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BOUNDED HOLOMORPHIC FUNCTION WITH SOME BOUNDARY BEHAVIOR IN THE UNIT BALL OF C^n

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1. Introduction

As is well known, the following theorem holds ([3]).

Fatou's theorem. *If $f(z)$ is a bounded holomorphic function in the unit disk of $\boldsymbol{C}$, then the limit*

$$\lim_{r\to 1} f(r\zeta)$$

exists for almost every point ζ on the unit circle of $\boldsymbol{C}$.

A similar theorem for bounded holomorphic functions in the unit ball of $\boldsymbol{C}^n$ for $n \geq 2$ also has been proved in [6]. These theorems show a kind of mildness of bounded holomorphic functions, which was one of the back-grounds of the inner function conjecture (for the details, see [6]). Later the existence of an inner function was proved by Aleksandrov [1] and Löw [4], which causes interest in the bounded holomorphic functions with wild boundary behavior along a radius of the ball.

As a study of boundary behavior along a radius of functions f defined in the unit disk or ball, we consider the following set:

$$\bigcap_{T<1} \overline{\{f(t\zeta) : T < t < 1\}},$$

where ζ is a boundary point. This set is called *the radial cluster set* of $f(z)$ at ζ. When the limit of $f(z)$ along the radius terminating at ζ exists, this set consists of one point. In [5], the author has shown that various sets appear as the radial cluster sets of holomorphic functions.

In this paper we deal with the following problem:
Does there exist a bounded holomorphic function in the unit ball of arbitrary dimension whose radial cluster set is "big" at every point belonging to a given subset of the boundary of the unit ball?

Our main theorem states the following as an answer to this problem:

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Main Theorem. *Let* $\{\zeta_k\}_{k=1}^{m}$ *be an arbitrary discrete subset of the boundary of the unit ball of* $\boldsymbol{C}^n$, *where* $1 \leq m \leq +\infty$, $n \geq 1$ *and* $\zeta_k \neq \zeta_l$ *if* $k \neq l$. *Then there exists a bounded holomorphic function* $f(z)$ *in the unit ball of* $\boldsymbol{C}^n$ *whose radial cluster set at* ζ_k

$$\bigcap_{T<1} \overline{\{f(t\zeta_k) : T < t < 1\}}$$

contains a closed disk of positive radius for all k.

This gives also some type of counter part of Fatou's theorem of arbitrary dimension. We prove Main theorem in Section 3.

2. Construction of the basic function

Lemma 1. *Let* $(2\pi\boldsymbol{T})^n = \boldsymbol{R}^n/(2\pi\boldsymbol{Z})^n$ *be a torus of dimension* n, *and let* $[x_1, x_2, \dots, x_n] \in (2\pi\boldsymbol{T})^n$ *denote the residue class of modulus* $(2\pi\boldsymbol{Z})^n$ *to which* $(x_1, x_2, \dots, x_n)$ *belongs. For* $(\omega_1, \omega_2, \dots, \omega_n) \in \boldsymbol{R}^n$, *define the map*

$$\varphi : [0, +\infty) \ni t \mapsto [x_1 + 2\pi\omega_1 t, x_2 + 2\pi\omega_2 t, \dots, x_n + 2\pi\omega_n t] \in (2\pi\boldsymbol{T})^n$$

for arbitrarily fixed $(x_1, x_2, \dots, x_n) \in \boldsymbol{R}^n$. *Then the image of* φ *is dense in* $(2\pi\boldsymbol{T})^n$ *if and only if* $\omega_1, \omega_2, \dots, \omega_n$ *are linearly independent over* $\boldsymbol{Z}$.

This map is classically known as *Kronecker's flow*. See, for example, [2] for Lemma 1.

Lemma 2. *Let* K *and* L *be positive real numbers which are linearly independent over* $\boldsymbol{Z}$. *Define*

$$F(z) = \exp(2\pi i K z) + \exp(2\pi i L z) \quad \text{for } z \in \boldsymbol{C}.$$

Then $\{F(t) : t \geq 0\}$ *is a dense subset of* $\{w \in \boldsymbol{C} : |w| \leq 2\}$ *and*

$$|F(z)| \leq \exp(-2\pi K \operatorname{Im} z) + \exp(-2\pi L \operatorname{Im} z) \quad \text{for } z \in \boldsymbol{C}.$$

Proof. We first consider $F(z)$ for $z = t$ in $[0, +\infty)$:

$$F(t) = \exp(2\pi i K t) + \exp(2\pi i L t).$$

For all $w \in \{w \in \boldsymbol{C} : |w| \leq 2\}$, there exist $w_1, w_2 \in \boldsymbol{C}$ such that $|w_1| = |w_2| = 1$ and $w = w_1 + w_2$. Since the image of the map $t \mapsto [2\pi K t, 2\pi L t] \in (2\pi\boldsymbol{T})^2$ is dense in $(2\pi\boldsymbol{T})^2$ by Lemma 1, we can find $(\exp(2\pi i K t), \exp(2\pi i L t))$ which is arbitrarily near to (w_1, w_2) in $\boldsymbol{C}^2$. This proves the first statement.

By the triangle inequality,

$$\begin{aligned} |F(z)| &\leq |\exp(2\pi i K(\operatorname{Re} z + i \operatorname{Im} z))| + |\exp(2\pi i L(\operatorname{Re} z + i \operatorname{Im} z))| \\ &= \exp(-2\pi K \operatorname{Im} z) + \exp(-2\pi L \operatorname{Im} z). \end{aligned}$$

Q.E.D.

Throughout this paper, we use the following notation:

$$\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w_k}, \quad |z| = \sqrt{\langle z, z\rangle} \quad \text{for } z = (z_1, \dots, z_n), w = (w_1, \dots, w_n) \in \boldsymbol{C}^n;$$

$$B_n = \{z \in \boldsymbol{C}^n : |z| < 1\}, \quad \partial B_n = \{z \in \boldsymbol{C}^n : |z| = 1\} \quad \text{and} \quad \Delta = B_1.$$

Lemma 3. *Let K and L be as in Lemma 2. For an arbitrary point $\zeta \in \partial B_n$, define*

$$\tilde{f}(z) = \exp(-2\pi i K \log(1 - \langle z, \zeta\rangle)) + \exp(-2\pi i L \log(1 - \langle z, \zeta\rangle)), \quad z \in B_n,$$

where the argument θ of the logarithm is taken as $-\pi < \theta \le \pi$. Then $\tilde{f}(z)$ is holomorphic in B_n and $\|\tilde{f}\| = \sup_{z \in B_n} |f(z)| \le \exp(\pi^2 K) + \exp(\pi^2 L)$. Moreover, the image of the radius of B_n terminating at ζ by $\tilde{f}(z)$ is a dense subset of $\{w \in \boldsymbol{C} : |w| \le 2\}$.

Proof. Since $|\langle z, \zeta\rangle| < 1$ for all z in B_n by Schwarz inequality, the function $\langle \cdot, \zeta\rangle$ maps B_n onto Δ, and hence, the image of B_n by $1 - \langle \cdot, \zeta\rangle$ is $\{z \in \boldsymbol{C} : |z - 1| < 1\}$. Since the argument is chosen as in Lemma 3, then we have

$$-\frac{\pi}{2} < \arg(1 - \langle z, \zeta\rangle) < \frac{\pi}{2}$$

and $-\log(1 - \langle z, \zeta\rangle)$ is a single valued holomorphic function in z, which shows that $\tilde{f}(z)$ is holomorphic in B_n. On the other hand, we easily have

$$\begin{aligned} |\tilde{f}(z)| &\le \exp(2\pi K \arg(1 - \langle z, \zeta\rangle)) + \exp(2\pi L \arg(1 - \langle z, \zeta\rangle)) \\ &< \exp(\pi^2 K) + \exp(\pi^2 L) \end{aligned}$$

by Lemma 2 and the fact that $\arg(1 - \langle z, \zeta\rangle) < \pi/2$. Note that the radius of B_n terminating at ζ is the set $\{t\zeta : 0 \le t < 1\}$. Therefore, by Lemma 2 and

$$\tilde{f}(t\zeta) = \exp(-2\pi i K \log(1 - t)) + \exp(-2\pi i L \log(1 - t)),$$

we conclude that the image of the radius terminating at ζ is a dense subset of $\{w \in \boldsymbol{C} : |w| \le 2\}$. Q.E.D.

Remarks. (1) For an arbitrary positive real number M, let

$$h(z) = \frac{M}{\exp(\pi^2 K) + \exp(\pi^2 L)} \tilde{f}(z).$$

Suppose that K and L are positive real numbers which are linearly independent over $\boldsymbol{Z}$. Then, by Lemma 3, $\|h\| \le M$ and the image of the radius of B_n terminating at ζ is a dense subset of the closed disk

$$\left\{ w \in \boldsymbol{C} : |w| \le \frac{2M}{\exp(\pi^2 K) + \exp(\pi^2 L)} \right\}.$$

(2) The function $h(z)$ has the only singularity ζ as a function on the closure $\overline{B_n}$ of B_n. Clearly, $h(z)$ is a continuous function in $\overline{B_n}\backslash\{\zeta\}$ and $|h(z)| \leq M$ for all $z \in \overline{B_n}\backslash\{\zeta\}$, since $-\pi/2 < \arg(1-\langle z,\zeta\rangle) < \pi/2$ for all $z \in \overline{B_n}\backslash\{\zeta\}$.

(3) The radial cluster set of $h(z)$ at ζ is a closed disk as in (1), namely

$$\bigcap_{T<1} \overline{\{h(t\zeta) : T < t < 1\}} = \left\{w \in \boldsymbol{C} : |w| \leq \frac{2M}{\exp(\pi^2 K) + \exp(\pi^2 L)}\right\}.$$

3. Proof of the main theorem

We first prove the following lemma on the radial cluster set:

Lemma 4. *Let $g_1(z)$ and $g_2(z)$ be functions which are defined in B_n and let Λ be the radial cluster set of $g_1(z)$ at a point $\zeta \in \partial B_n$. Suppose that $g_2(z)$ has the radial limit α at the point ζ, i.e.,*

$$\lim_{t\to 1} g_2(t\zeta) = \alpha.$$

Then the radial cluster set of $g_1 + g_2$ at ζ is $\Lambda + \alpha$.

Proof. Let ε be an arbitrary positive real number and p an arbitrary point in Λ. By assumptions of Lemma 4, there exist a sequence $\{t_k\}_{k=1}^{\infty}$ with $t_k \uparrow 1$ as $k \to \infty$ and a natural number N such that if $k > N$, then

$$|g_1(t_k\zeta) - p| < \frac{\varepsilon}{2} \quad \text{and} \quad |g_2(t_k\zeta) - \alpha| < \frac{\varepsilon}{2}.$$

Hence if $k > N$, then

$$|(p+\alpha) - (g_1+g_2)(t_k\zeta)| \leq |p - g_1(t_k\zeta)| + |\alpha - g_2(t_k\zeta)| < \varepsilon,$$

which yields that $p+\alpha$ is a point of the radial cluster set of g_1+g_2 at ζ. Consequently, we see

$$\Lambda + \alpha \subset \bigcap_{T<1} \overline{\{(g_1+g_2)(t\zeta) : T < t < 1\}}.$$

Conversely, suppose that s is a point of the radial cluster set of g_1+g_2 at ζ, which is not in $\Lambda+\alpha$. Then $r := s - \alpha \notin \Lambda$, and so we can find some positive number d such that $\{w : |w-r| \leq 2d\} \cap \Lambda = \phi$. Take an arbitrary monotone increasing sequence $\{q_k\}_{k=1}^{\infty}$ of non-negative numbers which converges to 1. Then, there exist a natural number N_d such that $|g_1(q_k\zeta) - r| \geq 2d$ if $k > N_d$. Thus if $k > N_d$, then

$$\begin{aligned}|(g_1+g_2)(q_k\zeta) - s| &= |g_1(q_k\zeta) + g_2(q_k\zeta) - (r+\alpha)| \\ &= |(g_1(q_k\zeta) - r) + (g_2(q_k\zeta) - \alpha)| \\ &\geq ||g_1(q_k\zeta) - r| - |g_2(q_k\zeta) - \alpha||.\end{aligned}$$

Clearly $|g_2(q_k\zeta) - \alpha| < \varepsilon$ for an arbitrary positive number ε if k is suffeciently large. If we choose ε less than d,

$$|(g_1 + g_2)(q_k\zeta) - s| \geq 2d - \varepsilon > d,$$

which contradicts the fact that s is a point of the radial cluster set of $g_1 + g_2$ at ζ. Thus we see

$$\bigcap_{T<1} \overline{\{(g_1 + g_2)(t\zeta) : T < t < 1\}} \subset \Lambda + \alpha.$$

Therefore we conclude that

$$\bigcap_{T<1} \overline{\{(g_1 + g_2)(t\zeta) : T < t < 1\}} = \Lambda + \alpha.$$

Q.E.D.

Proof of the main theorem. Let M be an arbitrary positive number, and let

$$f_k(z) = \frac{M}{\exp(\pi^2 K) + \exp(\pi^2 L)}(\exp(-2\pi i K \log(1 - \langle z, \zeta_k \rangle)) + \exp(-2\pi i L \log(1 - \langle z, \zeta_k \rangle)))$$

for z in B_n, where K and L are positive real numbers which are linearly independent over $\boldsymbol{Z}$. We choose the branch of the logarithm as in Lemma 3. Note that each $f_k(z)$ is the same function as $h(z)$ in Remarks if we substitute ζ_k for ζ. Every $f_k(z)$ is holomorphic in B_n.

When m is finite, let

$$f(z) = \frac{1}{m}\sum_{k=1}^{m} f_k(z).$$

Then $f(z)$ is holomorphic in B_n and $\|f\| \leq M$ by (1) in Remarks. We write

$$f(z) = \frac{1}{m} f_1(z) + \frac{1}{m}\sum_{k=2}^{m} f_k(z).$$

The radial cluster set of the first term at ζ_1 is the closed disk

$$\left\{ w \in \boldsymbol{C} : |w| \leq \frac{1}{m} \cdot \frac{2M}{\exp(\pi^2 K) + \exp(\pi^2 L)} \right\},$$

and the second term is continuous at ζ_1, which are derived by (2) and (3) in Remarks. Hence, by Lemma 4, the radial cluster set of $f(z)$ at ζ_1 is the closed disk

$$\left\{ w \in \boldsymbol{C} : \left| w - \frac{1}{m}\sum_{k=2}^{m} f_k(\zeta_1) \right| \leq \frac{1}{m} \cdot \frac{2M}{\exp(\pi^2 K) + \exp(\pi^2 L)} \right\}.$$

We can also derive the same result at each ζ_p for $2 \le p \le m$, by decomposing $f(z)$ into sum

$$f(z) = \frac{1}{m} f_p(z) + \frac{1}{m} \sum_{k=1, k \neq p}^{m} f_k(z).$$

These show that the radial cluster set of $f(z)$ at ζ_k is a closed disk for every k with $1 \le k \le m$.

When m is infinite, let

$$f(z) = \sum_{k=1}^{\infty} a_k f_k(z),$$

where $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive numbers satisfying $\sum_{k=1}^{\infty} a_k = 1$. Then

$$|f(z)| \le \sum_{k=1}^{\infty} |a_k f_k(z)| \le \sum_{k=1}^{\infty} a_k \|f_k\| \le M \sum_{k=1}^{\infty} a_k = M < +\infty,$$

which shows that $f(z)$ is a bounded holomorphic function in B_n.

Next, we see the radial cluster set of $f(z)$ at ζ_p for $p \ge 1$. Without loss of generality, we may assume $p = 1$. Let

$$F_n(z) = \sum_{k=1}^{n} a_k f_k(z) \quad \text{for } n = 1, 2, \ldots.$$

Since $F_n(z)$ absolutely and uniformly converges to $f(z)$ on B_n, there exists a natural number N_ε for any positive ε such that

(1) $$\|f - F_n\| < \varepsilon \quad \text{for } n > N_\varepsilon$$

and

(2) $$\|F_m - F_n\| < \varepsilon \quad \text{for } m, n > N_\varepsilon.$$

Note that

$$\bigcap_{T<1} \overline{\{a_1 f_1(t\zeta_1) : T < t < 1\}} = \left\{ w \in \boldsymbol{C} : |w| \le \frac{2a_1 M}{\exp(\pi^2 K) + \exp(\pi^2 L)} \right\}.$$

Set

$$\delta = \frac{1}{10} \cdot \frac{2a_1 M}{\exp(\pi^2 K) + \exp(\pi^2 L)}.$$

By (2), we have a natural number N such that

(3) $$\|F_n - F_{N+1}\| < \delta \quad \text{for } n > N + 1.$$

Decompose $F_{N+1}(z)$ as

$$F_{N+1}(z) = a_1 f_1(z) + \sum_{k=2}^{N+1} a_k f_k(z).$$

Then, the second term is continuous at ζ_1, and radial cluster set of $a_1 f_1(z)$ at ζ_1 is a closed disk as above. Hence, by Lemma 4, the radial cluster set of F_{N+1} at ζ_1 is the closed disk

$$\bar{D}_{N+1} = \left\{ w \in \boldsymbol{C} : \left| w - \sum_{k=2}^{N+1} a_k f_k(\zeta_1) \right| \leq \frac{2a_1 M}{\exp(\pi^2 K) + \exp(\pi^2 L)} \right\}.$$

Also, when $n > N+1$, the radial cluster set of $F_n(z)$ at ζ_1 is the closed disk

$$\bar{D}_n = \left\{ w \in \boldsymbol{C} : \left| w - \sum_{k=2}^{n} a_k f_k(\zeta_1) \right| \leq \frac{2a_1 M}{\exp(\pi^2 K) + \exp(\pi^2 L)} \right\}.$$

Notice that each $\bar{D}_n$ and $\bar{D}_{N+1}$ have the same radius 10δ. We also note that, if $n > N+1$, the distance between the centers of $\bar{D}_n$ and $\bar{D}_{N+1}$ is

$$\left| \sum_{k=2}^{n} a_k f_k(\zeta_1) - \sum_{k=2}^{N+1} a_k f_k(\zeta_1) \right| = \left| \sum_{k=N+2}^{n} a_k f_k(\zeta_1) \right|.$$

On the other hand, by (3) we have

$$\left| \sum_{k=N+2}^{n} a_k f_k(t\zeta_1) \right| = |F_n(t\zeta_1) - F_{N+1}(t\zeta_1)| < \delta$$

for all t with $0 \leq t < 1$. Since $\sum_{k=N+2}^{n} a_k f_k(z)$ is continuous at ζ_1 for every $n > N+1$, letting $t \to 1$, we obtain that

$$\left| \sum_{k=N+2}^{n} a_k f_k(\zeta_1) \right| \leq \delta.$$

Hence the distance between the centers of the disks $\bar{D}_{N+1}$ and $\bar{D}_n$ is less than or equel to δ for $n > N+1$. This indicates that every $\bar{D}_n$ contains the closed disk

$$\bar{D} = \left\{ w \in \boldsymbol{C} : \left| w - \sum_{k=2}^{N+1} a_k f_k(\zeta_1) \right| \leq \delta \right\}$$

when $n > N+1$. This fact shows that the closure of the image of the radius of B_n terminating at ζ_1 by $F_n(z)$ contains $\bar{D}$.

Take an arbitrary point $\alpha \in \bar{D}$ and an arbitrary positive number ε. By (1), we can choose some natural number N_ε such that

$$|f(t\zeta_1) - F_n(t\zeta_1)| < \frac{\varepsilon}{2} \tag{4}$$

for all t with $0 \leq t < 1$ and all $n > N_\varepsilon$. Set

$$n = \max\{N+1, N_\varepsilon\} + 1.$$

Then, since α belongs to $\bar{D}_n$, there exists a sequence $\{t_k\}_{k=1}^{\infty}$ with $t_k \uparrow 1$ and a natural number A_ε such that

$$|F_n(t_k\zeta_1) - \alpha| < \frac{\varepsilon}{2} \quad \text{for } k > A_\varepsilon. \tag{5}$$

Thus, for $k > A_\varepsilon$, by (4) and (5), we have

$$\begin{aligned}|f(t_k\zeta_1) - \alpha| &= |f(t_k\zeta_1) - F_n(t_k\zeta_1) + F_n(t_k\zeta_1) - \alpha| \\ &\le |f(t_k\zeta_1) - F_n(t_k\zeta_1)| + |F_n(t_k\zeta_1) - \alpha| < \varepsilon.\end{aligned}$$

This shows that α is contained in the radial cluster set of $f(z)$ at ζ_1. Hence

$$\bigcap_{T<1} \overline{\{f(t\zeta_1) : T < t < 1\}} \supset \bar{D}.$$

Q.E.D.

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