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ON THE ŁOJASIEWICZ EXPONENT AND NEWTON POLYHEDRON

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Abstract

The object of this paper is to give an estimation of the Łojasiewicz exponent of the gradient of a holomorphic function under Kouchnirenko's nondegeneracy condition, using information from the Newton polyhedron.

Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a germ of holomorphic function. The Łojasiewicz exponent of gradient of f, $L(f)$ is by definition

$$L(f) := \inf\{\lambda > 0 : |\text{grad } f| \geq \text{const. } |x|^{\lambda} \text{ near zero}\}.$$

It is well-known that $L(f) < \infty$ if and only if f has an isolated singularity at the origin. Chang and Lu [1] proved that for any integer r greater than $L(f)$, f is a C^0-sufficient, r-jet in holomorphic functions, i.e., adding to the function f monomials of order greater than $L(f)$ does note change its topological type. Originally this was proved by Kuo and Kuiper in the real case (see [4, 5]). Teissier [9] showed that C^0-sufficiency degree of f (i.e., the minimal integer r such that f is C^0-sufficient, r-jet) is equal to $[L(f)] + 1$, where $[L(f)]$ denotes the integral part of $L(f)$. We were motived by the work of Lichtin [7] and Fukui [2] who used the Newton polyhedron of f to give an estimation of $L(f)$, where f is non-degenerate in the sense of Kouchnirenko. In this note, following this procedure, we estimate the Łojasiewicz exponent of gradient $L(f)$ (Theorem 1 below). However, our estimations are based on other ideas, more precisely, we use the Kouchnirenko's theorem [3] on the Newton number and the geometric characterization of μ-constancy in [6, 9].

1. Newton polyhedron, main results

Now we recall some basic notions about the Newton polyhedron (see [3, 8] for details) and state the main result. Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be an analytic function defined by a convergent power series $\sum_{\nu} c_{\nu} x^{\nu}$. Also, let $\mathbf{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbf{R}^n, x_i \geq 0, i = 1, \dots, n\}$ and $\mathbf{Z}^n_+ = \mathbf{Z}^n \cap \mathbf{R}^n_+$. A Newton poly-

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hedron $\Gamma_+(f) \subset \mathbf{R}^n$ is defined by the convex hull of $\{v + \mathbf{R}^n_+ \mid c_v \neq 0\}$, and $\Gamma(f)$ be the union of the compact faces of $\Gamma_+(f)$. Define f_γ by $\sum_{v \in \gamma} c_v x^v$ for γ face of $\Gamma(f)$. We say that f is non-degenerate in Kouchnirenko's sense if, for any γ face of $\Gamma(f)$, the equations $\dfrac{\partial f_\gamma}{\partial x_1} = \cdots = \dfrac{\partial f_\gamma}{\partial x_n} = 0$ have no common solution on $x_1 \cdots x_n \neq 0$. The power series f is said to be convenient if $\Gamma_+(f)$ meets each of the coordinate axes. We let $\Gamma_-(f)$ denote the compact polyhedron which is the cone over $\Gamma(f)$ with the origin as a vertex. When f is convenient, the Newton number $v(f)$ is defined as $v(f) = n!V_n - (n-1)!V_{n-1} + \cdots + (-1)^{n-1}V_1 + (-1)^n$, where V_n is the n-dimensional volumes of $\Gamma_-(f)$ and for $1 \leq k \leq n-1$, V_k is the sum of the k-dimensional volumes of the intersection of $\Gamma_-(f)$ with the coordinate planes of dimension k. The Newton number may also be defined for non-convenient analytic function (see [3]). Finally, we let

$$\begin{aligned} a_j &= 1 + f(e_j) \quad \text{for } j = 1, \ldots, n, \\ r_j(f) &= \min\{m \in \mathbf{Z}_+ - \{0\} \mid v(f) = v(f + a_j x_j^m)\}, \quad \text{and} \\ r(f) &= \max\{r_j(f) \mid j = 1, \ldots, n\}, \end{aligned} \tag{1.1}$$

where e_j denotes the j-th unit row vector $(0, \ldots, 0, 1, 0, \ldots, 0)$.

Now we can state the main result.

THEOREM 1. *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be an analytic function having an isolated singularity at the origin. Suppose that f is non-degenerate in the sense of Kouchnirenko. Then $r(f) - 2 < L(f) \leq r(f) - 1$.*

2. Proof of the theorem

First we show that $L(f) > r(f) - 2$. So suppose now that $L(f) \leq r(f) - 2$, and modulo a permutation of coordinates in $\mathbf{C}^n$ we may assume $r(f) = r_1$. Because the non-degenerate condition of Kouchnirenko is an open condition (see [3, 8] for details), we can find an analytic family $F(x,t) = f(x) + \gamma(t)x_1^{r_1 - 1}$ such that $F(x, 0) = f(x)$ and $F_t(x) = F(x,t)$ is non-degenerate in Kouchnirenko's sense for each t. Since $L(f) \leq r_1 - 2$, then there exists a positive c such that $|\text{grad } f| \geq c|x|^{r_1 - 2}$ in a neighbourhood U of 0. Also, for t sufficiently small so that $|\gamma(t)| \leq \dfrac{c}{2}$, we have

$$|\text{grad } F(x,t)| \geq |\text{grad } f| - |\gamma(t)x_1^{r_1 - 2}| \geq \frac{c}{2}|x|^{r_1 - 2} \quad \text{as } x \in U.$$

Then, we get

$$\left|\frac{\partial F}{\partial t}(x,t)\right| = \left|\frac{\partial \gamma}{\partial t}(t)x_1^{r_1 - 1}\right| \ll |x|^{r_1 - 2} \lesssim |\text{grad } F(x,t)| \quad \text{as } (x,t) \to (0,0). \tag{2.1}$$

It follows from the geometric characterization of Lê and Saito [6] that F_t is μ-constant, where μ denotes the Milnor number. This fact, together with the Kouchnirenko's theorem [3], (i.e., the nondegeneracy condition implies $\mu(F_t) = \nu(F_t)$), gives $\nu(f(x)) = \nu(f(x) + \gamma(t)x_1^{r_1-1})$, which contradicts the definition of r_1 in (1.1).

In order to complete the proof of the theorem we need the following lemma.

LEMMA 2. *For any subset* $J \subset \{1, \dots, n\}$, *we have*

$$\nu(f) = \nu\left(f + \sum_{j \in J} a_j x_j^{r_j}\right).$$

Proof. First note that if $\#J = 1$, one finds this lemma by definition of r_j in (1.1). We will prove this lemma only for $\#J = 2$, the general case can be proved in a similar way. Let $J = \{j_1, j_2\}$, then it easy to see that $\Gamma_-(f) = \Gamma_-(f + a_{j_1}x_{j_1}^{r_{j_1}}) \cup \Gamma_-(f + a_{j_2}x_{j_2}^{r_{j_2}})$ is a polyhedral decomposition of $\Gamma_-(f)$, and $\Gamma_-(f + a_{j_1}x_{j_1}^{r_{j_1}} + a_{j_2}x_{j_2}^{r_{j_2}}) = \Gamma_-(f + a_{j_1}x_{j_1}^{r_{j_1}}) \cap \Gamma_-(f + a_{j_2}x_{j_2}^{r_{j_2}})$. Then, we have

$$\nu(\Gamma_-(f)) = \nu(\Gamma_-(f + a_{j_1}x_{j_1}^{r_{j_1}})) + \nu(\Gamma_-(f + a_{j_2}x_{j_2}^{r_{j_2}})) - \nu(\Gamma_-(f + a_{j_1}x_{j_1}^{r_{j_1}} + a_{j_2}x_{j_2}^{r_{j_2}}))$$

$$\nu(f) = \nu(f + a_{j_1}x_{j_1}^{r_{j_1}}) + \nu(f + a_{j_2}x_{j_2}^{r_{j_2}}) - \nu(f + a_{j_1}x_{j_1}^{r_{j_1}} + a_{j_2}x_{j_2}^{r_{j_2}}).$$

Thus, the assumption that $\nu(f) = \nu(f + a_j x_j^{r_j})$ implies $\nu(f) = \nu(f + a_{j_1}x_{j_1}^{r_{j_1}} + a_{j_2}x_{j_2}^{r_{j_2}})$. □

Now we are ready to prove that $r(f) - 1$ is an upper bound for the Łojasiewicz exponent $L(f)$. Define an analytic family $F(x,t) = f(x) + \sum_{j=1}^{n} \gamma_j(t)x_j^{r_j}$ such that $F(x,0) = f(x)$ and $F_t(x) = F(x,t)$ is non-degenerate in Kouchnirenko's sense for each t. This is again possible because of the non-degenracy condition of Kouchnirenko is an open condition (see [3, 8]). Recall that $\nu(F_t) = \mu(F_t)$ by Kouchnerinko [3], it follows from the above lemma that F_t is μ-constant. According to Teissier, ([9] Remarque 5 and [10], Chap. II), the μ-constancy of F_t implies that

$$L(f) = L(F_0) \leq L(F_t). \tag{2.2}$$

On the other hand, fix $t \in \mathbf{C}$. So we can find from Yoshinaga's theorem ([11], Theorem 1.7) that F_t is non-degenerate in Kouchnirenko's sense, if and only if there exists a positive ε such that

$$\sum_{i=1}^{n} \left| x_i \frac{\partial F_t}{\partial x_i} \right| \geq \varepsilon \sum_{\alpha \in \mathrm{ver}(F_t)} |x^\alpha| \quad \text{as } x \text{ near } 0, \tag{2.3}$$

where $\mathrm{ver}(F_t) = \{\alpha : \alpha \text{ is a vertex of } \Gamma(F_t)\}$. But, $r_j e_j \in \mathrm{ver}(F_t)$ for $t \neq 0$, the axial vertices of $\Gamma_+(F_t)$ (recall that e_j denotes the j-th unit row vector), which implies that $L(F_t) \leq r(f) - 1$ for $t \neq 0$. Together with (2.2), this completes the proof of theorem.

Remark 3. *The above inequality* (2.3) *can be proved directly by an argument, based on the curve selection lemma.*

We conclude with several examples.

Example 4. *Consider the map germ* $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ *given by* $f(x, y) = xy^8 + x^2y^3 + yx^7 + x^p + y^q$, *where* $p, q \geq 14$. *It is not hard to see that* $r_1(f) = 11$ *and* $r_2(f) = 13$. *It follows from Theorem* 1 *that* $11 < L(f) \leq 12$ *and so* f *is a* C^0*-sufficient*, 13*-jet*. *For the comparison, we note that from the Lichtin and Fukui results we have* $L(f) \leq \max\{p, q\} - 1$. *Their estimation, depens on the choice of the axial vertices* $(p, 0)$ *and* $(q, 0)$.

Example 5. *Let* $f : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ *given by* $f(x, y, z) = x^2(x + y)^2 + x(x + y)^4 + x^5 + (x + y + z)^5$. *This function is degenerate in Kouchnirenko's sense. However, by a linear transformation* $X = x$, $Y = x + y$ *and* $Z = x + y + z$ *we obtain a non-degenerate in Kouchnirenko's sense* $\tilde{f}(X, Y, Z) = X^2Y^2 + XY^4 + X^5 + Z^5$ *with the same value of the Łojasiewicz exponent of the gradient. Moreover, by the formula of Newton number, it is not difficult to compute that*

$$\begin{aligned} \mu(\tilde{f}) = \nu(\tilde{f}) = \nu(\tilde{f} + Y^6) = 48, \\ \nu(\tilde{f} + X^4) = \nu(\tilde{f} + Y^5) = 44 \quad \textit{and} \\ \nu(\tilde{f} + Z^4) = 36. \end{aligned}$$

Hence, we get $r_1(\tilde{f}) = 5$, $r_2(\tilde{f}) = 6$ *and* $r_3(\tilde{f}) = 5$. *Thus, from the above Theorem* 1 *we have* $4 < L(f) = L(\tilde{f}) \leq 5$. *In this case the Fukui result gives* $L(\tilde{f}) \leq 5$.

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