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PICARD VALUES AND DERIVATIVES OF MEROMORPHIC FUNCTIONS*

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Abstract

Let f be a transcendental meromorphic function and R be a rational function, $R \not\equiv 0$, and let k be a positive integer. In this paper, we obtain some results concerning the zeros of $f^{(k)} - R$, which generalize and improve related results of Wang-Fang and Bergweiler-Pang.

1. Introduction

In 1998, Wang and Fang [11] proved the following results.

Theorem A. *Let f be a transcendental meromorphic function and $k \in \mathbf{N}$. If f has only zeros of multiplicity at least $k+1$ and poles of multiplicity at least* 2, *then, for each $c \in \mathbf{C}\backslash\{0\}$, $f^{(k)} - c$ has infinitely many zeros.*

Theorem B. *Let f be a transcendental meromorphic function and $k \in \mathbf{N}$. If f has only zeros of multiplicity at least* 3, *then, for each $c \in \mathbf{C}\backslash\{0\}$, $f^{(k)} - c$ has infinitely many zeros.*

A natural problem arises: *What can we say if the constant c in the above results is replaced by a rational function $R(z) \not\equiv 0$?* (see [1]).

In 2000, Fang [3] considered the fixed points of f' and obtained

Theorem C. *Let f be a transcendental meromorphic function. If f has only multiple zeros and poles, then $f' - z$ has infinitely many zeros.*

Recently, Bergweiler and Pang [2] proved the following result, which is a significant improvement of Theorem C.

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Theorem D. *Let f be a transcendental meromorphic function, and let R be a rational function, $R \not\equiv 0$. Suppose that all zeros and poles of f are multiple, except possibly finite many, then $f' - R$ has infinitely many zeros.*

In this paper, we borrow the idea of Bergweiler and Pang [2] and prove the following results with a simpler proof.

Theorem 1. *Let f be a transcendental meromorphic function, and let R be a rational function, $R \not\equiv 0$, $k \in \mathbf{N}$. Suppose that all zeros of f have multiplicity at least $k+1$ and all poles of f have multiplicity at least 2, except possibly finite many. Then $f^{(k)} - R$ has infinitely many zeros.*

Theorem 2. *Let f be a transcendental meromorphic function, and let R be a rational function, $R \not\equiv 0$, $k \in \mathbf{N}$. Suppose that all zeros of f have multiplicity at least $k+2$, except possibly finite many. Then $f^{(k)} - R$ has infinitely many zeros.*

It seems reasonable to conjecture that the conclusion of Theorem 1 or Theorem 2 still holds under a considerably weaker condition that the zeros of f have multiplicity at least 3. In this direction, we have the following result.

Theorem 3. *Let f be a transcendental meromorphic function, $k \in \mathbf{N}$, and let P be a polynomial, $P \not\equiv 0$. If all zeros of f have multiplicity at least 3, except possibly finite many, then $f^{(k)} - P$ has infinitely many zeros.*

We shall use some standard notations and results from Nevanlinna theory (see [4, 9, 12]).

2. Some Lemmas

Now we recall some definitions. If there exists a curve $\Gamma \subset \mathbf{C}$ tending to ∞ such that $f(z) \to a$ as $z \to \infty$, $z \in \Gamma$, we call that a is an *asymptotic value* of f. A meromorphic function f is called a *Julia exceptional function* if $f^{\#}(z) = O(1/|z|)$ as $z \to \infty$. Here, as usual, $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ is the spherical derivative of f. It follows easily from the Ahlfors-Shimizu form of the Nevanlinna characteristic function that if f is a Julia exceptional function, then $T(r, f) = O((\log r)^2)$ as $r \to \infty$. The following result concerning functions satisfying this growth condition is due to Hayman [5, 6].

Lemma 1. *Let f be an entire function satisfying*

$$\log M(r, f) = O((\log r)^2)$$

as $r \to \infty$. Then $\log|f(re^{i\theta})| \sim \log M(r, f)$ as $r \to \infty$ for almost every $\theta \in [0, 2\pi]$.

The following result is due to Lehto and Virtanen [7].

LEMMA 2. *A transcendental Julia exceptional function does not have an asymptotic value.*

LEMMA 3. *Let f be a transcendental meromorphic function and $k \in \mathbf{N}$, and let R be a rational function satisfying $R(z) \sim cz^d$ as $z \to \infty$, with $c \in \mathbf{C}\backslash\{0\}$ and $d \in \mathbf{Z}$. Suppose that $f^{(k)} - R$ has only finitely many zeros and that $T(r, f) = O((\log r)^2)$ as $r \to \infty$. Set $g := f(z)/z^{d+k}$. Then g has an asymptotic value.*

Proof. Since $f^{(k)} - R$ has only finitely many zeros, there exists a polynomial $P \not\equiv 0$ such that $h = P/(f^{(k)} - R)$ is entire. By standard results in Nevanlinna theory, we have

$$\log M(r, h) \le 3T(2r, h) \le 3T(2r, f^{(k)}) + O(\log r)$$

and

$$T(2r, f^{(k)}) \le (k+1)T(2r, f) + O(\log r)$$

as $r \to \infty$. Thus $\log M(r, h) = O((\log r)^2)$. By Lemma 1, there exists $\theta \in [0, 2\pi]$ such that $|h(re^{i\theta})|r^{-m} \to \infty$ $(r \to \infty)$ for any $m \in \mathbf{Z}$. Let $m = \deg(P) + 2 + |d|$. It follows that

$$|f^{(k)}(re^{i\theta}) - R(re^{i\theta})| = \left|\frac{P(re^{i\theta})}{h(re^{i\theta})}\right| \le \frac{1}{r^{2+|d|}}$$

for sufficiently large r, say $r \ge r_0$. Hence

$$\int_{r_0}^{r} (f^{(k)}(te^{i\theta}) - R(te^{i\theta}))\, dt$$

tends to a finite limit as $r \to \infty$. We consider three cases.

CASE 1. $d > -1$. We have

$$\lim_{r\to\infty} \frac{f^{(k-1)}(re^{i\theta})}{\frac{c}{d+1}(re^{i\theta})^{d+1}} = 1.$$

Hence

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \frac{f(z)}{z^{d+k}} = \lim_{z\to\infty} \frac{f'(z)}{(d+k)z^{d+k-1}} = \cdots$$
$$= \lim_{z\to\infty} \frac{f^{(k-1)}(z)}{(d+k)(d+k-1)\cdots(d+2)z^{d+1}} = \frac{c}{(d+1)(d+2)\cdots(d+k)}.$$

CASE 2. $d < -1$. We have

$$f^{(k-1)}(z) = a + \frac{c}{d+1} z^{d+1} + O(|z|^d)$$

for some $a \in \mathbf{C}$ as $z \to \infty$. For $k = 1$, we have either $\lim_{z\to\infty} g(z) = \infty$ or $\lim_{z\to\infty} g(z) = c/(d+1)$ (see [2]). For $k \geq 2$, by an elemental calculation, we have that

(a) if $-k \leq d < -1$, then $\lim_{z\to\infty} g(z) = \infty$;
(b) if $d < -k$, then either $\lim_{z\to\infty} g(z) = \infty$ or

$$\lim_{z\to\infty} g(z) = \frac{c}{(d+1)(d+2)\cdots(d+k)}.$$

Case 3. $d = -1$. We have

$$\lim_{z\to\infty} \frac{f^{(k-1)}(z)}{c \log|z|} = 1.$$

Then

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \frac{f(z)}{z^{k-1}} = \lim_{z\to\infty} \frac{f'(z)}{(k-1)z^{k-2}} = \cdots = \lim_{z\to\infty} \frac{f^{(k-1)}(z)}{(k-1)!} = \infty.$$

This completes the proof of Lemma 3. □

Lemma 4 ([11]). *Let f be a meromorphic function of finite order in the plane, b nonzero complex numbers, and k a positive integer. If all zeros of f are of order at least $k+1$ and all poles of f are multiple, and $f^{(k)}(z) \neq b$, then $f(z)$ is a constant.*

Lemma 5 ([11]). *Let f be a meromorphic function of finite order in the plane, b nonzero complex numbers, and k a positive integer. If all zeros of f are of order at least $k+2$ and $f^{(k)}(z) \neq b$, then $f(z)$ is a constant.*

The next is one up-to-date version of Zalcaman's lemma due to Pang and Zalcman [8].

Lemma 6. *Let k be a positive integer and let $\mathscr{F}$ be a family of functions meromorphic in a domain D, such that each function $f \in \mathscr{F}$ has only zeros of multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If $\mathscr{F}$ is not normal at $z_0 \in D$, then, for each $0 \leq \alpha \leq k$, there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathscr{F}$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^{\alpha}} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$. Moreover, g has order at most 2.

3. Proof of Theorems

Proof of Theorem 1. Suppose that $f^{(k)} - R$ has finitely many zeros. We assume that $R(z) \sim cz^d$ as $z \to \infty$, with $c \in \mathbf{C} \setminus \{0\}$ and $d \in \mathbf{Z}$. Define

$$g(z) := f(z)/z^{d+k}.$$

If g is a Julia exceptional function, then $T(r, g) = O((\log r)^2)$ and hence $T(r, f) = O((\log r)^2)$ as $r \to \infty$. Thus, by Lemma 3, g has an asymptotic value. But g has not an asymptotic value by Lemma 2, a contradiction.

Thus g is not a Julia exceptional function, and then there exists a sequence (a_n) in $\mathbf{C}$ such that $a_n \to \infty$ and $a_n g^{\#}(a_n) \to \infty$ as $n \to \infty$. Let $D = \{z \in \mathbf{C} : |z - 1| < 1/2\}$, and set

$$\mathscr{F} = \left\{ g_n(z) := g(a_n z) z^{d+k} = \frac{f(a_n z)}{a_n^{d+k}}, n = 1, 2, 3 \ldots, z \in D \right\}.$$

Since

$$g_n^{\#}(1) = \frac{|a_n g'(a_n) + (d+k) g(a_n)|}{1 + |g(a_n)|^2} \geq |a_n| g^{\#}(a_n) - \frac{|d+k|}{2} \to \infty$$

as $n \to \infty$, we know that $\mathscr{F}$ is not normal at 1. Obviously, for sufficiently large n, all zeros of g_n in D have multiplicity at least $k+1$ and all poles of g_n are multiple. Then by Lemma 6, we can find sequences (n_j), (z_j) and (ρ_j) satisfying $n_j \in \mathbf{N}$, $n_j \to \infty$, $z_j \in D$, $z_j \to 1$, $\rho_j > 0$ and $\rho_j \to 0$ such that

$$G_{n_j}(\zeta) = \frac{g_{n_j}(z_j + \rho_j \zeta)}{\rho_j^k} = \frac{f(a_{n_j}(z_j + \rho_j \zeta))}{\rho_j^k a_{n_j}^{d+k}} \to G(\zeta)$$

locally uniformly with respect to the spherical metric, where $G(\zeta)$ is a non-constant meromorphic function in $\mathbf{C}$, all of whose zeros have multiplicity at least $k+1$. In particular, G is of order at most 2. By Hurwitz's theorem, G has only multiple poles.

Since $R(z) \sim cz^d$ as $z \to \infty$, we have

$$G_{n_j}^{(k)}(\zeta) - \frac{R(a_{n_j}(z_j + \rho_j \zeta))}{a_{n_j}^d} \to G^{(k)}(\zeta) - c$$

and (for j sufficiently large)

$$G_{n_j}^{(k)}(\zeta) - \frac{R(a_{n_j}(z_j + \rho_j \zeta))}{a_{n_j}^d} = \frac{f^{(k)}(a_{n_j}(z_j + \rho_j \zeta)) - R(a_{n_j}(z_j + \rho_j \zeta))}{a_{n_j}^d} \neq 0.$$

It follows from Hurwitz's theorem that either $G^{(k)} \neq c$ or $G^{(k)} \equiv c$ on $\mathbf{C}$. But $G^{(k)} \not\equiv c$ since all zeros of G have multiplicity at least $k+1$. Thus $G^{(k)} \neq c$. By Lemma 4, $G(\zeta)$ must be a constant, a contradiction. This completes the proof of Theorem 1. □

Proof of Theorem 2. Theorem 2 can be proved by using Lemma 5 and the same argument as in Theorem 1. We here omit the details. □

Proof of Theorem 3. For $k = 1$, the conclusion comes from Theorem 2. Now we assume $k \geq 2$. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ $(a_i \in \mathbf{C}, i = 0, 1, \ldots, n, a_n \neq 0)$. Suppose that $f^{(k)} - P$ has finitely many zeros. Then

$$N\left(r, \frac{1}{f^{(k)} - P}\right) = S(r, f). \tag{1}$$

By the logarithmic derivative theorem, we have

$$\begin{aligned}
&m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f^{(k)} - P}\right) \\
&\quad = m\left(r, \frac{f^{(k+n)}}{f} \frac{1}{f^{(k+n)}}\right) + m\left(r, \frac{(f^{(k)} - P)^{(n)}}{f^{(k)} - P} \frac{1}{(f^{(k)} - P)^{(n)}}\right) \\
&\quad \leq m\left(r, \frac{1}{f^{(k+n)}}\right) + m\left(r, \frac{1}{f^{(k+n)} - a_n n!}\right) + S(r, f) \\
&\quad = m\left(r, \frac{1}{f^{(k+n)}} + \frac{1}{f^{(k+n)} - a_n n!}\right) + S(r, f) \\
&\quad \leq m\left(r, \frac{1}{f^{(k+n+1)}}\right) + S(r, f) \\
&\quad \leq T(r, f^{(k+n+1)}) - N\left(r, \frac{1}{f^{(k+n+1)}}\right) + S(r, f). \\
&\quad \leq T(r, f^{(k)}) + (n+1)\overline{N}(r, f) - N\left(r, \frac{1}{f^{(k+n+1)}}\right) + S(r, f).
\end{aligned}$$

Thus

$$T(r, f) \leq (n+1)\overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - P}\right) - N\left(r, \frac{1}{f^{(k+n+1)}}\right) + S(r, f). \tag{2}$$

Using a inequality of Wang [10] and noting that $k \geq 2$, then, for every $\varepsilon > 0$, we have

$$\begin{aligned}
(n+1)\overline{N}(r, f) + N\left(r, \frac{1}{f}\right) &\leq (k+n-1)\overline{N}(r, f) + N\left(r, \frac{1}{f}\right) \\
&\leq 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k+n+1)}}\right) + \varepsilon T(r, f) + S(r, f).
\end{aligned} \tag{3}$$

Combining (2) and (3), we obtain

$$T(r,f) \le 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-P}\right) + \varepsilon T(r,f) + S(r,f). \tag{4}$$

Recalling that the zeros of f are of order ≥ 3 and setting $\varepsilon = \frac{1}{6}$, from (1) and (4), we get

$$T(r,f) \le 6N\left(r,\frac{1}{f^{(k)}-P}\right) + S(r,f) = S(r,f),$$

which contradicts the fact that f is a transcendental meromorphic function. Theorem 3 is proved. □

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