial rotation around the origin. Hence the lift is the identity and $T' = T$. If $T'(o) = o$, then T' must be the identity of S_{75}. Thus $\mathrm{Aut}(S_{75}) = \mathbf{Z}_2$.

Next, we shall show that S_{75} is hyperelliptic. For this purpose, it is sufficient to find $2g+2=8$ fixed points of T, that is, T is the hyperelliptic involution. As mentioned before, we have a lift γ of T such that $\gamma(z)=(\zeta-z)/(1-\bar{\zeta}z)$. A fixed point of T is the projection of a fixed point of $B\gamma$, where B is an element of $\langle A_1,\ldots,A_{30}\rangle$ of P_{75}. We shall give B and the fixed points of $B\gamma$ located in P_{75} below, where fixed points are approximate values.

B	Fixed points of $B\gamma$ in P_{75}
id	0.3560
$A_{1,19}$	$-0.4254-0.5081i$
$A_{2,22}$	$-0.0413-0.7147i$
$A_{3,24}$	$0.2680-0.7160i$
$A_{27,6}$	$0.2680+0.7160i$
$A_{28,8}$	$-0.0413+0.7147i$
$A_{29,11}$	$-0.4254+0.5081i$
$A_{30,15}$	-0.6358

As a consequence, T has 8 fixed points on S_{75}. Furthermore they are the Weierstrass points.

Similarly we can show that S_{483}, S_{489}, S'_{489}, S_{594}, S_{785}, S_{879} are hyperelliptic and they have the group of automorphisms $\mathbf{Z}_2$. As reference we shall give the tables with respect to the Weierstrass points for these surfaces.

B	Fixed points of $B\gamma$ in P_{483}
id	$0.5349i$
$A_{5,11}$	$-0.4841+0.6249i$
$A_{10,4}$	$0.4841+0.6249i$
$A_{1,15}A_{6,3}$	$-0.7833+0.1067i$
$A_{14,30}A_{9,12}$	$0.7833+0.1067i$
$A_{25,21}A_{7,28}$	$-0.2992-0.7317i$
$A_{20,24}A_{8,17}$	$0.2992-0.7317i$
$A_{26,23}A_{7,28}$	$-0.6994i$

B	Fixed points of $B\gamma$ in P_{489}
id	$0.5349i$
$A_{5,11}$	$-0.4841+0.6249i$
$A_{6,13}$	$-0.6057+0.3497i$
$A_{9,2}$	$0.6057+0.3497i$
$A_{25,21}A_{7,28}$	$-0.2992-0.7317i$
$A_{20,24}A_{8,17}$	$0.2992-0.7317i$
$A_{26,23}A_{7,28}$	$-0.6994i$
$A_{15,1}A_{8,17}$	$0.7833+0.1067i$

B	Fixed points of $B\gamma$ in P_{594}
id	$0.2767i$
$A_{3,13}$	$-0.7206 + 0.3455i$
$A_{5,16}$	$-0.7299 - 0.1318i$
$A_{7,21}$	$-0.2193 - 0.6527i$
$A_{8,24}$	$0.2193 - 0.6527i$
$A_{10,29}$	$0.7299 - 0.1318i$
$A_{12,2}$	$0.7206 + 0.3455i$
$A_{19,23}A_{7,21}$	$-0.8271i$

B	Fixed points of $B\gamma$ in P_{785}
id	0.4644
$A_{1,21}$	$-0.1458 - 0.6110i$
$A_{3,26}$	$0.5635 - 0.5575i$
$A_{27,4}$	$0.5635 + 0.5575i$
$A_{29,9}$	$-0.1458 + 0.6110i$
$A_{30,15}$	-0.5515
$A_{13,19}A_{30,15}$	$-0.5571 - 0.5163i$
$A_{17,11}A_{30,15}$	$-0.5571 + 0.5163i$

B	Fixed points of $B\gamma$ in P_{879}
id	$0.3986i$
$A_{4,12}$	$-0.6211 + 0.4961i$
$A_{6,16}$	$-0.6743 - 0.1030i$
$A_{7,20}$	$-0.3247 - 0.5246i$
$A_{8,25}$	$0.3247 - 0.5246i$
$A_{9,29}$	$0.6743 - 0.1030i$
$A_{11,3}$	$0.6211 + 0.4961i$
$A_{18,23}A_{7,20}$	$-0.7783i$

S_{184}: Let o and p be the centers of extremal disks. We know that S_{184} has an involution T obtained by $\gamma(z) = (\zeta - z)/(1 - \bar{\zeta}z)$, where $\zeta = (2\sin 4\beta)i/\tanh R$. First we shall show that T has 4 fixed points by considering the fixed points of $B\gamma$, where B is an element of $\langle A_1, \dots, A_{30}\rangle$ of P_{184}. By Lemma 6 it is enough to consider B as a product of at most two side-pairings. Then we see that there are 4 points z_1, z_2, z_3, z_4 in P_{184} respectively fixed by some $B\gamma$.

B	Fixed points of $B\gamma$ in P_{184}
id	$z_1 \approx 0.5349i$
$A_{9,2}$	$z_2 \approx 0.6057 + 0.3497i$

$A_{10,4}$	$z_3 \approx 0.4841 + 0.6249i$
$A_{25,21}A_{7,28}$	$z_4 \approx -0.2992 - 0.7317i$

Hence T has exactly 4 fixed points.

From the side-pairings of P_{184}, we see that the rotation of $2\pi/3$ around the origin induces an automorphism σ of S_{184} of order 3. Note that the fixed points of σ are o and p and that a non-trivial automorphism fixing o is σ or σ^2. We shall show that T and σ generate the dihedral group D_3 of order 6, namely, they satisfy a relation $\sigma T \sigma T = 1$. Since $\sigma T \sigma T$ fixes o, it follows that $\sigma T \sigma T = 1, \sigma$ or σ^2. If $\sigma T \sigma T = \sigma$, then $\sigma = 1$, a contradiction. If $\sigma T \sigma T = \sigma^2$, then $T \sigma T = \sigma$. For a fixed point q of T, this relation implies that $\sigma(q)$, $\sigma^2(q)$ are also fixed by T. Since q, $\sigma(q)$, $\sigma^2(q)$ are distinct, the fourth fixed point of T which is distinct from q, $\sigma(q)$, $\sigma^2(q)$ must be fixed by σ, that is, o or p, a contradiction. Hence the relation $\sigma T \sigma T = 1$ holds. Any automorphism T' of S_{184} satisfies $T' \in \langle \sigma \rangle$ or $T' \in T\langle \sigma \rangle$ according as $T'(o) = o$ or $T'(o) = p$, respectively. Consequently we have $\mathrm{Aut}(S_{184}) = D_3$.

The involutions of S_{184} are T, σT, and $\sigma^2 T$. If q is a fixed point of σT, then T fixes $\sigma(q)$, so that $\sigma(q)$ is in $\{p_1, p_2, p_3, p_4\}$, the set of fixed points of T. Namely, $q = \sigma^2(p_j)$ for some j. Conversely, it is clear that $\sigma^2(p_j)$ is a fixed point of σT. Therefore σT has exactly 4 fixed points. In the same manner $\sigma^2 T$ has exactly 4 fixed points. Since every involution has just 4 fixed points, it is not the hyperelliptic involution. Hence we see that S_{184} is non-hyperelliptic. The pre-images of the fixed points of T, σT or $\sigma^2 T$ in P_{184} are shown in Figure 4.

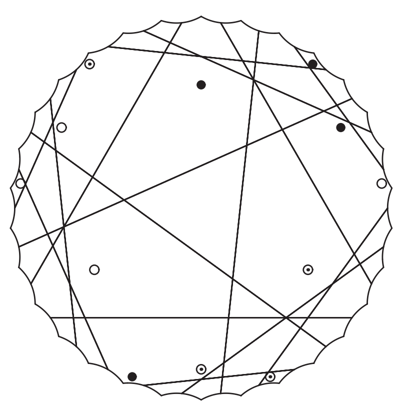

FIGURE 4. Pre-images of the fixed points of $T(\bullet)$, $\sigma T(\odot)$, or $\sigma^2 T(\circ)$

Similarly S'_{184}, S_{202}, S'_{202}, S_{437}, S'_{437}, S_{481}, S_{614}, and S_{617} are non-hyperelliptic and they have the group of automorphisms D_3. As reference we shall give the tables with respect to the fixed points of T.

B	Fixed points of $B\gamma$ in P_{202}
id	z_1
$A_{9,2}$	z_2
$A_{20,24}A_{8,17}$	$e^{4\pi i/3}z_3$
$A_{4,30}A_{9,2}$	$e^{2\pi i/3}z_4$

B	Fixed points of $B\gamma$ in P_{437}
id	z_1
$A_{6,13}$	$e^{2\pi i/3}z_2$
$A_{10,4}$	z_3
$A_{24,30}A_{9,22}$	$e^{2\pi i/3}z_4$

B	Fixed points of $B\gamma$ in P_{481}
id	z_1
$A_{26,23}A_{7,28}$	$e^{4\pi i/3}z_2$
$A_{20,24}A_{8,17}$	$e^{4\pi i/3}z_3$
$A_{25,21}A_{7,28}$	z_4

B	Fixed points of $B\gamma$ in P_{614}
id	z_1
$A_{26,23}A_{7,28}$	$e^{4\pi i/3}z_2$
$A_{10,4}$	z_3
$A_{5,11}$	$e^{4\pi i/3}z_4$

B	Fixed points of $B\gamma$ in P_{617}
id	z_1
$A_{26,23}A_{7,28}$	$e^{4\pi i/3}z_2$
$A_{30,14}A_{7,28}$	$e^{2\pi i/3}z_3$
$A_{15,1}A_{8,17}$	$e^{2\pi i/3}z_4$

We shall consider S_{498} and S_{138} as examples of a surface with a unique extremal disk (Figure 5).

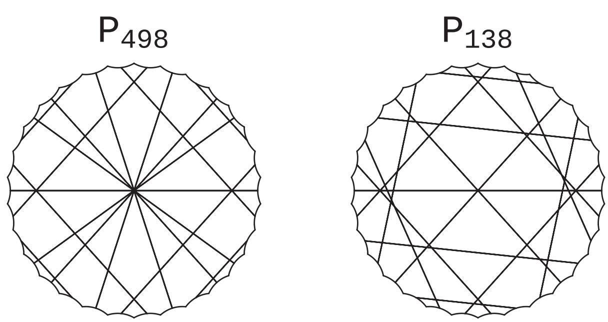

FIGURE 5. P_{498} and P_{138}

S_{498}: From the side-pairings of P_{498} it follows that there exists a unique non-trivial automorphism of S_{498} induced by the rotation of angle π around the origin. Since it is an involution with 8 fixed points, S_{498} is hyperelliptic and $\mathrm{Aut}(S_{498}) = \mathbf{Z}_2$.

S_{138}: Similarly we see that $\mathrm{Aut}(S_{138}) = \mathbf{Z}_2$ and the involution of S_{138} has only 4 fixed points. Hence S_{138} is non-hyperelliptic.

In the same way we obtain hyperelliptic extremal surfaces of genus 3.

Theorem 7. *The hyperelliptic surfaces in* $\mathscr{S}$ *are* S_{75}, S_{483}, S_{489}, S'_{489}, S_{498}, S_{499}, S'_{499}, S_{500}, S_{570}, S_{594}, S_{785}, *and* S_{879}.

4. Main results

We shall classify the extremal surfaces of genus 3 according to our results in the preceding sections. Here, the surfaces S_{481}, S_{500}, and S_{137} have already appeared in [4, 5].

Theorem 8. *The surfaces in* $\mathscr{S}$ *are classified as follows*:

1. *Extremal surfaces with two extremal disks*: *there are* 16 *surfaces* (12 *surfaces up to conformal or anti-conformal equivalence*).

(1) *Hyperelliptic surfaces*

S	The centers of extremal disks	Aut S
S_{75}	$\pi(0),\ \pi\left(\dfrac{2\sin 3\beta}{\tanh R}\right)$	$\mathbf{Z}_2$
S_{483}	$\pi(0),\ \pi\left(\dfrac{2\sin 4\beta}{\tanh R}i\right)$	$\mathbf{Z}_2$
$S_{489},\ S'_{489}$	$\pi(0),\ \pi\left(\dfrac{2\sin 4\beta}{\tanh R}i\right)$	$\mathbf{Z}_2$
S_{594}	$\pi(0),\ \pi\left(\dfrac{\sin\beta}{\tanh R\sin 2\beta}i\right)$	$\mathbf{Z}_2$
S_{785}	$\pi(0),\ \pi\left(\dfrac{1}{2\tanh R\sin 7\beta}\right)$	$\mathbf{Z}_2$
S_{879}	$\pi(0),\ \pi\left(\dfrac{\sin 2\beta}{\tanh R\sin 3\beta}i\right)$	$\mathbf{Z}_2$

(2) *Non-hyperelliptic surfaces*

S	The centers of extremal disks	Aut S
$S_{184},\ S'_{184}$	$\pi(0),\ \pi\left(\dfrac{2\sin 4\beta}{\tanh R}i\right)$	D_3

S_{202}, S'_{202}	$\pi(0)$, $\pi\left(\frac{2 \sin 4\beta}{\tanh R} i\right)$	D_3
S_{437}, S'_{437}	$\pi(0)$, $\pi\left(\frac{2 \sin 4\beta}{\tanh R} i\right)$	D_3
S_{481}	$\pi(0)$, $\pi\left(\frac{2 \sin 4\beta}{\tanh R} i\right)$	D_3
S_{614}	$\pi(0)$, $\pi\left(\frac{2 \sin 4\beta}{\tanh R} i\right)$	D_3
S_{617}	$\pi(0)$, $\pi\left(\frac{2 \sin 4\beta}{\tanh R} i\right)$	D_3

2. *Extremal surfaces with a unique extremal disk*

(1) *Surfaces with a non-trivial automorphism*

(i) *Hyperelliptic surfaces*

S	Aut S
S_{498}, S_{499}, S_{500}, S_{570}, S'_{499}	$\mathbf{Z}_2$

(ii) *Non-hyperelliptic surfaces*: *there are* 97 *surfaces* (52 *surfaces up to conformal or anti-conformal equivalence*).

S	Aut S
S_{138}, S_{139}, S_{140}, S_{141}, S_{317}, S_{344}, S_{345}, S_{357}, S_{358}, S_{359}, S_{360}, S_{361}, S_{362}, S_{377}, S_{378}, S_{379}, S_{380}, S_{398}, S_{401}, S_{402}, S_{405}, S_{427}, S_{428}, S_{432}, S_{439}, S_{512}, S_{513}, S_{571}, S_{584}, S_{585}, S_{669}, S_{671}, S_{672}, S_{746}, S_{748}, S_{807}, S_{808}, S_{809}, S_{810}, S_{831}, S_{832}, S_{834}, S_{835}, S_{837}, S_{850}, S_{852}, S_{874}, S_{875}, S_{926}, S_{927}, S'_{138}, S'_{139}, S'_{140}, S'_{141}, S'_{344}, S'_{345}, S'_{357}, S'_{358}, S'_{359}, S'_{360}, S'_{361}, S'_{362}, S'_{377}, S'_{378}, S'_{379}, S'_{380}, S'_{398}, S'_{401}, S'_{402}, S'_{405}, S'_{432}, S'_{439}, S'_{512}, S'_{513}, S'_{570}, S'_{571}, S'_{584}, S'_{585}, S'_{671}, S'_{746}, S'_{748}, S'_{807}, S'_{808}, S'_{809}, S'_{810}, S'_{831}, S'_{832}, S'_{835}, S'_{850}, S'_{852}, S'_{874}, S'_{875}, S'_{926}, S'_{927}	$\mathbf{Z}_2$
S_{315}, S'_{315}	$\mathbf{Z}_3$
S_{316}	$\mathbf{Z}_6$

(2) *Surfaces only with the trivial automorphism*: *there are* 1608 *surfaces* (859 *surfaces up to conformal or anti-conformal equivalence*).

S	Aut S
The other surfaces in $\mathscr{S}$	$\{1\}$

Acknowledgements. The author would like to thank the referee for the careful reading of the first submitted version of this paper.

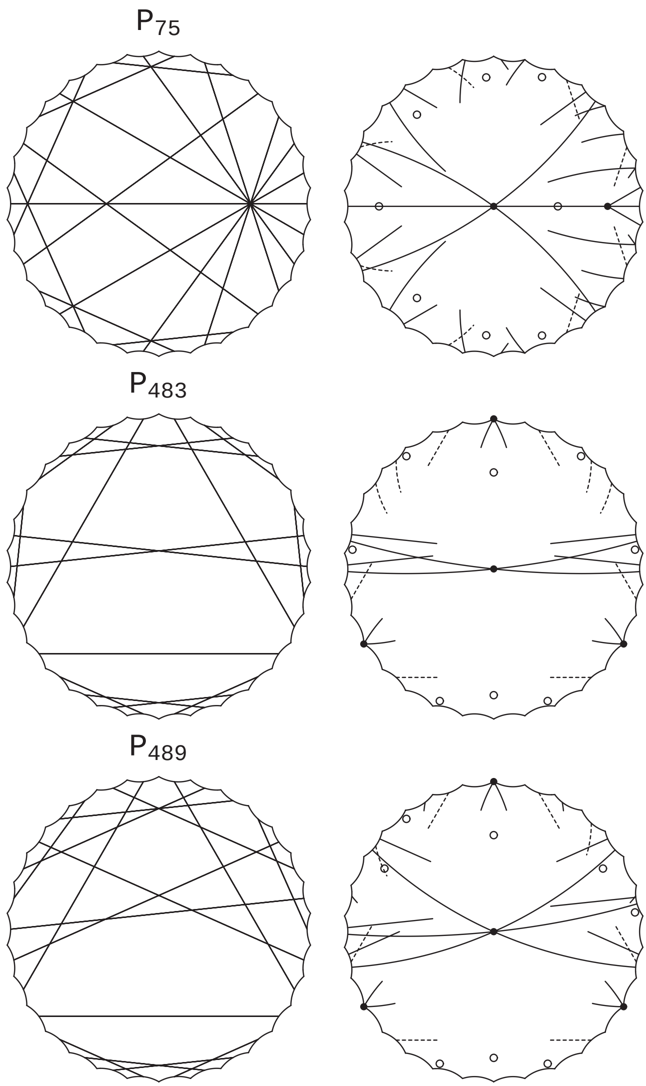

FIGURE 6. Side-pairings, the centers of extremal disks (•) and the Weierstrass points (∘) for hyperelliptic surfaces with two extremal disks

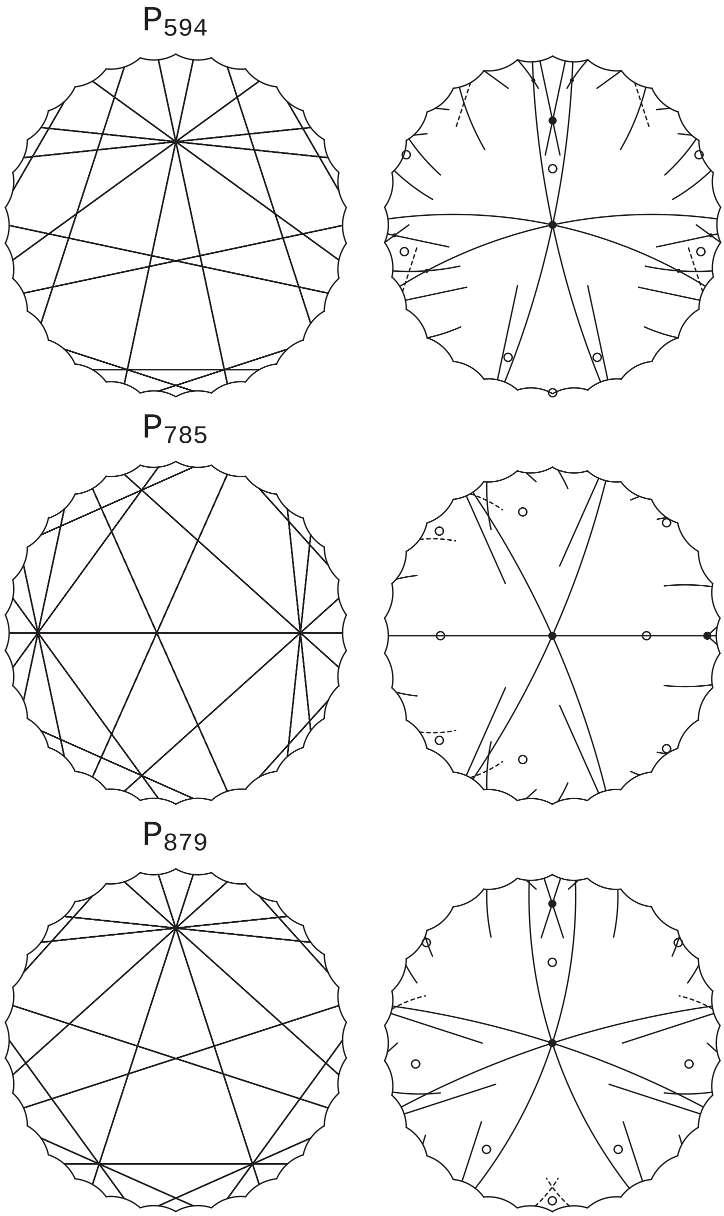

FIGURE 7. Side-pairings, the centers of extremal disks (•) and the Weierstrass points (∘) for hyperelliptic surfaces with two extremal disks

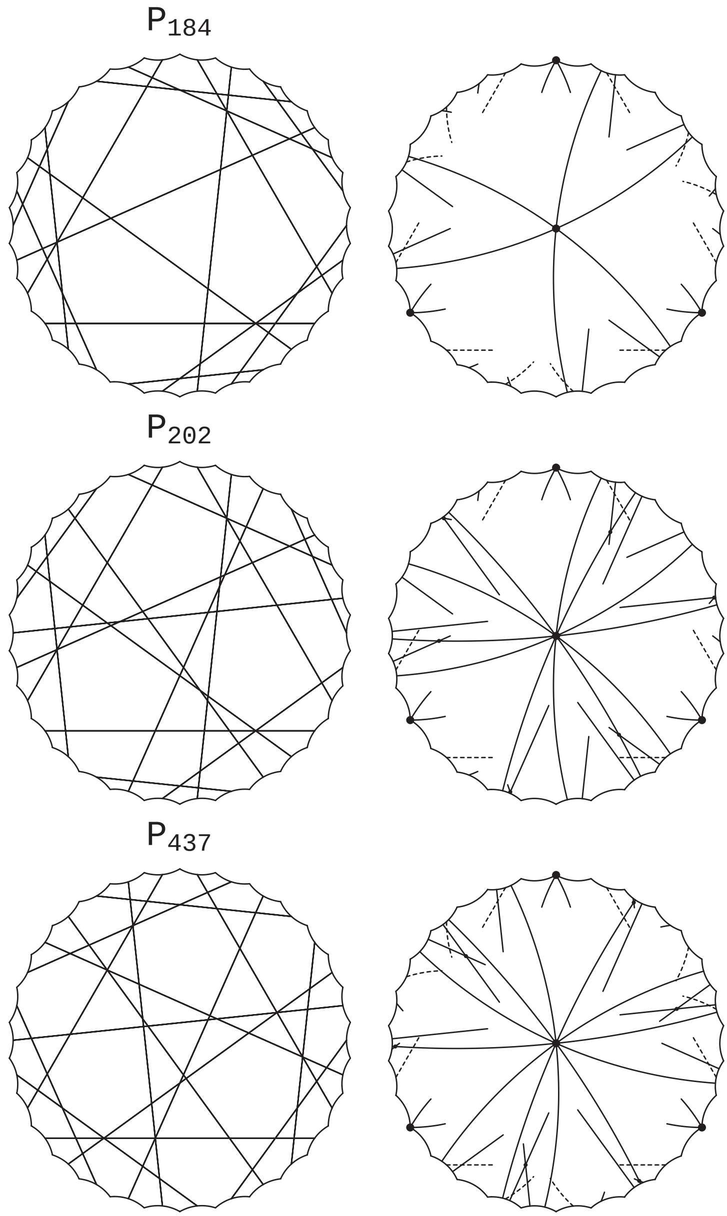

FIGURE 8. Side-pairings and the centers of extremal disks (•) for non-hyperelliptic surfaces with two extremal disks

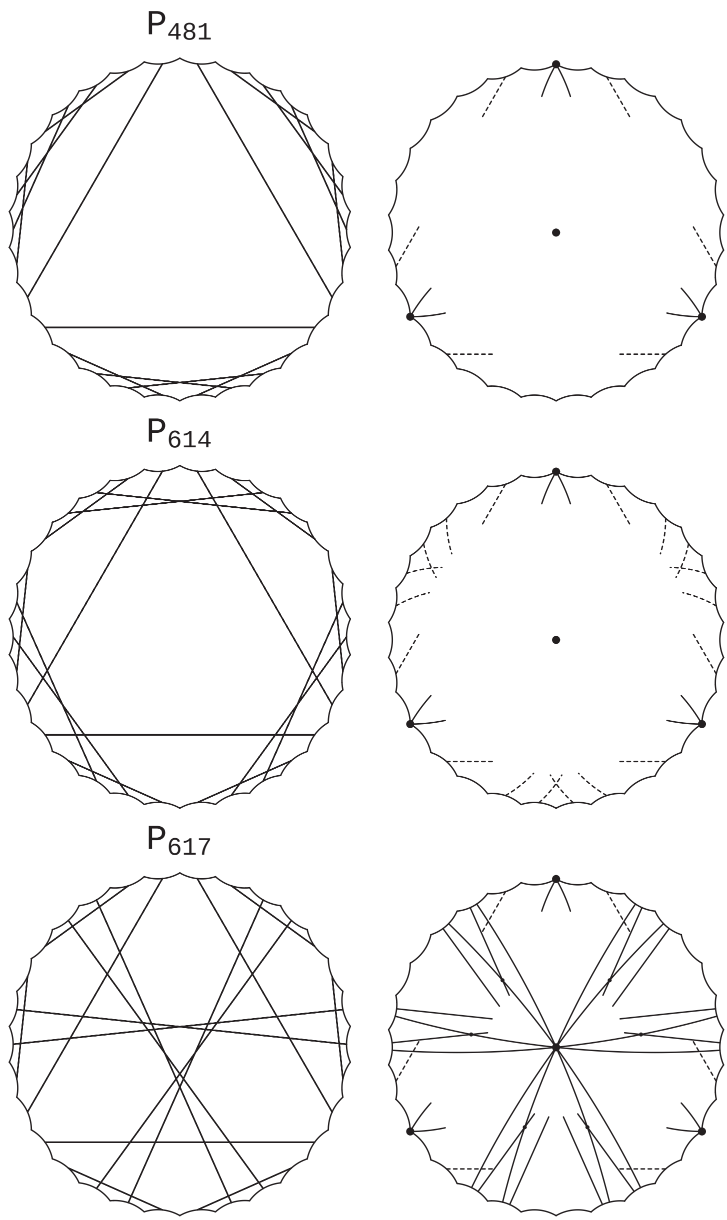

FIGURE 9. Side-pairings and the centers of extremal disks (•) for non-hyperelliptic surfaces with two extremal disks

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