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# ON A CERTAIN HOLOMORPHIC CURVE EXTREMAL FOR THE DEFECT RELATION

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## 1. Introduction

Let $f = [f_1, \dots, f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the $n$-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1, \dots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\},$$

where $n$ is a positive integer.

We use the following notations:

$$\|f(z)\| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a} = (a_1, \dots, a_{n+1}) \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$

$$\|\boldsymbol{a}\| = (|a_1|^2 + \cdots + |a_{n+1}|^2)^{1/2},$$
$$(\boldsymbol{a}, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1},$$
$$(\boldsymbol{a}, f(z)) = a_1 f_1(z) + \cdots + a_{n+1} f_{n+1}(z).$$

The characteristic function of $f$ is defined as follows (see [11]):

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log\|f(re^{i\theta})\| \, d\theta - \log\|f(0)\|.$$

Due to Cartan ([1]), we have the following relation:

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \max_{1 \le j \le n+1} |f_j(re^{i\theta})| \, d\theta + O(1). \tag{1}$$

We suppose throughout the paper that $f$ is transcendental; that is to say,

$$\lim_{r \to \infty} \frac{T(r, f)}{\log r} = \infty$$

and that $f$ is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

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It is well-known that $f$ is linearly non-degenerate over $\boldsymbol{C}$ if and only if the Wronskian $W = W(f_1, \dots, f_{n+1})$ of $f_1, \dots, f_{n+1}$ is not identically equal to zero.

For meromorphic functions in the complex plane we use the standard notations of Nevanlinna theory of meromorphic functions ([4], [5]).

For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$, we write

$$m(r, \boldsymbol{a}, f) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|\boldsymbol{a}\| \, \|f(re^{i\theta})\|}{|(\boldsymbol{a}, f(re^{i\theta}))|} \, d\theta,$$

$$N(r, \boldsymbol{a}, f) = N\left(r, \frac{1}{(\boldsymbol{a}, f)}\right).$$

We then have the first fundamental theorem:

$$T(r, f) = m(r, \boldsymbol{a}, f) + N(r, \boldsymbol{a}, f) + O(1) \tag{2}$$

([11], p. 76). We call the quantity

$$\delta(\boldsymbol{a}, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \boldsymbol{a}, f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, \boldsymbol{a}, f)}{T(r, f)}$$

the deficiency (or defect) of $\boldsymbol{a}$ with respect to $f$. We have

$$0 \le \delta(\boldsymbol{a}, f) \le 1$$

by (2) since $N(r, \boldsymbol{a}, f) \ge 0$ for $r \ge 1$ and $m(r, \boldsymbol{a}, f) \ge 0$ for $r > 0$.

Let $X$ be a subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in $N$-subgeneral position; that is to say, $\# X \ge N + 1$ and any $N + 1$ elements of $X$ generate $\boldsymbol{C}^{n+1}$, where $N$ is an integer satisfying $N \ge n$.

Cartan ([1], $N = n$) and Nochka ([6], $N > n$) gave the following

Theorem A (Defect relation). *For any $q$ elements $\boldsymbol{a}_j$ $(j = 1, \dots, q)$ of $X$,*

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le 2N - n + 1,$$

*where $2N - n + 1 \le q \le \infty$ (see also [2] or [3]).*

We are interested in the holomorphic curve $f$ for which the defect relation is extremal:

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1. \tag{3}$$

In [9] we proved the following theorem.

Theorem B. *Suppose that there are vectors $\boldsymbol{a}_j$ $(j = 1, \dots, q)$ in $X$ which satisfy* (3), *where $2N - n + 1 \le q \le \infty$. If $(n + 1, 2N - n + 1) = 1$, then there are at least*

$$\left[\frac{2N-n+1}{n+1}\right]+1$$

*vectors* $\boldsymbol{a} \in \{\boldsymbol{a}_j \ (j=1,\dots,q)\}$ *satisfying* $\delta(\boldsymbol{a}, f) = 1$.

Further, we improved this theorem in [10]. Namely, we weakened the condition "$(n+1, 2N-n+1) = 1$" in Theorem B to "$N > n = 2m$ $(m \in \boldsymbol{N})$" and obtained the same conclusion as in Theorem B.

In this paper we consider the holomorphic curve $f$ satisfying (3) from a different point of view.

Let

$$X_k(0) = \{\boldsymbol{a} = (a_1, \dots, a_{n+1}) \in X \mid a_k = 0\} \quad (1 \le k \le n+1).$$

Then, it is easy to see that

$$0 \le \# X_k(0) \le N$$

since $X$ is in $N$-subgeneral position.

Further we put (see Definition 1 in [7])

$$u_k(z) = \max_{1 \le j \le n+1, j \ne k} |f_j(z)|,$$

$$t_k(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \{\log u_k(re^{i\theta}) - \log u_k(e^{i\theta})\}\, d\theta,$$

and

$$\Omega_k = \limsup_{r\to\infty} \frac{t_k(r,f)}{T(r,f)}.$$

Proposition A (see [7]).

(a) $t_k(r, f)$ *is independent of the choice of reduced representation of* $f$.

(b) $t_k(r, f) \le T(r, f) + O(1)$.

(c) $N(r, 1/f_j) \le t_k(r, f) + O(1)$ $(j = 1, \dots, n+1, j \ne k)$.

(d) $0 \le \Omega_k \le 1$.

Our main purpose of this paper is to prove the following theorem:

Theorem. *Suppose that*

(i) $N > n \ge 2$;

(ii) *there are vectors* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(2N - n + 1 < q \le \infty)$ *satisfying* $\delta(\boldsymbol{a}_j, f) > 0$ $(j = 1, \dots, q)$ *and*

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1.$$

*If* $\Omega_k < 1$ *for some* $k$ $(1 \le k \le n+1)$, *then*

(a) $\#X_k(0) = N$;
(b) *there is a subset* $P \subset \{1, 2, \dots, q\}$ *satisfying*

$$\#P = N - n + 1, \quad d(P) = 1, \quad \delta(\boldsymbol{a}_j, f) = 1 \quad (j \in P)$$

*and*

$$X_k(0) \cap \{\boldsymbol{a}_j \mid j \in P\} = \phi,$$

*where* $d(P)$ *is the dimension of the vector space spanned by* $\{\boldsymbol{a}_j \mid j \in P\}$.
(c) *Any* $n$ *elements of* $X - \{\boldsymbol{a}_j \mid j \in P\}$ *are linearly independent.*

As an application of this theorem, we can prove the following result:

"*Let* $f$ *be any exponential curve. If* $N > n \geq 2$, *then*

$$\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f) < 2N - n + 1."$$

This result means that any exponential curve is not extremal for the defect relation when $N > n \geq 2$.

## 2. Preliminaries and lemma

We shall give some lemmas for later use. Let $f = [f_1, \dots, f_{n+1}]$, $X$ and $X_k(0)$ etc. be as in Section 1, $q$ any integer satisfying $2N - n + 1 < q < \infty$ and we put $Q = \{1, 2, \dots, q\}$.

Let $\{\boldsymbol{a}_j \mid j \in Q\}$ be a family of vectors in $X$. For a non-empty subset $P$ of $Q$, we denote

$$V(P) = \text{the vector space spanned by } \{\boldsymbol{a}_j \mid j \in P\} \quad \text{and} \quad d(P) = \dim V(P)$$

and we put

$$\mathcal{O} = \{P \subset Q \mid 0 < \#P \leq N + 1\}.$$

Lemma 1 ((2.4.3) in [3], p. 68). *For* $P \in \mathcal{O}$, $\#P - d(P) \leq N - n$.

For $\{\boldsymbol{a}_j \mid j \in Q\}$, let

$$\omega : Q \to (0, 1]$$

be the Nochka weight function given in [3, p. 72] and $\theta$ the reciprocal number of the Nochka constant given in [3, p. 72]. Then they have the following properties:

Lemma 2 (see [3], Theorem 2.11.4).
(a) $0 < \omega(j)\theta \leq 1$ *for all* $j \in Q$;
(b) $q - 2N + n - 1 = \theta(\sum_{j=1}^{q} \omega(j) - n - 1)$;
(c) $(N + 1)/(n + 1) \leq \theta \leq (2N - n + 1)/(n + 1)$;
(d) *If* $P \in \mathcal{O}$, *then* $\sum_{j \in P} \omega(j) \leq d(P)$.

*Note* 1. (c) of Lemma 2 can be refined as follows: $\dfrac{N}{n} \le \theta \le \dfrac{2N-n+1}{n+1}$.

*Proof.* When $\theta = (2N-n+1)/(n+1)$, there is nothing to prove as $N/n \le (2N-n+1)/(n+1)$.

When $\theta < (2N-n+1)/(n+1)$, there is an element $P \in \mathcal{O}$ satisfying

$$\theta = \frac{2N-n+1-\#P}{n+1-d(P)} \quad (1 \le d(P) \le n)$$

by the definition of $\theta$. By Lemma 1 we have

$$\theta = \frac{2N-n+1-\#P}{n+1-d(P)} \ge \frac{N+1-d(P)}{n+1-d(P)} \ge \frac{N}{n}$$

since $d(P) \ge 1$. □

Put

$$P_k(0) = \{j \in Q \mid \boldsymbol{a}_j \in X_k(0)\} \quad \text{and} \quad d_k = \sum_{j \in P_k(0)} \omega(j).$$

Then, we have the inequality

$$d_k \le n \tag{4}$$

since $d_k \le d(P_k(0))$ by Lemma 2(d) and $d(P_k(0)) \le n$ by the definition of $X_k(0)$.

Lemma 3 (Defect relation) (see Theorem 3 in [8]). *For any* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$, *we have the following inequalities*:

(I) $\sum_{j=1}^{q} \omega(j)\delta(\boldsymbol{a}_j, f) \le d_k + 1 + (n-d_k)\Omega_k$;

(II) $\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le 2N - n + 1 - \dfrac{N}{n}(n-d_k)(1-\Omega_k)$.

By applying Lemma 2 and Note 1 to (I) we obtain (II) as usual. (II) is an amelioration of Theorem 3 (II) in [8].

*Remark* 1. This is an amelioration of Theorem A. Since $\Omega_k \le 1$ and $d_k \le n$ we have the inequalities:

$$d_k + 1 + (n-d_k)\Omega_k \le n+1 \quad \text{and}$$
$$2N - n + 1 - N(n-d_k)(1-\Omega_k)/n \le 2N - n + 1.$$

Lemma 4. *For any* $\boldsymbol{a} \in X_k(0)$, $\delta(\boldsymbol{a}, f) \ge 1 - \Omega_k$.

*Proof.* For $\boldsymbol{a} \in X_k(0)$ we have the inequality

$$|(\boldsymbol{a}, f(z))| \le K u_k(z)$$

for a positive constant $K$ by the definitions of $X_k(0)$ and $u_k(z)$. From this inequality we have the inequality

$$N(r, \boldsymbol{a}, f) = \frac{1}{2\pi}\int_0^{2\pi} \log|(\boldsymbol{a}, f(re^{i\theta}))|\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \log u_k(re^{i\theta})\, d\theta + \log K$$

$$= t_k(r, f) + O(1) \quad (r > 0),$$

from which we obtain the inequality $\delta(\boldsymbol{a}, f) \ge 1 - \Omega_k$. □

Lemma 5 ([9], Lemma 3). *Suppose that* $N > n$. *For* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ *the maximal deficiency sum*

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1$$

*holds if and only if the following two relations hold*:

1) $(1 - \theta\omega(j))(1 - \delta(\boldsymbol{a}_j, f)) = 0 \ (j = 1, \dots, q)$;
2) $\sum_{j=1}^{q} \omega(j)\delta(\boldsymbol{a}_j, f) = n + 1$.

Corollary 1 ([9], Corollary 1(I)). *Suppose that* $N > n$ *and that for* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$, *the equality*

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1$$

*holds. If* $\theta\omega(j) < 1$ *for some* $j$, *then* $\delta(\boldsymbol{a}_j, f) = 1$.

Definition 1 ([9], Definition 1). We put

$$\lambda = \min_{P \in \mathcal{O}} \frac{d(P)}{\#P}.$$

Then, $\lambda$ has the following property.

Lemma 6 ([9], Proposition 2). $1/(N - n + 1) \le \lambda \le (n + 1)/(N + 1)$.

*Remark* 2. (a) If $\lambda < (n + 1)/(2N - n + 1)$, then $\lambda = \min_{1 \le j \le q} \omega(j)$ and $\omega(j) = \lambda$, $\theta\omega(j) < 1$ $(j \in P_0)$ for an element $P_0 \in \mathcal{O}$ satisfying $\lambda = d(P_0)/\#P_0$.

(b) If $\lambda \ge (n + 1)/(2N - n + 1)$, then $\omega(j) = 1/\theta = (n + 1)/(2N - n + 1)$ $(j = 1, \dots, q)$.

In fact, the first assertion of (a) is given in the proof of Proposition 2.4.4 ([3], p. 68) with the definition of $\omega(j)$ ([3], p. 72). For the second assertion of (a), as $\omega(j) = \lambda$ $(j \in P_0)$ by the definition of $\omega$ and $(n + 1)/(2N - n + 1) \le 1/\theta$, we have the conclusion.

(b) See the definitions of $\omega(j)$ and $\theta$ ([3], p. 72).

Lemma 7 ([9], Corollary 2). *For* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$, *we have the inequality*

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le \min\left(2N - n + 1, \frac{n+1}{\lambda}\right).$$

### 3. Proof of Theorem when $q < \infty$

Let $f$, $X$, $X_k(0)$ and $\omega$ etc. be as in the previous sections and $q$ an integer satisfying

$$2N - n + 1 < q < \infty.$$

Throughout this section we suppose that
(i) $N > n \ge 2$;
(ii) there are vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ satisfying $\delta(\boldsymbol{a}_j, f) > 0$ $(j = 1, \dots, q)$ and

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1;$$

(iii) $\Omega_k < 1$ for some $k$ $(1 \le k \le n+1)$.

Proposition 1. $X_k(0) \subset \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$.

*Proof.* If there exists a vector $\boldsymbol{a} \in X_k(0)$ satisfying $\boldsymbol{a} \notin \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$, then $\delta(\boldsymbol{a}, f) > 0$ by Lemma 4 and (iii), and so by Theorem A we have the inequality

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le 2N - n + 1 - \delta(\boldsymbol{a}, f) < 2N - n + 1,$$

which is a contradiction to our assumption (ii). □

Proposition 2. $d_k = n$.

*Proof.* From Lemma 3(II) and the assumption (ii) we have the inequality

$$(1 - \Omega_k)(n - d_k) \le 0.$$

Then, by the assumption (iii) and (4), we obtain the equality $d_k = n$. □

Proposition 3. (a) $\theta = N/n$, (b) $\#P_k(0) = N$ *and* (c) $\theta\omega(j) = 1$ $(j \in P_k(0))$.

*Proof.* As $X$ is in $N$-subgeneral position, we have $\#X_k(0) \le N$. From Proposition 2 and Lemma 2(a)

$$\text{(A)} \quad \theta n = \sum_{j \in P_k(0)} \theta\omega(j) \le \sum_{j \in P_k(0)} 1 = \#P_k(0) = \#X_k(0) \le N,$$

so that we have $\theta \le N/n$. By Note 1 we obtain $\theta = N/n$.

Combining this result with the inequality (A), we have

$$\# P_k(0) = N \quad \text{and} \quad \theta\omega(j) = 1 \quad (j \in P_k(0)). \qquad \square$$

Corollary 2. $\lambda < (n+1)/(2N - n + 1)$.

*Proof.* By Lemma 7 and the assumption (ii), we have

$$\lambda \leq \frac{n+1}{2N-n+1}.$$

If $\lambda = (n+1)/(2N-n+1)$, then by Remark 2(b) and Proposition 3(a)

$$\theta = \frac{2N-n+1}{n+1} = \frac{N}{n},$$

which is a contradiction, since $N/n < (2N-n+1)/(n+1)$ when $N > n \geq 2$. This implies that our corollary holds. $\square$

Put

$$P_1 = \{ j \mid \theta\omega(j) < 1, 1 \leq j \leq q \}.$$

Then,

$$P_1 \cap P_k(0) = \phi \tag{5}$$

by Proposition 3(c). We have the following

Proposition 4. $N - n + 1 \leq \# P_1 < 2N - n + 1$.

*Proof.* (a) From Lemma 2(b), we have

$$q - (2N - n + 1) = \theta\left(\sum_{j=1}^{q} \omega(j) - n - 1\right) = \sum_{j \notin P_1} \theta\omega(j) + \sum_{j \in P_1} \theta\omega(j) - \theta n - \theta.$$

Here, as $\theta\omega(j) = 1$ for $j \notin P_1$ we have that

$$\sum_{j \notin P_1} \theta\omega(j) = q - \# P_1$$

and by Proposition 3(a) we have

$$\sum_{j \in P_1} \theta\omega(j) - \theta n - \theta = -N + \frac{N}{n}\left(\sum_{j \in P_1} \omega(j) - 1\right).$$

Combining these three equalities we obtain

$$(\mathrm{B}_1) \quad q - (2N - n + 1) = q - \# P_1 - N + \frac{N}{n}\left(\sum_{j \in P_1} \omega(j) - 1\right).$$

Here, as $1/(N-n+1) \leq \lambda \leq \omega(j)$ $(j \in P_1)$ due to Lemma 6, Corollary 2 and Remark 2(a) we obtain the inequality

$$(\mathrm{B}_2) \quad q - \#P_1 - N + \frac{N}{n}\left(\sum_{j \in P_1} \omega(j) - 1\right) \geq q - \#P_1 - N + \frac{N}{n}\left(\frac{\#P_1}{N-n+1} - 1\right).$$

From $(\mathrm{B}_1)$ and $(\mathrm{B}_2)$ we have the inequality

$$q - (2N - n + 1) \geq q - \#P_1 - N + \frac{N}{n}\left(\frac{\#P_1}{N-n+1} - 1\right),$$

which reduces to the inequality

$$(N - n + 1 - N/n)(\#P_1 - N + n - 1) \geq 0.$$

As

$$N - n + 1 - \frac{N}{n} = \frac{(N-n)(n-1)}{n} > 0$$

by the assumption (i), we have

$$\#P_1 \geq N - n + 1.$$

(b) As $\delta(\boldsymbol{a}_j, f) = 1$ $(j \in P_1)$ by Corollary 1, from Propositions 1, 3(b) and the assumption (ii) we have

$$\#P_1 < 2N - n + 1$$

as $P_1 \cap P_k(0) = \phi$ by (5). □

Let $P_0$ be an element of $\mathcal{O}$ satisfying

$$\frac{d(P_0)}{\#P_0} = \lambda,$$

where $\lambda = \min_{P \in \mathcal{O}} d(P)/\#P$. Then, $\omega(j) = \lambda$ $(j \in P_0)$ and

$$\phi \neq P_0 \subset P_1 \tag{6}$$

since $\theta\lambda < 1$ by Corollary 2 and Remark 2(a).

Proposition 5. (a) $\#P_0 = N - n + 1$, (b) $d(P_0) = 1$ and (c) $\omega(j) = \lambda = 1/(N-n+1)$ $(j \in P_0)$.

*Proof.* By Proposition 3(a), $\theta$ is equal to $N/n$, which is smaller than $(2N - n + 1)/(n+1)$. By the definition of $\theta$, there exists a set $P \in \mathcal{O}$ satisfying

$$P_0 \subset P, \quad 1 \leq d(P) \leq n$$

and

$$\theta = \frac{2N - n + 1 - \#P}{n + 1 - d(P)} = \frac{N}{n}.$$

By Proposition 3(a) and Lemma 1 we have the inequality

$$0 = \theta - \frac{N}{n} = \frac{2N - n + 1 - \#P}{n + 1 - d(P)} - \frac{N}{n} = \frac{(N-n)(n-1) + Nd(P) - n\#P}{n(n+1-d(P))}$$
$$\geq \frac{(N-n)(d(P)-1)}{n(n+1-d(P))} \geq 0,$$

which implies that

$$d(P) = 1 \quad \text{and} \quad \#P = N - n + 1.$$

By Lemma 2(d), Remark 2(a) with Corollary 2 and Lemma 6 we obtain the inequality

$$1 = d(P) \geq \sum_{j \in P} \omega(j) \geq (N - n + 1)\lambda \geq 1$$

and we have

$$\lambda = \frac{1}{N - n + 1} = \omega(j) \quad (j \in P).$$

By the choice of $P_0$, $1 \leq d(P_0) \leq d(P) = 1$ and so we have

$$d(P_0) = 1 \quad \text{and} \quad \#P_0 = N - n + 1.$$ □

Proposition 6. $P_1 = P_0$.

*Proof.* By Lemma 2(b) we have the equality

$$q - (2N - n + 1) = \theta\left(\sum_{j=1}^{q} \omega(j) - n - 1\right)$$
$$= \theta\left(\sum_{j \notin P_k(0) \cup P_0} \omega(j) + \sum_{j \in P_k(0) \cup P_0} \omega(j) - n - 1\right).$$

Here, as $P_k(0) \cap P_0 = \phi$ by (5) and (6), $\sum_{j \in P_k(0)} \omega(j) = d_k = n$ (Proposition 2) and $\sum_{j \in P_0} \omega(j) = 1$ (Proposition 5(a), (c)), we have

$$\sum_{j \in P_k(0) \cup P_0} \omega(j) = \sum_{j \in P_k(0)} \omega(j) + \sum_{j \in P_0} \omega(j) = n + 1,$$

so that we have the equality

$$(\mathrm{C}_1) \quad q - (2N - n + 1) = \theta \sum_{j \notin P_k(0) \cup P_0} \omega(j).$$

As $P_0 \subset P_1$, $\theta\omega(j) = 1$ for $j \notin P_1$ and $\theta\omega(j) < 1$ for $j \in P_1$ by the definition of $P_1$, $\#P_k(0) = N$ (Proposition 3(b)), $\#P_0 = N - n + 1$ (Proposition 5(a)) and $P_k(0) \cap P_1 = \phi$ by (5), we have

$$(\mathrm{C}_2)\quad \theta \sum_{j \notin P_k(0) \cup P_0} \omega(j) = q - (2N - n + 1) - \#(P_1 - P_0) + \theta \sum_{j \in P_1 - P_0} \omega(j)$$

$$= q - (2N - n + 1) - \sum_{j \in P_1 - P_0} (1 - \theta\omega(j)),$$

From $(\mathrm{C}_1)$ and $(\mathrm{C}_2)$ we have the equality

$$\sum_{j \in P_1 - P_0} (1 - \theta\omega(j)) = 0.$$

If $P_0 \subsetneqq P_1$, we have a contradiction since $1 - \theta\omega(j) > 0$ for $j \in P_1$. This means that $P_1 = P_0$ must hold. □

Proposition 7. *Any $n$ elements of $X - \{\boldsymbol{a}_j \mid j \in P_0\}$ are linearly independent. In particular, any $n$ elements of $X_k(0)$ are linearly independent.*

*Proof.* Let $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ be any $n$ elements of $X - \{\boldsymbol{a}_j \mid j \in P_0\}$. Then, the set

$$\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\} \cup \{\boldsymbol{a}_j \mid j \in P_0\}$$

contains $n + 1$ linearly independent elements since $X$ is in $N$-subgeneral position. As $d(P_0) = 1$, $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ must be linearly independent. As $X_k(0) \subset X - \{\boldsymbol{a}_j \mid j \in P_0\}$, we have the last assertion. □

Summarizing Propositions from 1 through 7, we obtain the following

Theorem 1. *Suppose that*
(i) $N > n \geq 2$;
(ii) *there are vectors* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(2N - n + 1 < q < \infty)$ *satisfying*

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1.$$

*If $\Omega_k < 1$ for some $k$ $(1 \leq k \leq n + 1)$, then*
(a) $X_k(0) \subset \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$ *and* $\#X_k(0) = N$;
(b) *there is a subset $P \subset Q$ satisfying*

$$\#P = N - n + 1, \quad d(P) = 1, \quad \delta(\boldsymbol{a}_j, f) = 1 \quad (j \in P)$$

*and*

$$X_k(0) \cap \{\boldsymbol{a}_j \mid j \in P\} = \phi;$$

(c) *any $n$ elements of $X - \{\boldsymbol{a}_j \mid j \in P\}$ are linearly independent.*

## 4. Proof of Theorem when $q = \infty$

Let $[f_1, \ldots, f_{n+1}]$, $X$, $X_k(0)$, $\theta$ and $\omega$ etc. be as in Section 1, 2 or 3. From Theorem A, it is easy to see that the set

$$\{\boldsymbol{a} \in X \mid \delta(\boldsymbol{a}, f) > 0\}$$

is at most countable and

$$\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f) \le 2N - n + 1.$$

In this section we consider a holomorphic curve $f$ with an infinite number of vectors $\boldsymbol{a}_j \in X$ such that

$$\delta(\boldsymbol{a}_j, f) > 0 \quad (j = 1, 2, 3, \ldots).$$

We put

$$\boldsymbol{N} = \{1, 2, 3, \ldots\} \quad \text{(the set of positive integers)};$$
$$Y = \{\boldsymbol{a}_j \mid j \in \boldsymbol{N}\};$$
$$\mathcal{O}_\infty = \{P \subset \boldsymbol{N} \mid 0 < \#P \le N + 1\}$$

and for any non-empty, finite subset $P$ of $\boldsymbol{N}$, we use

$$V(P) \quad \text{and} \quad d(P)$$

as in Section 2.

Definition 2 (see [9], p. 144). We put

$$\lambda_\infty = \min_{P \in \mathcal{O}_\infty} \frac{d(P)}{\#P}.$$

Note that the set $\{d(P)/\#P \mid P \in \mathcal{O}_\infty\}$ is a finite set. We have the followings ([9], p. 144):

$(a_\infty)$ $1/(N - n + 1) \le \lambda_\infty \le (n + 1)/(N + 1)$;

$(b_\infty)$ (the inequality (12) in [9])

$$\sum_{j=1}^{\infty} \delta(\boldsymbol{a}_j, f) \le (n + 1)/\lambda_\infty.$$

From now on throughout this section we suppose that

(i) $N > n \ge 2$;

(ii) there exists a subset $Y = \{\boldsymbol{a}_j \mid j \in \boldsymbol{N}\}$ of $X$ satisfying $\delta(\boldsymbol{a}_j, f) > 0$ and

$$\sum_{j=1}^{\infty} \delta(\boldsymbol{a}_j, f) = 2N - n + 1; \tag{7}$$

(iii) $\Omega_k < 1$ for some $k$ $(1 \le k \le n + 1)$.

Note that we obtain the inequality

$$\lambda_\infty \leq (n+1)/(2N-n+1) \tag{8}$$

from (7) and $(b_\infty)$. Let $P_0$ be an element of $\mathcal{O}_\infty$ satisfying

$$\frac{d(P_0)}{\# P_0} = \lambda_\infty$$

and let $\varepsilon$ be any positive number satisfying

$$0 < \varepsilon < (N-n)(1-\Omega_k)/(N-n+1)(n+1). \tag{9}$$

We restrict the number $\varepsilon$ as in (9) for the forthcoming Propositions from 8 through 13 to hold.

Now, for the number $\varepsilon$ in (9), there exists $p \in \boldsymbol{N}$ satisfying $\{1, 2, \ldots, p\} \supset P_0$, $p > 2N - n + 1$ and

$$2N - n + 1 - \varepsilon < \sum_{j=1}^{p} \delta(\boldsymbol{a}_j, f). \tag{10}$$

For an integer $q$ not less than $p$, we put

$$Q = \{1, 2, \ldots, q\}.$$

Note that $2N - n + 1 < q < \infty$. For this $Q$, we use $\theta_q$, $\omega_q$ and $\lambda_q$ instead of $\theta$, $\omega$ and $\lambda$ in Section 2 respectively. Note that

$$\lambda_q = \lambda_\infty \tag{11}$$

since $Q \supset P_0$. Further we obtain the following inequalities from the equality (2) in [9]:

$$n + 1 - \frac{\varepsilon}{\theta_q} < \sum_{j=1}^{q} \omega_q(j) \delta(\boldsymbol{a}_j, f) \tag{12}$$

$$\sum_{j=1}^{q} (1 - \theta_q \omega_q(j))(1 - \delta(\boldsymbol{a}_j, f)) < \varepsilon. \tag{13}$$

From now on we put $\varepsilon_1 = \varepsilon/(1 - \Omega_k)$ for simplicity. Then,

$$0 < \varepsilon_1 < (N-n)/(N-n+1)(n+1). \tag{14}$$

Proposition 8. $X_k(0) \subset \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q\}$.

*Proof.* If there exists a vector $\boldsymbol{a} \in X_k(0)$ satisfying $\boldsymbol{a} \notin \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q\}$, then by Lemma 4, Theorem A and (10) we have the inequality

$$2N-n+1-\varepsilon<\sum_{j=1}^{q}\delta(\boldsymbol{a}_j,f)\le 2N-n+1-\delta(\boldsymbol{a},f)\le 2N-n+1-(1-\Omega_k)$$
$$<2N-n+1-\varepsilon$$

as $p \le q$ and $\varepsilon < 1-\Omega_k$ from (9). This is a contradiction. □

We put

$$P_k(0)=\{j\in Q\,|\,\boldsymbol{a}_j\in X_k(0)\}\quad\text{and}\quad d_k(q)=\sum_{j\in P_k(0)}\omega_q(j).$$

Note that

(15) $$\#P_k(0)\le N\quad\text{and}\quad d_k(q)\le d(P_k(0))\le n.$$

In fact, we have $\#P_k(0)=\#X_k(0)\le N$ as $X$ is in $N$-subgeneral position. We have $d_k(q)\le d(P_k(0))$ by Lemma 2(d) and $d(P_k(0))\le n$ by the definition of $X_k(0)$.

Proposition 9. $n-\varepsilon_1/\theta_q<d_k(q)$.

*Proof.* From (12) and Lemma 3(I) we have the inequality

$$n+1-\varepsilon/\theta_q<\sum_{j=1}^{q}\omega_q(j)\delta(\boldsymbol{a}_j,f)\le d_k(q)+1+(n-d_k(q))\Omega_k$$

from which we obtain

$$(n-d_k(q))(1-\Omega_k)<\varepsilon/\theta_q$$

and so $n-\varepsilon_1/\theta_q<d_k(q)$ as $\Omega_k<1$ and $\varepsilon_1=\varepsilon/(1-\Omega_k)$. □

Proposition 10. (a) $\theta_q<(N+\varepsilon_1)/n$ *and* (b) $\#P_k(0)=N$.

*Proof.* From Proposition 9, Lemma 2(a) and (15), we have the inequality

$$\text{(D)}\quad \theta_q(n-\varepsilon_1/\theta_q)<\theta_q d_k(q)=\theta_q\sum_{j\in P_k(0)}\omega_q(j)\le\#P_k(0)\le N,$$

from which we obtain $\theta_q<(N+\varepsilon_1)/n$ easily. Next, from (D) and Note 1, we obtain that $N-\varepsilon_1<\#P_k(0)\le N$, so that $\#P_k(0)=N$ as $\varepsilon_1<1$ from (14). □

Corollary 3. (a) $\theta_q\lambda_q<1$ *and* (b) $\lambda_q<(n+1)/(2N-n+1)$.

*Proof.* (a) From (8), (11) and Proposition 10(a), we have

$$\theta_q \lambda_q < \frac{N + \varepsilon_1}{n} \frac{n+1}{2N - n + 1}$$

and by (14) and the assumption (i) it is easy to see that

$$\frac{N + \varepsilon_1}{n} \frac{n+1}{2N - n + 1} < 1.$$

We have (a) of this corollary.

(b) By (8) and (11), we have $\lambda_q \leq (n+1)/(2N-n+1)$. If $\lambda_q$ is equal to $(n+1)/(2N-n+1)$, then by Remark 2(b) we have $\theta_q \lambda_q = 1$, which contradicts (a) of this corollary. We have (b) of this corollary. □

Put

$$P_1 = \{ j \in Q \mid \theta_q \omega_q(j) < 1, j \notin P_k(0) \}.$$

Note that

$$P_1 \cap P_k(0) = \phi. \tag{16}$$

Proposition 11. $N - n + 1 \leq \# P_1$.

*Proof.* From Lemma 2(b) and (16) we have the equality

$$(\mathrm{E}_1) \quad q - (2N - n + 1) = \theta_q \left\{ \sum_{j=1}^{q} \omega_q(j) - n - 1 \right\}$$

$$= \theta_q \left\{ \sum_{j \in P_k(0)} \omega_q(j) + \sum_{j \in P_1} \omega_q(j) + \sum_{j \notin P_k(0) \cup P_1} \omega_q(j) - n - 1 \right\}$$

and by Proposition 9

$$> \theta_q \left\{ \sum_{j \in P_1} \omega_q(j) + \sum_{j \notin P_k(0) \cup P_1} \omega_q(j) - (1 + \varepsilon_1/\theta_q) \right\}.$$

Here, by $(a_\infty)$, (11) and Remark 2(a) with Corollary 3(b) we have

$$\sum_{j \in P_1} \omega_q(j) \geq \frac{\# P_1}{N - n + 1}$$

and as $\theta_q \omega_q(j) = 1$ for $j \notin P_k(0) \cup P_1$ by Lemma 2(a) and the definition of $P_1$, we have

$$\theta_q \sum_{j \notin P_k(0) \cup P_1} \omega_q(j) = q - \# P_k(0) - \# P_1,$$

so that we have the inequality

$$(\mathrm{E}_2)\quad \text{the last term of } (\mathrm{E}_1) \geq \frac{\theta_q \# P_1}{(N-n+1)} + q - \# P_k(0) - \# P_1 - \theta_q - \varepsilon_1.$$

From $(\mathrm{E}_1)$ and $(\mathrm{E}_2)$ we obtain the following inequality by Proposition 10(b)

$$\# P_1\left(1 - \frac{\theta_q}{N-n+1}\right) > N - n + 1 - \theta_q - \varepsilon_1,$$

which reduces to the inequality

$$\# P_1(N-n+1-\theta_q) > (N-n+1)(N-n+1-\theta_q-\varepsilon_1).$$

Here, by Proposition 10(a) and by the fact that $0 < \varepsilon_1 < 1$ from (14) we have the inequality

$$N-n+1-\theta_q > N-n+1-\frac{N+\varepsilon_1}{n} = \frac{(N-n)(n-1)-\varepsilon_1}{n} > 0$$

as $N > n \geq 2$ (the assumption (ii)), so that we have

$$\begin{aligned}
\# P_1 &> (N-n+1)\left(1 - \frac{\varepsilon_1}{N-n+1-\theta_q}\right) \\
&> (N-n+1)\left(1 - \frac{\varepsilon_1}{(N-n+1-(2N-n+1)/(n+1))}\right) \\
&= (N-n+1)\left(1 - \frac{(n+1)\varepsilon_1}{(N-n)(n-1)}\right) \\
&> N-n
\end{aligned}$$

by Lemma 2(c) and (14). This means that $\# P_1 \geq N-n+1$. □

Proposition 12. (a) $\# P_0 = N-n+1$, (b) $d(P_0) = 1$ *and* (c) $\theta_q = N/n$.

*Proof.* By the definition of $\theta_q$ and the choice of $P_0$, there exists a set $P$ satisfying

$$P_0 \subset P, \quad 1 \leq d(P) \leq n$$

and

$$\theta_q = \frac{2N-n+1-\# P}{n+1-d(P)}. \tag{17}$$

By Proposition 10(a), (17) and Lemma 1 we have the inequality

$$\text{(F)}\quad 0 > \theta_q - (N+\varepsilon_1)/n = \theta_q - N/n - \varepsilon_1/n$$
$$= \frac{(N-n)(n-1) + Nd(P) - n\#P}{n(n+1-d(P))} - \frac{\varepsilon_1}{n}$$
$$\geq \frac{(N-n)(d(P)-1)}{n(n+1-d(P))} - \frac{\varepsilon_1}{n}.$$

First we prove that $d(P) = 1$. Suppose that $d(P) \geq 2$. Then, from (F) we have the inequality

$$\frac{\varepsilon_1}{n} > \frac{N-n}{n(n-1)},$$

which reduces to the inequality

$$\varepsilon_1 > (N-n)/(n-1),$$

which contradicts (14). This means that $d(P)$ must be equal to 1.

As $d(P) = 1$, we have from (17) and Note 1 that

$$\theta_q = \frac{2N-n+1-\#P}{n} \geq \frac{N}{n},$$

from which we have that $\#P \leq N-n+1$. On the other hand, as

$$\theta_q = \frac{2N-n+1-\#P}{n} < \frac{N+\varepsilon_1}{n}$$

by Proposition 10(a), we have the following inequality by (14)

$$\#P > N-n+1-\varepsilon_1 > N-n+1-\frac{N-n}{(N-n+1)(n+1)} > N-n.$$

We have that $\#P = N-n+1$. Substituting $\#P = N-n+1$ and $d(P) = 1$ in (17) we obtain that $\theta_q = N/n$.

Next, by Lemma 2(d), $(a_\infty)$, (11) and Remark 2(a) with Corollary 3(b) we have the inequality

$$1 = d(P) \geq \sum_{j\in P} \omega_q(j) \geq (N-n+1)\lambda_q \geq 1$$

since $d(P) = 1$ as is proved above, so that we have

$$\lambda_q = \frac{1}{N-n+1} = \omega_q(j) \quad (j \in P).$$

As $1 \leq d(P_0) \leq d(P) = 1$, we have $d(P_0) = 1$. By the choice of $P_0$, we have the equality

$$\frac{1}{\# P_0} = \frac{d(P_0)}{\# P_0} = \lambda_q = \frac{1}{N-n+1},$$

from which we have that $\# P_0 = N - n + 1$. □

COROLLARY 4. $\lambda_q = \lambda_\infty = 1/(N-n+1) = \omega_q(j)$ $(j \in P_0)$.

PROPOSITION 13. (a) $P_1 = P_0$ *and* (b) $d_k(q) = n$.

*Proof.* First we note that

$$\theta_q \omega_q(j) = \frac{N}{n(N-n+1)} < 1 \quad (j \in P_0) \tag{18}$$

as $\theta_q = N/n$ (Proposition 12(c)) and $\omega_q(j) = 1/(N-n+1)$ for $j \in P_0$ (Corollary 4). Next, we prove that $P_0 \cap P_k(0) = \phi$. Suppose to the contrary that $P_0 \cap P_k(0) \neq \phi$. As $d(P_0) = 1$, we have $P_0 \subset P_k(0)$. Then, by Propositions 9, 12(a), Corollary 4 and Lemma 2(a) we have

$$\begin{aligned} n - \varepsilon_1/\theta_q < d_k(q) &= \sum_{j \in P_k(0)} \omega_q(j) = \sum_{j \in P_0} \omega_q(j) + \sum_{j \in P_k(0) - P_0} \omega_q(j) \\ &\leq 1 + \frac{\#(P_k(0) - P_0)}{\theta_q}. \end{aligned}$$

By Propositions 10(b), 12(a) and 12(c) the last term of this inequality is equal to

$$1 + \frac{(n-1)n}{N},$$

so that we have the inequality

$$\frac{(n-1)(N-n)}{n} < \varepsilon_1.$$

This contradicts (14). This implies that

$$P_0 \cap P_k(0) = \phi. \tag{19}$$

(18) and (19) mean that $P_0 \subset P_1$. By Lemma 2(b) we have the equality

$$\begin{aligned} q - (2N - n + 1) &= \theta_q \left( \sum_{j=1}^{q} \omega_q(j) - n - 1 \right) \\ &= \theta_q \left( \sum_{j \notin P_k(0) \cup P_0} \omega_q(j) + \sum_{j \in P_k(0) \cup P_0} \omega_q(j) - n - 1 \right). \end{aligned}$$

Here, as $P_k(0) \cap P_0 = \phi$, $\sum_{j \in P_k(0)} \omega_q(j) = d_k(q)$ (the definition of $d_k(q)$) and $\sum_{j \in P_0} \omega(j) = 1$ (Proposition 12(a), Corollary 4), we have

$$\sum_{j\in P_k(0)\cup P_0}\omega_q(j)=\sum_{j\in P_k(0)}\omega_q(j)+\sum_{j\in P_0}\omega_q(j)=d_k(q)+1,$$

so that we have the equality

$$(\mathrm{G}_1)\quad q-(2N-n+1)=\theta_q\sum_{j\notin P_k(0)\cup P_0}\omega_q(j)-\theta_q(n-d_k(q)).$$

As $P_0\subset P_1$, $\theta_q\omega_q(j)=1$ for $j\notin P_k(0)\cup P_1$ and $\theta_q\omega_q(j)<1$ for $j\in P_1$ by Lemma 2(a) and the definition of $P_1$, $\#P_k(0)=N$ (Proposition 10(b)), $\#P_0=N-n+1$ (Proposition 12(a)) and $P_k(0)\cap P_1=\phi$ by the definition of $P_1$, we have

$$\begin{aligned}(\mathrm{G}_2)\quad \theta_q\sum_{j\notin P_k(0)\cup P_0}\omega_q(j)&=q-(2N-n+1)-\#(P_1-P_0)+\theta_q\sum_{j\in P_1-P_0}\omega_q(j)\\&=q-(2N-n+1)-\sum_{j\in P_1-P_0}(1-\theta_q\omega_q(j)).\end{aligned}$$

From $(\mathrm{G}_1)$ and $(\mathrm{G}_2)$ we have the equality

$$q-(2N-n+1)=q-(2N-n+1)-\sum_{j\in P_1-P_0}(1-\theta_q\omega_q(j))-\theta_q(n-d_k(q)),$$

so that we have the equality

$$\sum_{j\in P_1-P_0}(1-\theta_q\omega_q(j))+\theta_q(n-d_k(q))=0.$$

If either $P_0\subsetneqq P_1$ or $d_k(q)<n$ holds, we have a contradiction since $\theta_q\omega_q(j)<1$ for $j\in P_1$ and $d_k(q)\le n$ by (15). This means that it must hold both $P_1=P_0$ and $d_k(q)=n$. □

PROPOSITION 14. *For any $j\in P_0$, $\delta(\boldsymbol{a}_j,f)=1$.*

*Proof.* Suppose to the contrary that

$$(20)\qquad \min_{j\in P_0}\delta(\boldsymbol{a}_j,f)=\delta<1.$$

Now, for any positive number $\varepsilon_2$ satisfying

$$(21)\qquad 0<\varepsilon_2<\min\left\{\left(1-\frac{N}{n(N-n+1)}\right)(1-\delta),\frac{(N-n)(1-\Omega_k)}{(N-n+1)(n+1)}\right\},$$

we choose $s\in\boldsymbol{N}$ satisfying $S=\{1,\dots,s\}\supset P_0$, $s\ge p$ and

$$(22)\qquad 2N-n+1-\varepsilon_2<\sum_{j=1}^{s}\delta(\boldsymbol{a}_j,f).$$

Note that $2N - n + 1 < s < \infty$. For this $S$ we use $\theta_s$, $\omega_s$ and $\lambda_s$ instead of $\theta$, $\omega$ and $\lambda$ in Section 2 respectively. Then, by the choice of $s$ the following relations hold from the results obtained in this section:

(a) $\lambda_s = \lambda_\infty = 1/(N - n + 1) = \omega_s(j)$ for $j \in P_0$ (Corollary 4);

(b) $\theta_s = N/n$ (Proposition 12(c)).

By the equality (2) in the proof of Lemma 3 in [9], Lemma 3, Remark 1 and (22) we obtain

$$\sum_{j=1}^{s}(1 - \theta_s\omega_s(j))(1 - \delta(\boldsymbol{a}_j, f)) < \varepsilon_2$$

so that for any $j \in S$

$$(1 - \theta_s\omega_s(j))(1 - \delta(\boldsymbol{a}_j, f)) < \varepsilon_2.$$

By the definition of $\delta$, (a) and (b) given above we have the inquality

$$\left(1 - \frac{N}{n(N - n + 1)}\right)(1 - \delta) < \varepsilon_2,$$

which is a contradiction to (21). This means that $\delta = 1$ and we completes the proof of this proposition. □

As in Proposition 7, we have the following

Proposition 15. *Any $n$ elements of $X - \{\boldsymbol{a}_j \mid j \in P_0\}$ are linearly independent.*

Summarizing Propositions from 8 through 15 given above we obtain the following

Theorem 2. *Suppose that*

(i) $N > n \geq 2$;

(ii) *there are an infinite number of vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots \in X$ satisfying $\delta(\boldsymbol{a}_j, f) > 0$ $(j \in \boldsymbol{N})$ and*

$$\sum_{j=1}^{\infty}\delta(\boldsymbol{a}_j, f) = 2N - n + 1.$$

*If $\Omega_k < 1$ for some $k$ $(1 \leq k \leq n + 1)$, then*

(a) *$X_k(0) \subset \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots\}$ and $\# X_k(0) = N$;*

(b) *there is a subset $P$ of $\boldsymbol{N}$ satisfying*

$$\# P = N - n + 1, \quad d(P) = 1, \quad \delta(\boldsymbol{a}_j, f) = 1 \quad (j \in P)$$

*and*

$$X_k(0) \cap \{\boldsymbol{a}_j \mid j \in P\} = \phi;$$

(c) *any $n$ elements of $X - \{\boldsymbol{a}_j \mid j \in P\}$ are linearly independent.*

## 5. Application

In this section we shall apply the result obtained in Section 3 to exponential curves. For any $n+1$ distinct complex numbers $\mu_1, \mu_2, \dots, \mu_{n+1}$ we define a holomorphic curve $f_e$ by

$$f_e = [e^{\mu_1 z}, e^{\mu_2 z}, \dots, e^{\mu_{n+1} z}].$$

We call it an exponential curve ([11], p. 94). It is easy to see that $f_e$ is transcendental and non-degenerate. We use the notations $X_k(0)$, $\Omega_k$ etc. given in Section 1 in this section. We denote by $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_{n+1}$ the standard basis of $\boldsymbol{C}^{n+1}$.

Let $D$ be the convex polygon spanned around the $n+1$ points $\mu_1, \mu_2, \dots, \mu_{n+1}$ and $\ell$ the length of the polygon, where $\ell = 2|\mu_j - \mu_k|$ if the polygon reduces to a segment with the endpoints $\mu_j$ and $\mu_k$.

Lemma 8 ([11], pp. 95–98). $T(r, f_e) = (\ell/2\pi)r + O(1)$.

Lemma 9. $\#\{k \,|\, \Omega_k < 1; 1 \le k \le n+1\} \ge 2$.

*Proof.* (a) The case when $D$ is an $n+1$-gon.

In this case, the points $\mu_1, \mu_2, \dots, \mu_{n+1}$ are the vertices of $D$. We number without loss of generality the vertices $\mu_j$ $(j = 1, \dots, n+1)$ in asending sequence as one goes arround $D$ in the positive direction. For any $k$ $(1 \le k \le n+1)$, the $n$-gon $D_k$ with the vertices $\mu_1, \dots, \mu_{k-1}, \mu_{k+1}, \dots, \mu_{n+1}$ is convex. Let $\ell_k$ be the length of the circumference of $D_k$. By the representation (1) of $T(r, f)$ due to Cartan given in Introduction, by the definition of $t_k(r, f)$ and by Lemma 8 we have

$$t_k(r, f_e) = \frac{\ell_k}{2\pi} r + O(1),$$

and so we have

$$\Omega_k = \limsup_{r \to \infty} \frac{t_k(r, f)}{T(r, f)} = \frac{\ell_k}{\ell} < 1$$

since $\ell_k < \ell$ as is easily seen.

(b) The case when $D$ is an $m+1$-gon $(2 \le m \le n-1)$.

We may suppose without loss of generality that the vertices of $D$ are $\mu_1, \mu_2, \dots, \mu_{m+1}$. The other points $\mu_{m+2}, \dots, \mu_{n+1}$ are on the circumference of $D$ or inside $D$.

For any $k$ $(1 \le k \le m+1)$, let $D_k$ be the convex polygon surrounding the points $\mu_1, \dots, \mu_{k-1}, \mu_{k+1}, \dots, \mu_{n+1}$ and let $\ell_k$ be the length of the circumference of $D_k$. Then as in (a), we have

$$t_k(r, f_e) = \frac{\ell_k}{2\pi} r + O(1),$$

and so we have

$$\Omega_k = \limsup_{r\to\infty} \frac{t_k(r,f)}{T(r,f)} = \frac{\ell_k}{\ell} < 1$$

since $\ell_k < \ell$ as is easily seen by an application of the triangle inequality.

(c) The case when $D$ reduces to a segment $L$.

We may suppose without loss of generality that

(i) $\mu_1$ and $\mu_{n+1}$ are the endpoints of $L$;

(ii) The points $\mu_1, \mu_2, \dots, \mu_{n+1}$ are in ascending sequence as one goes from $\mu_1$ to $\mu_{n+1}$ on $L$.

Then, as in (a) we have

$$t_1(r,f_e) = \frac{1}{\pi}|\mu_2 - \mu_{n+1}|r + O(1), \quad t_{n+1}(r,f_e) = \frac{1}{\pi}|\mu_1 - \mu_n|r + O(1)$$

and

$$T(r,f_e) = \frac{1}{\pi}|\mu_1 - \mu_{n+1}|r + O(1),$$

from which we obtain

$$\Omega_1 = \frac{|\mu_2 - \mu_{n+1}|}{|\mu_1 - \mu_{n+1}|} < 1 \quad \text{and} \quad \Omega_{n+1} = \frac{|\mu_1 - \mu_n|}{|\mu_1 - \mu_{n+1}|} < 1.$$

□

Lemma 10. 1) $\#\{\boldsymbol{a} \in X \mid \delta(\boldsymbol{a}, f_e) > 0\} \le N(n+1)$.

2) $\delta(\boldsymbol{a}, f_e) = 1$ *if and only if* $\boldsymbol{a} = a\boldsymbol{e}_k$ $(a \neq 0)$ *for some* $k$ $(1 \le k \le n+1)$ *and for some nonzero constant* $a$.

*Proof.* 1) Let $\boldsymbol{a} = (a_1, a_2, \dots, a_{n+1})$ be an element of $X$ satisfying $\delta(\boldsymbol{a}, f_e) > 0$. Then, at least one of $a_1, a_2, \dots, a_{n+1}$ is equal to zero.

In fact, suppose to the contrary that $a_j \neq 0$ $(j = 1, \dots, n+1)$. Then $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_{n+1}$ and $\boldsymbol{a}$ are in general position and by Theorem A for $N = n$, we have

$$\sum_{j=1}^{n+1} \delta(\boldsymbol{e}_j, f_e) + \delta(\boldsymbol{a}, f_e) \le n+1,$$

from which we have $\delta(\boldsymbol{a}, f_e) = 0$ since $\delta(\boldsymbol{e}_j, f_e) = 1$ $(j = 1, \dots, n+1)$.

This means that

$$\{\boldsymbol{a} \in X \mid \delta(\boldsymbol{a}, f_e) > 0\} \subset \bigcup_{k=1}^{n+1} X_k(0)$$

and as $X$ is in $N$-subgeneral position, $\#X_k(0) \le N$ $(k = 1, \dots, n+1)$. Due to these facts we reach to the fact that

$$\#\{\boldsymbol{a} \in X \mid \delta(\boldsymbol{a}, f_e) > 0\} \le N(n+1).$$

2) If $\boldsymbol{a} = a\boldsymbol{e}_k$ $(a \neq 0)$, then it is trivial that $\delta(\boldsymbol{a}, f_e) = 1$.
Conversely, suppose that

$$\boldsymbol{a} = a_{j_1}\boldsymbol{e}_{j_1} + \cdots + a_{j_m}\boldsymbol{e}_{j_m} \quad (a_{j_1} \neq 0, \dots, a_{j_m} \neq 0; 2 \leq m \leq n).$$

Let

$$g_e = [e^{\alpha_1 z}, \dots, e^{\alpha_m z}] \quad (\alpha_p = \mu_{j_p} \ (p = 1, \dots, m)).$$

Then, $g_e$ is a transcendental and non-degenerate exponential curve and by Lemma 8

$$T(r, g_e) = \frac{\ell'}{2\pi} r + O(1),$$

where $(0 <)\ell'(\leq \ell)$ is the length of the convex polygon spanned around the points $\alpha_1, \dots, \alpha_m$.

As $N(r, \boldsymbol{a}, f_e) = N(r, \boldsymbol{a}, g_e)$ and

$$\delta(\boldsymbol{a}, g_e) = 1 - \limsup_{r \to \infty} \frac{N(r, \boldsymbol{a}, g_e)}{T(r, g_e)} = 0$$

by 1) of this lemma, we have

$$\begin{aligned}
\delta(\boldsymbol{a}, f_e) &= 1 - \limsup_{r \to \infty} \frac{N(r, \boldsymbol{a}, f_e)}{T(r, f_e)} \\
&= 1 - \limsup_{r \to \infty} \frac{N(r, \boldsymbol{a}, g_e)}{T(r, g_e)} \cdot \frac{T(r, g_e)}{T(r, f_e)} \\
&= 1 - \frac{\ell'}{\ell} < 1.
\end{aligned}$$

□

Using these lemmas we obtain the following

THEOREM 3. *When $N > n \geq 2$, for any exponential curve $f_e$*

$$\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f_e) < 2N - n + 1.$$

*Proof.* Suppose to the contrary that there exists an exponential curve $f_e$ satisfying

$$\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f_e) = 2N - n + 1.$$

Then, as the number of $\boldsymbol{a} \in X$ satisfying $\delta(\boldsymbol{a}, f_e) > 0$ is finite by Lemma 10-1), let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q$ be the elements of $X$ satisfying

$$\delta(\boldsymbol{a}_j, f_e) > 0 \quad (j = 1, \dots, q)$$

and

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f_e) = 2N - n + 1, \tag{23}$$

where $2N - n + 1 \le q < \infty$.

(I) The case when $q = 2N - n + 1$.

In this case, as $q = 2N - n + 1$ and $\delta(\boldsymbol{a}_j, f_e) \le 1$ we obtain from (23) that

$$\delta(\boldsymbol{a}_j, f_e) = 1 \quad (j = 1, \dots, 2N - n + 1).$$

By Lemma 10-2), for each $j = 1, \dots, 2N - n + 1$ there exists some $k$ $(1 \le k \le n + 1)$ satisfying $\boldsymbol{a}_j = \alpha_j \boldsymbol{e}_k$.

Put for each $k = 1, \dots, n + 1$

$$x_k = \#\{\boldsymbol{a}_j \mid \boldsymbol{a}_j = \alpha_j \boldsymbol{e}_k; \alpha_j \ne 0, 1 \le j \le 2N - n + 1\}.$$

Then, by (23) and $q = 2N - n + 1$ we have

$$\sum_{k=1}^{n+1} x_k = 2N - n + 1. \tag{24}$$

As $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{2N-n+1}$ are in $N$-subgeneral position and $2N - n + 1 > N + 1$, it must hold that $1 \le x_k$ for each $k$ and

$$\sum_{k=1}^{n+1} x_k - x_p \le N, \quad (p = 1, 2, \dots, n + 1). \tag{25}$$

Summing up $n + 1$ inequalities of (25), we obtain

$$n \sum_{k=1}^{n+1} x_k \le N(n + 1). \tag{26}$$

From (24) and (26) we obtain the inequality

$$n(2N - n + 1) \le N(n + 1),$$

from which we have the inequality

$$(N - n)(n - 1) \le 0,$$

which is impossible since $N > n \ge 2$.

(II) The case when $2N - n + 1 < q < \infty$.

By Lemma 9 we may suppose that

$$\Omega_\mu < 1 \quad \text{and} \quad \Omega_\nu < 1 \quad (1 \le \mu \ne \nu \le n + 1).$$

By Theorem 1 for $k = \mu$

(a) $X_\mu(0) \subset \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$ and $\#X_\mu(0) = N$;

(b) There exists a subset $P$ of $Q = \{1, 2, \dots, q\}$ satisfying

$$\#P = N - n + 1, \quad d(P) = 1, \quad \delta(\boldsymbol{a}_j, f_e) = 1 \quad (j \in P)$$

and

$$X_\mu(0) \cap \{\boldsymbol{a}_j \mid j \in P\} = \phi.$$

Note that $\#P = N - n + 1 \geq 2$. By Lemma 10-2) and (b) given above we obtain that

$$\boldsymbol{a}_j = \beta_j \boldsymbol{e}_\mu \quad (j \in P).$$

This means that $\boldsymbol{a}_j \in X_\nu(0)$ $(j \in P)$, and so if we choose $n$ vectors containing at least two vectors of $\{\boldsymbol{a}_j \mid j \in P\}$ from $X_\nu(0)$, they are linearly dependent. On the other hand, by Theorem 1(c) for $k = \nu$, any $n$ elements of $X_\nu(0)$ must be linearly independent. This is a contradiction.

From (I) and (II) we have that there is no exponential curve $f_e$ satisfying

$$\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f_e) = 2N - n + 1.$$

We complete the proof of this theorem. □

*Remark* 3. When $n = 1$, there is an example of exponential curve $f_e$ which satisfies (23) for any $N \geq 2$. Put $f_e = [e^z, e^{2z}]$ and

$$X = \{\boldsymbol{a}_j = j\boldsymbol{e}_1 \ (j = 1, 2, \dots, N), \boldsymbol{a}_j = j\boldsymbol{e}_2 \ (j = N+1, N+2, \dots, 2N)\}.$$

Then, $X$ is in $N$-subgeneral position and

$$\sum_{j=1}^{2N} \delta(\boldsymbol{a}_j, f_e) = 2N.$$

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