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KODAI MATH. J.
28 (2005), 73–91

VANISHING THETANULLS FOR SOME DIHEDRAL AND CYCLIC COVERINGS OF RIEMANN SURFACES

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Abstract

Let $W_g \to W_z$ be a ramified p-sheeted covering of Riemann surfaces of genus g and z, $(z > 0)$ where p is an odd prime. Assume that the Galois group is either dihedral or cyclic. Assume, moreover, that the covering is *full*; that is, there us an integral divisor E, of degree $2r$ on W_z which lifts to be canonical on W_g. Then $g = rp + 1$, where $r \geq 1$. Clearly, W_g admits 2^{2z} half-canonical linear series of dimension at least $r - z$ arising from divisors on W_z whose double is E. *Theorem 1* Of these 2^{2z} half-canonical linear series u_z $(= 2^{z-1}(2^z - 1))$ have dimension at least $r - z + 1$. *Theorem 2* Let W_g $(g = 3r + 1, r \geq 3)$ admit four half canonical linear series, three of dimension $r - 1$, and one of dimension r, whose sum is bi-canonical, where the half-canonical linear series of dimension r is unique. Then W_g is a full elliptic-trigonal Riemann surface. (This characterizes the cases $z = 1$, $p = 3$, $g \geq 10$)

1. Introduction

Let $\pi_{gz} : W_g \to W_z$ be an m-sheeted covering of Riemann surfaces of genus g and z. In this paper W_z will always have positive genus. Then W_g has a property not shared by all Riemann surfaces of genus g. This special property may be reflected in some special property of the Jacobian of W_g. We shall be interested in the vanishing properties of the theta function at half-periods (vanishing theta nulls.) By Riemann's solution to the Jacobi inversion problem, this means that we will be interested in the existence of non-generic half-canonical linear series on W_g. [8] Of course, we do not expect these half-canonical linear series to exist very often, but they do occur in the case $m = 2$. [1] We consider the case $m = 3$. If π_{gz} is unramified then non-generic half-canonical linear series exist, but with ramification one would not expect this to be true in general. The case of interest in this paper is described in the following definition.

Definition. The covering $\pi_{gz} : W_g \to W_z$ is said to be *full* if there exists a linear series g_{2r}^{2r-z} on W_z which lifts to be canonical on W_g. □

AMS Mathematical Subject Classification. Primary 14H30, Secondary 30F10.
Received February 20, 2004.

Since two-sheeted coverings are always full, this is a generalization of the case $m = 2$ since the existence of non-generic half-canonical linear series on W_g is almost always obvious.

If π_{gz} is three sheeted and non-cyclic then the Galois group is the dihedral group of order six. It turns out that much of the analysis in the case $m = 3$, works equally well in the case of p-sheeted dihedral coverings where p is an odd prime.

If π_{gz} is full, $r \geq z$, and the lift of g_{2r}^{2r-z} is canonical, then W_g now admits 2^{2z} half canonical linear series due to the fact that $g_{2r}^{2r-z} \equiv 2g_r^{r-z}$ in 2^{2z} different ways. However, in the case of full p-sheeted ramified dihedral or cyclic coverings a certain number, u_z, of these half-canonical linear series have dimension greater than that of g_r^{r-z}. This is covered in Sections 5 and 6. ($u_z = 2^{z-1}(2^z - 1) =$ number of odd theta characteristics in dimension z.)

For $p = 3$, $z = 1$, $g \geq 10$, we show when the existence of the four non-generic half-canonical linear series characterizes full elliptic-trigonal Riemann surfaces. (Section 7) However, we are unable to distinguish between the dihedral and cyclic cases by the methods of this paper.

Section 3 concerns the existence of full ramified p-sheeted dihedral coverings and gives a useful characterization. Section 6 concerns the cyclic case. Section 4 gives a very brief account of Weierstrass points for the case $p = 3$. (We know of no generalization for $p > 3$.) Section 2 on preliminary results is arranged so that the last part of this section is only needed for Section 7.

It is possible for a W_{10} to cover tori in three sheets in four different ways. (For W_{3r+1}, $r > 3$, a three-sheeted covering of a torus is unique.) If those four coverings are full then it can be shown that W_{10} admits an elementary abelian group of order 27. Thus the existence of this group is characterized by certain vanishing properties of the theta function. In this case our inability to distinguished between the cyclic and dihedral cases is overcome by the abundance of coverings. The proof involves an extensive examination of the inequality of Castelnuove-Severi and will be presented in a sequel to this paper.

2. Definitions, classical theorems, preliminary results

W_g will always stand for a compact Riemann surface of genus g. K_g will stand for the canonical linear series g_{2g-2}^{g-1}. The field of meromorphic functions on W_g will be denoted $M(W_g)$. If $f \,\varepsilon\, M(W_g)$ then $(f)_a$ will stand for the a-places of f counted with multiplicity so that the divisor of f, (f), is $(f)_0 - (f)_\infty$. If D is an integral divisor, $|D|$ will stand for the complete linear series of integral divisors linearly equivalent to D.

If $\pi : W_g \to W_z$ is a t-sheeted covering and D is a divisor of degree d on W_z then $\pi^{-1}(D)$ will denote the divisor of lifted points with ramification points counted according to multiplicity; consequently $\deg \pi^{-1}(D) = \text{td}$. If $P \,\varepsilon\, W_z$ then $\pi^{-1}(P)$, of degree t, will be called a *complete fiber* of π. If D is on W_g then $\pi(D)$ is the image of D of degree d.

The fibers of a t-sheeted covering $W_g \to W_q$ will be denoted $\gamma_t(q)$, and we will describe the covering by saying that W_g *admits* a $\gamma_t(q)$. If $q = 0$ then $\gamma_t(0)$ is a g_t^1. If $t = 2$ then W_g is said to be q-hyperelliptic ($q = 0$, hyperelliptic; $q = 1$, elliptic-hyperelliptic.) If $t = 3$ W_g is said to be q-trigonal ($q = 0$, trigonal; $q = 1$, elliptic-trigonal) A linear series g_n^r on W_g will be said to be *compounded* of $\gamma_t(q)$ if the divisors of the non-fixed points of g_n^r are lifted from divisors on W_q; that is, they are unions of divisors in $\gamma_t(q)$.

If $P(X)$ is an irreducible polynomial of degree n in $M(W_z)[X]$ then this polynomial defines an algebraic extension of $M(W_z)$ which lives on a Riemann surface, W_g, covering W_z in n sheets. $M(W_z)$ is isomorphic to a subfield of $M(W_g)$ of index n. In this context $M(W_z)$ will stand for the field on W_z or for its isomorphic image on W_g. No confusion should result. If $P(X) = X^n - f$ then we will say that $f^{(1/n)}$ *defines* the cyclic covering $W_g \to W_z$.

With one exception (Theorem 6.1) p will always stand for an odd prime. A p-sheeted dihedral covering $\pi_{gz} : W_g \to W_z$ will have two types of ramification points. Those of multiplicity p will be called *total*. Those of multiplicity 2 will be called *ordinary*. The complete fibers of π_{gz} are of three types: a single total ramification point, $(p-1)/2$ ordinary ramifications points together with a single unramified point, or p unramified points. Suppose π_{gz} has s total ramification points and ordinary ramification points over n points of W_z. Then the total ramification of the covering is $(p-1)s + ((p-1)/2)n$. The Riemann-Hurwitz formula gives

$$2g - 2 = p(2z - 2) + ((p-1)/2)(2s + n) \tag{2.1}$$

where n is always even. If the covering is full then $2s + n \equiv 0 \pmod{2p}$ since p divides $2g - 2$.

Let W_a be a Riemann surface of genus a admitting a group of automorphisms, G, isomorphic to the dihedral group of order $2p$. We will always write $G = \langle \psi, \varphi \rangle$ where $\Psi^p = \varphi^2 = e$. Let $W_g = W_a/\langle \varphi \rangle$, $W_h = W_a/\langle \psi \rangle$ and $W_z = W_a/G$. Then $a + 2z = 2g + h$. [7]

THEOREM 2.1 [7]. *Let g_n^r be a complete linear series on W_z. Let g_{2n}^{r+c} (resp g_{pn}^{r+b}) be the completion of the lift of g_n^r to W_h (resp W_g). Then the completion of g_n^r lifted to W_a is g_{2pn}^{r+c+2b}.* □

In the theorem let D_z be a divisor in g_n^r on W_z. On W_a let D_a be the divisor of degree $2pn$ which is the lift of D_z. The vector space of meromorphic functions on W_a which are multiples of D_a is a complex represention of G of dimension $r + c + 2b + 1$.

COROLLARY 2.2. *The multiplicity of the trivial representation is $r + 1$. The multiplicity of the non-trivial one dimentional representation is c. The sum of the multiplicities of the irreducible representations of dimension two is b.* □

We call attention to an important obvious fact.

Corollary 2.3. *If there is a divisor in* g_{2pn}^{r+c+2b} *which is not invariant under* $\langle\psi\rangle$ *then* b *is positive.* □

We now quote some classical theorems with abbreviations for later reference.

(RR) *Riemann-Roch* If g_n^r is complete then $r = n - g + i$ where i is the index of specialty.

(BN) *Brill-Noether* If g_n^r and h_m^s are complete and their sum is K_g, then $n - 2r = m - 2s$.

(Cliff) *Clifford's theorem* If g_n^r is special then $n - 2r \geq 0$. Equality in non-trivial cases implies W_g is hyperelliptic.

(CS) *Castelnuovo-Severi inequality* If W_g admits a $\gamma_m(s)$ and a $\gamma_n(t)$ and the two coverings admit no non-trivial common factorizations then

$$g \leq ms + nt + (m-1)(n-1)$$

and equality has further consequences.

We now state a series of results with either proofs, references, or whose proofs follow from standard techniques.

Lemma 2.4. *Let* $\pi : W_g \to W_z$ *be a* p*-sheeted covering with positive ramification. Suppose* A *and* B *are integral divisors on* W_z *so that* $\pi^{-1}(A) \equiv \pi^{-1}(B)$. *Then* $A \equiv B$. □

Lemma 2.5. *Let* A *and* B *be integral divisors on* W_g *so that* $2A \equiv pB$ *where* $\deg B \geq 2g$. *Then there exists an integral divisor* C *so that* $B = 2C$ *and* $A = pC$. □

The material in this section from now on will be needed only in Section 7.

Lemma 2.6. *Let* g_n^r *and* h_n^s *be two different linear series so that* $2g_n^r \equiv 2h_n^s$. *Then on an unramified two-sheeted covering,* W_{2g-1}, *of* W_g *there is a linear series* g_{2n}^{r+s+1} *which is the completion of the lift of* g_n^r *(and* h_n^s*). If one of the two linear series is simple then so also is* g_{2n}^{r+s+1}.

Proof. Let D and E be divisors in g_n^r and h_n^s respectively. Let f be a function whose divisor is $2D - 2E$. Then W_{2g-1} is defined by $\sqrt{f}$. If D' is the lift of D then the dimension of the multiples of D' is $(r+1) + (s+1)$. If g_n^r is simple there are more multiples of D' than there are of D. So g^{r+s+1} is simple. □

Lemma 2.7. *Let* W_g *admit* 4 *distinct linear series,* g_n^a, h_n^b, k_n^c, *and* l_n^d *so that*

i) $2g_n^a \equiv 2h_n^b \equiv 2k_n^c \equiv 2l_n^d$

ii) $g_n^a + h_n^b \equiv k_n^c + l_n^d$

Then there exists an unramified 4*-sheeted Galois covering* $W_{4g-3} \to W_g$ *(the Galois group is the four group) and on* W_{4g-3} *there exists a* $g^{a+b+c+d+3}$ *which is the completion of the lift of* g_n^a. *If one of the four linear series on* W_g *is simple so also is* $g^{a+b+c+d+3}$.

Proof. Apply the previous lemma twice. □

LEMMA 2.8. *Suppose W_h admits a $\gamma_2(g)$, $g \geq 6$, and is also elliptic-trigonal. Then W_g is elliptic-trigonal or trigonal.*

Proof. Since $h \geq 11$, W_h admits a unique $\gamma_3(1)$ (CS). The involution whose quotient is W_g must permute the divisors of $\gamma_3(1)$. Thus W_g admits a $\gamma_3(q)$ where $q = 1$ or 0. □

A linear series g^s_m will be said to *impose t* (linear) *conditions* on a complete g^r_n if $|g^r_n - g^s_m| = g^{r-t}_{n-m}$.

LEMMA 2.9 [3]. *Let g^r_{g-1} be half canonical. Let g^1_m be a linear series without fixed points where $m \leq 2r + 1$. Then g^1_m imposes at most $[m/2]$ conditions on g^r_{g-1}.* □

LEMMA 2.10 [3]. *Let g^s_m $(s \geq 2)$ be a simple linear series without fixed points with $m - s \leq 2r$. Then g^s_m imposes at most $[(m - s + 1)/2]$ conditions on any half-canonical g^r_{g-1}. Thus such a half-canonical g^r_{g-1} must be simple.* □

LEMMA 2.11 [3]. *Suppose W_g admits a $\gamma_2(q)$. Then every half-canonical g^{q+1}_{g-1} is compounded of $\gamma_2(q)$. (Thus g^r_{g-1} being simple and half-canonical implies that $r \leq q$.)* □

LEMMA 2.12 ([1], p 51). *Suppose W_g admits a $\gamma_2(q)$ and $r = [(g-1)/2] - q \geq 0$. Then W_g admits many half-canonical g^r_{g-1}'s. If g is odd the number is at least 4^q. If g is even the number is at least $(g + 2 - 2q)4^q$.* □

LEMMA 2.13. *Suppose W_g admits a $\gamma_3(q)$. Let D be an integral divisor of degree 3 so that $|2D|$ is compounded of $\gamma_3(q)$. Then D is in $\gamma_3(q)$.*

Proof. Let $D = x + y + z$. $2x + 2y + 2z$ is a union of 2 divisors in $\gamma_3(q)$. If D is not in $\gamma_3(q)$, say $2x + y$ is in $\gamma_3(q)$. But a divisor in $\gamma_3(q)$ is determined by any point in it, so $2x + y = y + 2z$ or $x = z$. D is in $\gamma_3(q)$ after all. □

LEMMA 2.14 [3]. *Suppose W_g admits a g^1_4. Let D be an integral divisor of degree 4 not in g^1_4, so that $|2D| = 2g^1_4 = g^2_8$. Then there exist two disjoint integral divisors of degree 2, P and Q, so that $D = P + Q$ and $|2P| = |2Q| = g^1_4$.* □

(The proof uses the same kind of reasoning as in Lemma 2.13.)

LEMMA 2.15. *Suppose W_{3r+1} $(r \geq 2)$ is trigonal and admits a half-canonical g^r_{3r} (necessarily compounded of g^1_3 by Lemma 2.9.) Then W_{3r+1} does not admit a complete half-canonical g^{r-1}_{3r}.*

Proof. Suppose the contrary. g_{3r}^{r-1} is compounded of g_3^1 so that $g_{3r}^{r-1} = (r-1)g_3^1 + D$ where D is not in g_3^1. Since $K_{3r+1} \equiv (2r)g_3^1$ we see that $2D \equiv 2g_3^1$. Since $3r+1 \geq 7$, $|2g_3^1| = g_6^2$. Lemma 2.13 gives the contradiction. □

Lemma 2.16 (Castelnuovo, [2]). *Let g_n^r be a simple linear series on W_g. $(n > 2r)$ Then $2g_n^r$ has dimension $3r - 1 + \varepsilon$, $\varepsilon \geq 0$.*

Lemma 2.17. *Suppose W_{3r+1} admits a simple half-canonical g_{3r}^r and a g_m^1 which imposes two conditions on g_{3r}^r. If $r > 5$ then $m = 4$.* □

Proof. Since g_{3r}^r is simple $m \geq 4$. Let D be an integral divisor of $r-2$ points in general position on W_{3r+1}. Then $g_{3r}^r - D$ $(= g_{2r+2}^2)$ gives a plane model for W_{3r+1} with $r-2$ singularities of degree $m-1$ and one of multiplicity $2r+2-m$ and possibly other singularities. Then

$$3r + 1 \leq [(2r+1)(2r) - (r-2)(m-1)(m-2) - (2r+2-m)(2r+1-m)]/2$$

which simplifies to: $m \leq 4r/(r-1)$. Since $r > 5$ we have $m \leq 4$. □

Lemma 2.18 [2]. *Let g_n^r and g_m^s be two distinct linear series where g_n^r is simple and $r \geq s$. Then $g_n^r + g_m^s$ has dimension at least $r + 2s$.* □

3. Existence

Any ramified dihedral covering of odd prime order p, $\pi_{gz} : W_g \to W_z$, arises in accordance with the following *procedure*. There is given a meromorphic function $y : W_z \to \mathbf{P}^1$. $\sqrt{y}$ $(:= Y)$ defines a two-sheeted covering $\pi_{hz} : W_h \to W_z$, ramified over the zeros and poles of y whose orders are odd. Let $Z = (1-Y)/(1+Y)$. Then a p-sheeted cyclic covering $\pi_{ah} : W_a \to W_h$ is defined by $Z^{(1/p)}$ $(:= U)$ and is ramified over those zeros and poles of Z whose orders are not divisible by p. The map $\pi_{az} := \pi_{hz} \circ \pi_{ah}$ is a Galois covering with Galois group, G, isomorphic to the dihedral group of order $2p$. $G = \langle \varphi, \psi \rangle$ where $\varphi^* U = U^{-1}$ and $\psi^* U = \omega U$, $\omega^p = 1$, and $\omega \neq 1$. Then $W_g = W_a / \langle \varphi \rangle$ and the covering map $\pi_{ag} : W_a \to W_g$ is the Galois closure of the p-sheeted dihedral covering $\pi_{gz} : W_g \to W_z$. We will prove the assertions in this paragraph only in the following special situation.

Theorem 3.1. *A ramified p-sheeted dihedral covering is full if and only if it arises by the above procedure with a y satisfying the following*:

(3.1) $(y)_0 = B_{zn}$, a divisor of n distinct points, n even

$(y)_\infty = 2C_{z\mu}$, $C_{z\mu}$ a divisor of degree μ $(= n/2)$

$(y)_1 = (p-2)D_{zs} + pD_{zt}$, D_{zs} a divisor of s distinct points, $\deg D_{zt} = t$

$(\deg y = n = 2\mu = (p-2)s + pt)$

Proof. Assuming the existence of such a y, we follow the above procedure keeping track of significant divisors.

Let $\sqrt{y}$ $(:= Y)$ define $\pi_{hz} : W_h \to W_z$. Let $Z = (1 - Y)/(1 + Y)$. Y and Z are in $M(W_h)$, and both are of degree n.

$$(Z)_1 = (Y)_0 = B_{hn}, \text{ the } n \text{ ramified points of } \pi_{hz}$$

$$(Z)_{-1} = (Y)_\infty = C_{h\mu}, \ \mu \text{ complete unramified fibers of } \pi_{hz};\ \deg C_{h\mu} = 2\mu = n$$

$$(Z)_0 = (Y)_1 = (p-2)D_{hs} + pD_{ht} \text{ where } D_{hs} \text{ and } D_{ht} \text{ lie above } D_{zs} \text{ and } D_{zt}$$

$$(Z)_\infty = (Y)_{-1} = (p-2)E_{hs} + pE_{ht} \text{ where } E_{hs} \text{ and } E_{ht} \text{ lie above } D_{zs} \text{ and } D_{zt}$$

Now $Z^{(1/p)}$ $(:= U)$ defines $\pi_{ah} : W_a \to W_h$, a p-sheeted cyclic covering. $W_h = W_a/\langle\psi\rangle$ where $\psi^* U = \omega U$, as before.

$$(U)\omega^i = B^i_{an} \text{ lying above } B_{hn},\ i = 0, 1, 2, \ldots, p-1.\ \deg B^i_{an} = n.$$

$$(U){-\omega^i} = C^i_{a\mu} \text{ lying above } C_{h\mu},\ i = 0, 1, 2, \ldots, p-1.\ \deg C^i_{a\mu} = n.$$

$$(U)_0 = (p-2)D_{as} + D_{at}$$

$$(U)_\infty = (p-2)E_{as} + E_{at}$$

where $D_{as} + E_{as}$ are the ramified points of π_{ah}, D_{at} and E_{at} are each composed of t complete unramified fibers of π_{ah}, and $D_{at} + E_{at}$ is composed of t complete fibers of π_{az}.

We may assume that the fixed points of φ are B^0_{an}. Now

$$K_a \equiv \pi^{-1}_{az}(K_z) + \sum_{i=0}^{p-1} B^i_{an} + (p-1)(D_{as} + E_{as})$$

By the linear equivalence of divisors where U takes different values

$$\sum_{i=1}^{p-1} B^i_{an} \equiv ((p-1)/2)[(p-2)(D_{as} + E_{as}) + (D_{at} + E_{at})]. \quad \text{Consequently}$$

$$K_a \equiv \pi^{-1}_{az}(K_z) + B^0_{an} + ((p-1)/2)[p(D_{as} + E_{as}) + (D_{at} + E_{at})]$$

$$\equiv B^0_{an} + \pi^{-1}_{az}(K_z + ((p-1)/2)[D_{zs} + D_{zt}]). \quad \text{But}$$

$$K_a \equiv B^0_{an} + \pi^{-1}_{ag}(K_g), \quad \text{so by Lemma 2.4}$$

$$K_g \equiv \pi^{-1}_{gz}(K_z + ((p-1)/2)[D_{zs} + D_{zt}]).$$

Thus $\pi_{gz} : W_g \to W_z$ is a full p-sheeted dihedral covering.

Now we assume for a ramified p-sheeted dihedral covering $\pi_{gz} : W_g \to W_z$ that $K_g \equiv \pi^{-1}_{gz}(G_0)$ for a fixed divisor G_0 on W_z. (By Lemma 2.4 any two such G_0's are linearly equivalent.) We must show that this covering arises from a meromorphic function y on W_z as described in the statement of the theorem. We first show that there are, in deed, ordinary ramification points.

Lemma 3.2. *Let $\pi_{gz} : W_g \to W_z$ be a ramified p-sheeted full covering with only total ramification points. Then the covering is cyclic.*

Proof of Lemma 3.2. Let B be the divisor of ramification points of π_{gz}. Then

$$K_g \equiv (p-1)B + \pi_{gz}^{-1}(K_z). \quad \text{But}$$
$$K_g \equiv \pi_{gz}^{-1}(G_0)$$

for a divisor G_0 on W_z. The right hand sides of these last two equivalences, being linearly equivalent, are the zeros and poles of a function in $M(W_g)$ whose pth power is a function in $M(W_z)$. Thus the covering is cyclic. □

Our covering π_{gz} is assured of ordinary ramification points. Let $\pi_{ag} : W_a \to W_g$ be the Galois closure of π_{gz}. Let the Galois group be $G = \langle \varphi, \psi \rangle$, $W_g = W_a/\langle \varphi \rangle$ and $W_h = W_a/\langle \psi \rangle$. Let B_{an}^0 be the ramification divisor of π_{ag}, a divisor of even degree n on W_a. Then

$$K_a \equiv B_{an}^0 + \pi_{ag}^{-1}(K_g)$$

Letting K_{ao} be a divisor in K_a invariant under G we have

$$K_{ao} \equiv B_{an}^0 + \pi_{az}^{-1}(G_0). \tag{3.2}$$

Let $L = \{f \,\varepsilon\, M(W_a) \,|\, (f) \geq \pi_{az}^{-1}(G_0) - K_{ao}\}$.

L is a complex representation for G. It contains a function (whose zeros and poles are the two sides of (3.2)) with a divisor not invariant under ψ. Thus it must contain an irreducible two dimensional representation $\langle f_1, f_2 \rangle$, where we may assume ($\tau^p = 1$, $\tau \neq 1$)

$$\psi^* f_1 = \tau f_1, \quad \psi^* f_2 = \tau^{-1} f_2, \quad \varphi^* f_1 = f_2, \quad \varphi^* f_2 = f_1.$$

Let $U = f_1/f_2$ and let $V = (1-U)/(1+U)$. Then $\varphi^* V = -V$ and so V defines π_{ag}.

Now assume that the degree of U (and therefore V) is n.

We assert that $B_{an}^0 = (V)_0$ or $(V)_\infty$. Observe that $B_{an}^0 \equiv (V)_0$. To prove the assertion let X be a function on W_a where $(X) = B_{an}^0 - (V)_0$. Since the zeros and poles of X are invariant under φ, $\varphi^* X = \lambda X$ where $\lambda = 1$ or -1. Assume $\lambda = 1$. Then X is invariant under φ, and X has B_{an}^0 for zeros, a divisor which contains no complete fiber of π_{ag}. Therefore X is constant and $B_{an}^0 = (V)_0$. (If $\lambda = -1$ replace X with VX and deduce that $B_{an}^0 = (V)_\infty$.)

We now assume that $B_{an}^0 = (V)_0$. Since $U = (1-V)/(1+V)$ we have $(U)_1 = B_{an}^0$. Let $\omega = \tau^2$. Then $\psi^* U = \omega U$. It follows that $(U)_{\omega} i = \psi^i(B_{an}^0)$ $(:= B_{an}^i)$ $i = 0, 1, \ldots, p-1$ give the p divisors, of degree n, of the ordinary ramification points of π_{az}.

Now $\psi^* U^p = U^p$ and so U defines π_{ah}, and the ramification points of π_{ah} are the zeros and poles of U whose orders are not divisible by p. Let $Z = U^p$, a function on W_h. Let

$$(Z)_0 = D_{h1} + 2D_{h2} + \cdots + (p-1)D_{h(p-1)} + pD_{hp}$$

where the D_{hi}, $i = 1, 2, \ldots p-1$ (but not necessarily p) are pairwise coprime divisors of points counted with multiplicity one. On W_a

$$(U)_0 = D_{a1} + 2D_{a2} + \cdots + (p-1)D_{a(p-1)} + D_{ap}$$

where the D_{ai}, $i = 1, 2, \ldots (p-1)$ are ramification points of π_{ah} lying above the D_{hi}, and D_{ap} is a set of ($\deg D_{hp}$) complete fibers of π_{ah}.

If Φ is the involution of W_h which lifts to φ and if

$$(Z)_\infty = E_{h1} + 2E_{h2} + \cdots + (p-1)E_{h(p-1)} + pE_{hp}$$

then Φ interchanges the D_{hi}'s and the E_{hi}'s and

$$(U)_\infty = E_{a1} + 2E_{a2} + \cdots + (p-1)E_{a(p-1)} + E_{ap}$$

with definitions analogous to those of the D_{ai}'s.

We now show that $D_{hi} = 0$ for $i \neq p-2$, p. Then setting $Y = (1-Z)/(1+Z)$, noting that $\Phi^* Y = -Y$, and setting $y = Y^2$, the proof will be complete for $\deg U = n$. (If in the assertion above, $B^0_{an} = (V)_\infty$ then set $y = Y^{-2}$.)

On W_a again note that $(U)_0 \equiv (U)_\infty \equiv (U)_1 \equiv B^i_{an}$ for all i.

$$K_a \equiv \sum_{i=0}^{p-1} B^i_{an} + \sum_{j=1}^{p-1} (p-1)(D_{aj} + E_{aj}) + \pi_{az}^{-1}(K_z)$$

$$\sum_{i=0}^{p-1} B^i_{an} \equiv B^0_{an} + ((p-1)/2)\left(\sum_{j=1}^{p-1} j(D_{aj} + E_{aj}) + (D_{ap} + E_{ap})\right) \quad \text{Therefore}$$

$$\begin{aligned} K_a \equiv{} & B^0_{an} + \pi_{az}^{-1}(K_z) + ((p-1)/2)(D_{ap} + E_{ap}) \\ & + \sum [((p-1)j)/2 + (p-1)](D_{aj} + E_{aj}) \end{aligned}$$

From the last sum extract the $j = p-2$ term and denote what is left by a primed summation.

$$\begin{aligned} K_a \equiv{} & B^0_{an} + \pi_{az}^{-1}(K_z) + ((p-1)/2)(D_{ap} + E_{ap}) + ((p-1)/2)p(D_{a(p-2)} \\ & + E_{a(p-2)}) + \sum{}'[\ ](D_{aj} + E_{aj}) \end{aligned}$$

where none of the integers in the brackets are divisible by p. Note that the second, third, and fourth terms denote sets of complete fibers of π_{az}, namely

$$\pi_{az}^{-1}(K_z + ((p-1)/2)(D_{zp} + D_{z(p-2)}))$$

for suitable divisors D_{zp} and $D_{z(p-2)}$ on W_z where π_{az} has total ramification points above $D_{z(p-2)}$ and is unramified above D_{zp}. Since

$$K_a \equiv B^0_{an} + \pi^{-1}_{ag}(K_g) \equiv B^0_{an} + \pi^{-1}_{az}(G_0),$$

descending to W_g by Lemma 2.4 we have

$$\pi^{-1}_{gz}(G_0) \equiv \pi^{-1}_{gz}(K_z + ((p-1)/2)(D_{zp} + D_{z(p-2)})) + \sum{}'[\ ]D_{gj}$$

for suitable divisors D_{gj} on W_g. The D_{gj} $(j < p)$ are the total ramification points for π_{gz}. If $\sum{}'[\ ]D_{gz}$ is not zero in the last equivalence then the right and left hand sides are the zeros and poles of a function on W_g whose pth power is a function in $M(W_z)$. This implies that π_{gz} is a cyclic covering. This contradiction proves the theorem in case the degree of U is n.

Now assume that the degree of U is less than n. Then

$$(f_1) = A_1 + F + \pi^{-1}_{az}(G_0) - K_{ao}.$$

$$(f_2) = A_2 + F + \pi^{-1}_{az}(G_0) - K_{ao}$$

where $(A_1, A_2) = 0$, $F \neq 0$, and F is invariant under φ. Again let $U = f_1/f_2$, $V = (1-U)/(1+U)$, $\varphi^* V = -V$ and V defines π_{ag}. Define X by $(X) = B^0_{an} - [(V)_0 + F]$. Then $\varphi^* X = \lambda X$, and $\lambda = 1$ or -1. If $\lambda = 1$, then X is invariant under φ and $B^0_{an} = (V)_0 + F$. If $\lambda = -1$, then VX is invariant under φ and $B^0_{an} = (V)_\infty + F$. Thus F consists of part of B^0_{an}.

We wish to show that $\operatorname{Deg} F = n/2$.

Assume $\lambda = 1$. (The case $\lambda = -1$ is handled in an analogous way.) $(U)_1 = (V)_0$ and so $B^0_{an} \equiv (U)_1 + F$. Define $B^1_{an} := \psi(B^0_{an})$. Then $B^1_{an} = \psi((U)_1) + \psi(F)$ and $(U)_\omega = \psi((U)_1)$. Consequently $(U)_1 + F \equiv (U)_\omega + \psi(F)$. Thus we have two functions, θ_1, θ_2 with divisors:

$$(\theta_1) = (U)_1 - (U)_\omega, \quad (\theta_2) = F - \psi(F)$$

and $B^0_{an} = (U)_1 + F$, the ramification divisor of φ. Apply CS to the two covers $\theta_1 : W_a \to \mathbf{P}^1$ and $\pi_{ag} : W_a \to W_g$. Since θ_1 cannot be in $M(W_g)$ we have $a \leq (\deg \theta_1 - 1) + 2g$. But $a = 2g - 1 + n/2$. Consequently $\deg \theta_1 \geq n/2$. Similarly $\deg \theta_2 \geq n/2$. Since $\deg \theta_1 + \deg \theta_2 = n$ we conclude that $\deg F = \deg A_1 = \deg A_2 = n/2$.

Now B^0_{an} is part of $(V)_0 + (V)_\infty$ since V defines π_{ag}. But $\deg F = \deg U = \deg V = n/2$; therefore $B^0_{an} = (V)_0 + (V)_\infty = (U)_1 + (U)_{-1} = (U^2)_1$, and $\psi^* U^2 = \omega^2 U^2$.

Thus U^2 satisfies the properties of U (when $\deg U = n$), and we may proceed with the rest of the proof as before to obtain our desired y. Similarly for the case $B^0_{an} = (V)_\infty + F$. This completes the proof of the theorem. □

Note that in the first part of the theorem where we proceeded from y to the full covering π_{gz}, we can achieve an irreducible two dimensional representation in L by defining f_1 and f_2 as follows:

$$(f_1) = (p-2)D_{as} + D_{ap} + \pi^{-1}_{az}(G_0) - K_{ao}$$

$$(f_2) = (p-2)E_{as} + E_{ap} + \pi^{-1}_{az}(G_0) - K_{ao}$$

Thus we see that for a given full covering π_{ag} there are at most $(\dim L)/2$ possible y's which give rise to it according to the procedure at the beginning of this section. This allows us to compute, naively, the dimension, α, in moduli space of such W_g's.

By the Riemann Hurwitz formula for $y : W_z \to \mathbf{P}^1$ we have

$$2z - 2 = -2n + n/2 + (p-3)s + (p-1)t + \alpha$$

Since $n = (p-2)s + pt$ we have $\alpha = 2z - 2 + (n/2) + (n+2s)/p$.

4. Weierstrass points $(p = 3)$

We draw attention to the fact that for full three-sheeted dihedral coverings (with g sufficiently large) ordinary ramification points are Weierstrass points.

Theorem 3. *If $W_g \to W_z$ is a full three-sheeted covering and $g \geq 4 + 6z$, then each ordinary ramification point of the covering is a Weierstrass point.*

Proof. Suppose $g^x_{(2g-2)/3}$ on W_z lifts to be canonical on W_g. If P is an ordinary ramification point then $x(2P)$ will be special if $2x \geq g$. Now $x \geq ((2g-2)/3) - z$, so a sufficient condition that P be a Weierstrass point is $2(((2g-2)/3) - z) \geq g$. □

5. Half-canonical linear series on dihedral coverings

Assuming a full ramified p-sheeted dihedral covering π_{gz}, there is a linear series g^{2r-z}_{2r} on W_z which lifts to be canonical on W_g, and so $2g - 2 = 2rp$. Since

$$2g - 2 = p(2z-2) + ((p-1)/2)(2s+n)$$

and $2p$ divides $2s + n$, we see that

$$2r \geq (2z-2) + (p-1) \geq 2z$$

since $p \geq 3$. Thus g^{2r-z}_{2r} is not special and is complete. Consequently $g^{2r-z}_{2r} \equiv 2g^{r-z}_r$ for 2^{2z} linear series g^{r-z}_r (which need not be complete), and so W_g admits 2^{2z} half-canonical linear series of dimension at least that of g^{r-z}_r.

Theorem 5.1. *Of those 2^{2z} half-canonical linear series u_z $(= 2^{z-1}(2^z - 1))$ have dimension greater than that of the corresponding g^{r-z}_r.*

Proof. Continue the notation of the previous sections.

(4.1) $$K_a \equiv B^0_{an} + \pi^{-1}_{ag}(K_g) \equiv B^0_{an} + \pi^{-1}_{az}(g^{2r-z}_{2r})$$

(4.2) $$K_g \equiv \pi^{-1}_{gz}(g^{2r-z}_{2r}) \equiv \pi^{-1}_{gz}(K_z) + B_{gn} + (p-1)D_{gs}$$

where D_{gs} is the set of s total ramification points of π_{gz}, and B_{gn} is the set of

$n(p-1)/2$ ordinary ramification points which lie in n stacks of $(p-1)/2$ points over the n points of B_{zn} on W_z. Map the elements represented by (4.2) back onto W_z by π_{gz}.

$$pg_{2r}^{2r-z} \equiv pK_z + ((p-1)/2)B_{zn} + (p-1)D_{zs}. \tag{4.3}$$

Now two-sheeted coverings are always full (Theorem 6.1), so there is a linear series, f, of degree $2z-2+n/2$ on W_z so that

$$K_h \equiv \pi_{hz}^{-1}(K_z) + B_{hn} \equiv \pi_{hz}^{-1}(f)$$

Mapping onto W_z by π_{hz} gives

$$2f \equiv 2K_z + B_{zn} \tag{4.4}$$

Rewriting (4.3):

$$pg_{2r}^{2r-z} \equiv K_z + ((p-1)/2)(2K_z + B_{zn} + 2D_{zs}),$$

and so

$$pg_{2r}^{2r-z} \equiv K_z + ((p-1)/2)(2f + 2D_{zs})$$

There are, generically, u_z half-canonical g_{z-1}^0's on W_z so that for any such g_{z-1}^0 the last equivalence can be written

$$pg_{2r}^{2r-z} \equiv 2[g_{z-1}^0 + ((p-1)/2)(f + D_{zs})]$$

By Lemma 2.5 there is a g_r^{r-z} so that

$$g_{2r}^{2r-z} \equiv 2g_r^{r-z} \quad \text{and} \quad g_{z-1}^0 + ((p-1)/2)(f + D_{zs}) \equiv pg_r^{r-z} \tag{4.5}$$

Considering π_{ah} we see that

$$K_a \equiv \pi_{ah}^{-1}(K_h) + (p-1)(D_{as} + E_{as}) \quad \text{or} \tag{4.6}$$

$$K_a \equiv \pi_{az}^{-1}(f) + (p-1)(D_{as} + E_{as})$$

Combining this with (4.1) we have

$$\pi_{az}^{-1}(f) + (p-1)(D_{as} + E_{as}) \equiv \pi_{az}^{-1}(2g_r^{r-z}) + B_{an}^0 \tag{4.7}$$

Multiply (4.7) by $(p-1)/2$, and substitute (4.5) eliminating $\pi_{az}^{-1}(((p-1)/2)f)$

$$\pi_{az}^{-1}(pg_r^{r-z} - ((p-1)/2)D_{zs} - g_{z-1}^0) + ((p-1)^2/2)(D_{as} + E_{as})$$
$$\equiv \pi_{az}^{-1}((p-1)g_r^{r-z}) + ((p-1)/2)B_{an}^0$$

or

$$\pi_{az}^{-1}(g_r^{r-z}) \equiv ((p-1)/2)B_{an}^0 + ((p-1)/2)(D_{as} + E_{as}) + \pi_{az}^{-1}(g_{z-1}^0) \tag{4.8}$$

since $\pi_{az}^{-1}(D_{zs}) \equiv p(D_{as} + E_{as})$.

Thus there is a divisor equivalent to $\pi_{az}^{-1}(g_r^{r-z})$ not invariant under $\langle\psi\rangle$. By Corollary 2.3, the dimension of $|\pi_{gz}^{-1}(g_z^{r-z})|$ must be greater than that of $|g_z^{r-z}|$. □

6. Full ramified cyclic coverings

Let W_z be a Riemann surface, and let D_z $(= x_1 + x_2 + \cdots + x_s)$ be a divisor on W_z of s distinct points $(s \geq 2)$. A cyclic covering of prime degree p, W_g, ramified over D_z is defined by $y^{(1/p)}$ where $y \varepsilon M(W_z)$ and

$$(y) = a_1x_1 + a_2x_2 + \cdots + a_sx_s + pC_z$$

where $0 < a_i < p$, $i = 1, 2, \ldots, s$ and $\sum a_i + p(\deg C_z) = 0$. We say that $\{a_1, a_2, \ldots a_s\}$ are the *rotation numbers* for the covering. Since $y^{(k/p)}$, $k \not\equiv 0$ (mod p) also defines W_g, the rotation numbers are only defined up to an integral multiple mod p. The rotation numbers are also defined topologically by a homomorphism from the fundamental group $\pi_1(W_z - D_z, \cdot)$ onto Z_p whose kernel corresponds to the covering. The rotation numbers are the values of the homomorphism on the paths that "circle" the a_i's.

THEOREM 6.1. *A ramified cyclic covering of prime degree p (≥ 2) is full if and only if the rotation numbers are all equal to one another.*

Proof. Set $s = tp$. $\mathrm{Deg}((p-1)D_z + pK_z) = p((p-1)t + (2z-2)) \geq p(2z)$ where $(p-1)t + 2z - 2 \geq z$. Consequently, on W_z there are p^{2z} linear series $|G|$ where

$$pG \equiv (p-1)D_z + pK_z \tag{6.1}$$

If $f \varepsilon M(W_z)$ has the right and left hand sides of (6.1) as zeros and poles then $f^{(1/p)}$ defines a covering $\pi : W_g \to W_z$ where $(h = f^{(1/p)},\ \pi(D_g) = D_z)$

$$(h) = (p-1)D_g + \pi^{-1}(K_z) - \pi^{-1}(G) \tag{6.2}$$

Since $K_g \equiv (p-1)D_g + \pi^{-1}(K_z)$ the covering is full with equal rotation numbers. But there are p^{2z} cyclic coverings with equal rotation numbers, so they all arise in this manner. Thus equal rotation numbers imply fullness.

For the converse reverse the argument starting with (6.2) multiplied by p and deduce (6.1), that is, that $h^p \varepsilon M(W_z)$ and $(h^p) = (p-1)D_z + pK_z - pG$. □

Now we consider half-canonical linear series on W_g arising from an equivalence $2H \equiv G$ for a solution G on W_z of (6.1). (Now $p \geq 3$) We have 2^{2z} possible H's with $|H| = g^r_{(g-1)/p}$ where $r \geq (g-1)/p - z$.

Let g^0_{z-1} be one of the u_z half-canonical linear series on W_z by (6.1) the equivalence

$$((p-1)/2)D_z + pg^0_{z-1} \equiv pH$$

has p^{2z} solutions $|H|$, one for each full cyclic covering $W_g \to W_z$ ramified over D_z On such a W_g we have

$$((p-1)/2)D_g + \pi^{-1}(g^0_{z-1}) \equiv \pi^{-1}(H) \tag{6.3}$$

$|\pi^{-1}(H)|$ has dimension greater than that of $|H|$ since the divisor on the left hand side of (6.3) is not lifted from W_z.

We see that the vanishing of the theta function at half periods (corresponding to the existence of half-canonical linear series) for a ramified full p-sheeted cyclic covering ($p \geq 3$) mimics those of full p-sheeted dihedral coverings.

Now we wish to show that in moduli space for genus g the full p-sheeted cyclic coverings of a W_z are in the closure of the space of full p-sheeted dihedral coverings of a W_z, assuming that

$$2g - 2 = p(2z - 2) + s(p - 1), \quad p|s, \quad s > 0.$$

On a Riemann surface of genus z, pick a divisor C of s distinct points. We want a divisor D of degree $2s/p$ so that $2C \equiv pD$. That such a D exists follows if $2s/p \geq z$. To insure that C and D are disjoint it is useful to have D move in a pencil, so we will assume $2s/p > z$. Thus we are assuming $4g - 4 > p^2 z + 3pz - 4p$. (If $p = 3$ and $z = 1$ then $2g > 5$). Assuming that C and D are disjoint, let $F \,\varepsilon\, M(W_z)$ have divisor: $(F) = pD - 2C$. Now let $y = 1 - \lambda F$, where the function y depends on the complex parameter λ ($\neq 0$.) Then

$$(y)_0 = (F)_{(1/\lambda)} := B_\lambda$$
$$(y)_\infty = (F)_\infty = 2C$$
$$(y)_1 = (F)_0 = pD$$

Note that $4/(1 - y) = 4/(\lambda F)$.

For all but a finite number of λ's ($\neq 0$) B_λ will consist of $2s$ distinct points so we shall always make this assumption as $\lambda \to 0$. For a fixed $\lambda \neq 0$ we follow the procedure of section 3 to obtain a full p-sheeted dihedral covering, W_g, of W_z with $(p - 1)/2$ ordinary ramification points lying over each of the $2s$ points of B_λ.

Again let $Y = \sqrt{y}$, $Z = (1 - Y)/(1 + Y)$, and $U = Z^{(1/p)}$. Now let $T = U + U^{-1}$, a function on W_g since T is invariant under φ. T satisfies an equation of the type

$$T^p + Q(T) = Z + Z^{-1}$$

where Q is a polynomial of degree less than p. For example

$$p = 3 \quad T^p + Q(T) = T^3 - 3T$$
$$p = 5 \quad T^p + Q(T) = T^5 - 5T^3 + 5T$$

But $Z + Z^{-1} = -2 + 4/(1 - y)$. Thus

$$T^p + Q(T) + 2 = 4/(\lambda F)$$

where F is independent of λ. Substituting $\lambda^{-(1/p)} T$ for T we have

$$T^p + \lambda Q(\lambda^{-(1/p)} T) + 2\lambda = 4/F.$$

Letting $\lambda \to 0$ this becomes $T^p = 4/F$, the equation of a full p-sheeted covering of W_z with total ramification points over the s distinct points of C. (We omit the details of passing to the limit.)

A slightly more complicated argument would deal with the case where C and D are not disjoint.

7. Full elliptic-trigonal Riemann surfaces

THEOREM 7.1. *Let $W_g \to W_1$ be a full three-sheeted covering of Riemann surfaces of genus g and one. ($g = 3r+1$, $r \geq 3$). Then W_g admits four complete half-canonical linear series g_{3r}^{r-1}, h_{3r}^{r-1}, k_{3r}^{r-1}, l_{3r}^{r} whose sum is bicanonical* (a quartet.) *The first three are composite being the lifts of linear series on W_1, and the fourth is fixed point free and simple being the completion of a lift of a linear series on W_1. l_{3r}^r is the unique linear series of degree $3r$ and dimension r or greater.*

Proof. Theorem 5.1 and the discussion in Section 6 insures the existence of the four linear series. l_{3r}^r, being the lift of a l_r^{r-1}, is fixed point free, and having more divisors than l_r^{r-1}, it is simple. If another linear series G_{3r}^r exists then $l_{3r}^r + G_{3r}^r$ has dimension at least $3r$ (Lemma 2.18) and so is canonical. Since l_{3r}^r is half-canonical, $l_{3r}^r = G_{3r}^r$. The same result for a $G_{3r}^{r+\varepsilon}$ $(\varepsilon \geq 0)$ now follows.

On W_1 let g_r^{r-1}, h_r^{r-1}, k_r^{r-1}, l_r^{r-1} be the linear series that lift to the four on W_g. On W_1 we have

$$g_r^{r-1} + h_r^{r-1} \equiv k_r^{r-1} + l_r^{r-1}$$

so on W_g we have

$$g_{3r}^{r-1} + h_{3r}^{r-1} \equiv k_{3r}^{r-1} + l_{3r}^{r}.$$

Since each linear series is half-canonical, their sum is bi-canonical. □

THEOREM 7.2. *Let W_g be a compact Riemann surface ($g = 3r+1$, $r \geq 3$) admitting a quartet g_{3r}^{r-1}, h_{3r}^{r-1}, k_{3r}^{r-1}, l_{3r}^{r}, where l_{3r}^r is the only half-canonical linear series on W_g of degree $3r$ and dimension r. Then W_g is a full elliptic-trigonal Riemann surface.*

Proof. We first show that l_{3r}^r is simple and without fixed points. Suppose l_{3r}^r is composite. By Lemma 2.15 W_g is not trigonal. Therefore, W_g admits a $\gamma_2(q)$ where W_q admits a complete $l_{(3r-f)/2}^r$ (where l_{3r}^r has f fixed points), and $l_{(3r-f)/2}^r$ is not special (Cliff.) Then $q = (r-f)/2$ (RR) by Lemma 2.12 W_g admits many half-canonical G_{3r}^r's. This contradiction insures that l_{3r}^r is simple. That l_{3r}^r is without fixed points follows from the fact that $2(l_{3r-1}^r) = g_{6r-2}^{3r-1+\varepsilon}$, $\varepsilon \geq 0$ (Lemma 2.16) and that W_g is not hyperelliptic (Cliff.)

By Lemma 2.7 there exists an unramified 4-sheeted Galois covering $W_{4g-3} \to W_g$ and W_{4g-3} admits a simple g_{12r}^{4r} ($12r \geq 36$.) By a theorem of Eisenbud–Harris [6, p 102] W_{4g-3} is trigonal, elliptic-trigonal, or admits a g_n^1

imposing two conditions on g^{4r}_{12r}. By Lemma 2.17 $n = 4$. It follows that W_g admits one of the three mutually exclusive alternatives.

Assuming that W_{12r+1} is elliptic-trigonal, it follows by Lemma 2.8 (applied twice) that W_{3r+1} is also. Fullness will follow from the following lemma.

LEMMA 7.3. *Let W_{3r+1} $(r \geq 3)$ admit a simple half-canonical l^r_{3r} and an elliptic-trigonal covering $\pi : W_{3r+1} \to W_1$. If a divisor D of l^r_{3r} contains two points of a fiber of π, then D is the lift of a divisor of degree r on W_1. Thus π is a full covering.*

Proof of Lemma 7.3. Let E be a divisor in $\gamma_3(1)$, two points of which are in D. Let F be another divisor in $\gamma_3(1)$ which has a point in D. Then $|E+F|$ is a g^1_6 on W_g (being the lift of a g^1_2) and so imposes three conditions on l^r_{3r} (Lemma 2.9.) Thus $(E+F, D) = E+F$. Now $|D| = l^r_{3r}$, $D-(E+F) = l^{r-3}_{3r-6}$, and so $|D-E| = l^{r-2}_{3r-3}$. Consequently, $l^{r-2}_{3r-3} - F = l^{r-3}_{3r-6}$. Thus F imposes one condition on l^{r-2}_{3r-3}, l^{r-2}_{3r-3} is the lift of a g^{r-2}_{r-1} on W_1, and the result follows. □

Since W_g is not trigonal, to complete the proof we must show that W_g does not admit a g^1_4. Suppose that W_g does admit a g^1_4. We will arrive at a contradiction.

We first show that if $r \geq 5$ then $l^r_{3r} - g^1_4$ $(= l^{r-2}_{3r-4})$ is simple and without fixed points. Remember that g^1_4 imposes at most two conditions on l^r_{3r}, and therefore, at most two conditions on any subseries of dimension at least 2. If l^{r-2}_{3r-4} is composite and compounded of g^1_4 then $l^{r-2}_{3r-4} \equiv (r-2)g^1_4 + D_f$, and so $3r-4 \geq 4r-8$, a contradiction. If l^{r-2}_{3r-4} is compounded of a $\gamma_t(q)$ then t divides 4, and so $t = 2$. As earlier in this proof, W_q admits a non-special $g^{r-2}_{(3r-4-f)/2}$ and $q = (r-f)/2$. Again this leads to too many half-canonical G^r_{3r}'s. If l^{r-2}_{3r-4} has a fixed point then so does l^r_{3r} $(= l^{r-2}_r + g^1_4)$, a contradiction.

We now proceed for all $r \geq 3$. For $r = 4$ we assume that $l^4_{12} - g^1_4$ $(= l^2_8)$ is simple. (Note that the preceding argument shows that if l^2_8 is composite then the only possibility is $|2g^1_4| = l^2_8$.) If $r = 3$ we assume that $l^3_9 - g^1_4$ $(= l^1_5)$ is without fixed points. (The only other possibility is that $l^1_5 = g^1_4 + x$, where x is a fixed point.)

By Lemma 2.10 l^{r-2}_{3r-4} imposes at most $r-1$ conditions on the other linear series in the quartet, and if $r \geq 4$ they are all simple as a consequence. (The author is indebted to L. Donohoe [5] for the basic idea in the following discussion as well as many of the details.)

Suppose that $g^{r-1}_{3r} \equiv l^{r-2}_{3r-4} + S$, where S, a divisor of degree 4, is not in g^1_4. By doubling this last equivalence, it follows that $|2S| \equiv 2g^1_4$, and Lemma 2.14 shows that $S = P + Q$ where $|2P| \equiv |2Q| \equiv g^1_4$, and $(P,Q) = 0$. Then

$$g^{r-1}_{3r} - P \equiv l^{r-2}_{3r-4} + Q \equiv l^r_{3r} - Q \tag{7.1}$$

Q imposes one or two conditions on l^r_{3r} since l^r_{3r} is without fixed points.

If Q imposes one condition on l^r_{3r} then P is a fixed divisor for g^{r-1}_{3r}.

Suppose Q imposes two conditions on l^r_{3r}. Then $l^r_{3r} - Q := l^{r-2}_{3r-2}$. By BN $l^r_{3r} + Q$ has dimension r. Since $g^{r-1}_{3r} \equiv l^{r-2}_{3r-4} + P + Q$, and $g^{r-1}_{3r} + P \equiv l^r_{3r} + Q$, we see that g^{r-1}_{3r} has Q as a fixed divisor and $g^{r-1}_{3r} - Q \equiv l^{r-2}_{3r-4} + P$.

In all cases g^{r-1}_{3r} has a divisor of fixed points of degree two, P_g, and there is another divisor Q_g $(2P_g \equiv 2Q_g \equiv g^1_4)$ so that

$$g^{r-1}_{3r} - P_g \equiv l^{r-2}_{3r-4} + Q_g \;\; (:= g^{r-1}_{3r-2})$$

By the same argument there exist P_h, Q_h (resp P_k, Q_k), divisors of degree two, so that P_h (resp P_k) is a divisor of fixed points for h^{r-1}_{3r} (resp k^{r-1}_{3r}) and

$$h^{r-1}_{3r} - P_h \equiv l^{r-2}_{3r-4} + Q_h \;\; (:= h^{r-1}_{3r-2})$$

$$k^{r-1}_{3r} - P_k \equiv l^{r-2}_{3r-4} + Q_k \;\; (:= k^{r-1}_{3r-2})$$

We claim that $g^{r-1}_{3r-2} = h^{r-1}_{3r-2} = k^{r-1}_{3r-2}$. For if the first two are not equal then $g^{r-1}_{3r-2} + h^{r-1}_{3r-2} \equiv g^{3r-3+\varepsilon}_{6r-4}$ which is special (Lemma 2.18, RR.) By Cliff and BN we see that $\varepsilon = 0$, and $g^{r-1}_{3r-2} + h^{r-1}_{3r-2} + g^1_4 \equiv K_g$. By replacing g^1_4 by $2P_g$ and noting that $g^{r-1}_{3r-2} + P_g$ is half canonical we see that $g^{r-1}_{3r-2} = h^{r-1}_{3r-2}$ after all. Call this linear series m^{r-1}_{3r-2}. Then we have:

$$\begin{aligned} m^{r-1}_{3r-2} &= g^{r-1}_{3r} - P_g \equiv l^{r-2}_{3r-4} + Q_g \\ &= h^{r-1}_{3r} - P_h \equiv l^{r-2}_{3r-4} + Q_h \\ &= k^{r-1}_{3r} - P_k \equiv l^{r-2}_{3r-4} + Q_k \end{aligned}$$

Consequently $Q_g = Q_h = Q_k$ $(:= Q)$ and $l^r_{3r} \equiv m^{r-1}_{3r-2} + Q$. Thus $2P_g \equiv 2P_h \equiv 2P_k \equiv 2Q$ and $P_g + P_h \equiv P_k + Q$. By Lemma 2.7 it follows that there is a smooth Galois covering $W_{4g-3} \to W_g$ and W_{4g-3} admits a g^3_8 which is necessarily composite. Thus W_{4g-3} is elliptic-hyperelliptic and W_g is q-hyperelliptic ($q = 0$ or 1.) We have reached the desired contradiction.

Now we must consider the exceptional cases.

For $r = 4$ assume $l^4_{12} \equiv 3g^1_4$. First we show that g^3_{12} is simple. If not then W_{13} admits a $\gamma_t(q)$ where t divides 4; thus $t = 2$. W_q admits a g^3_{6-f} (f fixed points for g^3_{12}) and W_q is hyperelliptic. Since l^4_{12} is simple, Lemma 2.11 implies that $q \geq 4$. But then $f = 0$, and $g^3_6 \equiv 3g^1_2$, and this gives $g^3_{12} \equiv 3g^1_4$, a contradiction. Thus g^3_{12} is simple.

If g^3_{12} has one fixed point, $g^3_{12} = g^3_{11} + x$, then by Lemma 2.10 $l^4_{12} - g^3_{11} = y$ and $2x \equiv 2y$, a contradiction. If g^3_{12} has two fixed points then $g^3_{12} = g^3_{10} + P$, $l^4_{12} - g^3_{10} = Q$, $2P \equiv 2Q \equiv g^1_4$, and so

$$g^3_{12} - P \equiv l^4_{12} - Q \tag{7.2}$$

If g^3_{12} has 3 fixed points then W_{13} admits a simple g^3_9. Since this implies that $g \leq 12$ we have a contradiction. If g^3_{12} has no fixed points then $g^3_{12} - g^1_4 = g^1_8$. $l^4_{12} - g^1_8 = l^0_4$ (Lemma 2.9), and $2l^0_4 \equiv 2g^1_4$. Thus $l^0_4 = P + Q$, $2P \equiv 2Q \equiv g^1_4$, $g^3_{12} \equiv g^1_8 + 2P$, $l^4_{12} \equiv g^1_8 + P + Q$, or

(7.3) $$g^3_{12} - P \equiv l^4_{12} - Q.$$

Exactly the same argument can be applied to h^3_{12} and k^3_{12} and we get equivalences analogous to (7.2) and (7.3). Then we can apply the argument following (7.1) and reach the conclusion that W_{13} does not admit a g^1_4.

The case $r = 3$ starts with $l^3_9 \equiv 2g^1_4 + x = g^1_4 + l^1_5$ where $l^1_5 = g^1_4 + x$. We first show that g^2_9 is simple. If g^2_9 is composite and W_{10} admits a g^1_4 (which is unique) then W_{10} admits a $\gamma_2(q)$ where W_q admits a g^2_4 and is hyperelliptic and by Lemma 2.11 $q \geq 3$. Thus $g^2_4 = 2g^1_2$, $g^2_9 \equiv 2g^1_4 + y$, and $2x \equiv 2y$. This contradiction shows that g^2_9 is simple.

If g^2_9 is simple it cannot admit only one fixed point ($g^2_9 = g^2_8 + x$, $l^3_9 \equiv g^2_8 + y$, $2x \equiv 2y$, contradiction, as before.) It cannot admit three fixed points ($l^3_9 - g^2_6 \equiv g^1_3$, Lemma 2.10.) If g^2_9 admits a fixed divisor P of degree 2, then $g^2_9 = g^2_7 + P$, $l^3_9 \equiv g^2_7 + Q$, $2P \equiv 2Q$, and we have

(7.4) $$l^3_9 - Q \equiv g^2_9 - P$$

If g^2_9 is without fixed points then $g^2_9 \equiv g^1_4 + g^0_5$. $2g^0_5 \equiv 2l^1_5 \equiv 2g^1_4 + 2x \equiv l^3_9 + x$. $2g^1_4$ $(\equiv g^2_8) \equiv K_{10} - 2l^1_5$. By Cliff $2l^1_5$ $(\equiv 2g^0_5 \equiv l^3_{10})$ has dimension 3 and so has x for a fixed point. Consequently, $x \,\varepsilon\, g^0_5$, $g^0_5 = g^0_4 + x$, $2g^0_4 + 2x \equiv 2l^1_5 \equiv 2g^1_4 + 2x$, or $2g^1_4 \equiv 2g^0_4$. $g^0_4 \equiv P + Q$, $2P \equiv 2Q$, $(P, Q) = 0$, and we have

(7.5) $$l^3_9 - Q \equiv g^2_9 - P$$

For each of the linear series g^2_9, h^2_9, k^2_9 we have formulas like (7.4) and (7.5), and we conclude the proof for the case $g = 10$ as in the proof following formula (7.1). For $r \geq 3$ the proof that W_{3r+1} does not admit a g^1_4 is complete. □

The case $g = 7$ was considered in [4].

By CS it follows that only for $g \leq 10$ can a W_g admit several elliptic-trigonal coverings. If W_{10} covers a torus in three sheets then by Lemma 2.4 there can be only one quartet lifted from this torus to W_{10}. Thus there is a one-to-one correspondence between full three-sheeted coverings $W_{10} \to W_1$ and quartets on W_{10}.

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